

Master Program in *Data Science and Business Informatics*

Statistics for Data Science

Lesson 10 - Moments. Functions of random variables

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Moments

- Let X be a continuous random variable with density function $f(x)$
- k^{th} moment of X , if it exists, is:

$$E[X^k] = \int_{-\infty}^{\infty} x^k f(x) dx$$

- $\mu = E[X]$ is the first moment of X
- k^{th} central moment of X is:

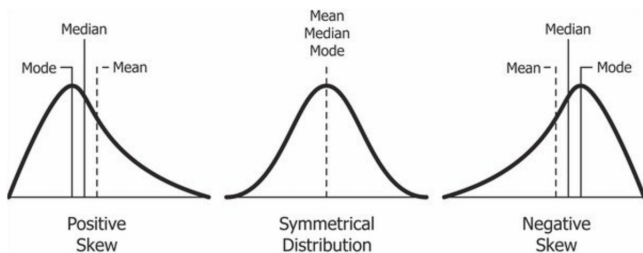
$$\mu_k = E[(X - \mu)^k] = \int_{-\infty}^{\infty} (x - \mu)^k f(x) dx$$

- $\sigma = \sqrt{E[(X - \mu)^2]}$ standard deviation is the square root of the second central moment
- k^{th} standardized moment of X is:

$$\tilde{\mu}_k = \frac{\mu_k}{\sigma^k} = E \left[\left(\frac{X - \mu}{\sigma} \right)^k \right]$$

Skewness

- $\tilde{\mu}_1 = E[(X-\mu)]/\sigma = 0$ since $E[X - \mu] = 0$
- $\tilde{\mu}_2 = E[(X-\mu)^2]/\sigma^2 = 1$ since $\sigma^2 = E[(X - \mu)^2]$
- $\tilde{\mu}_3 = E[(X-\mu)^3]/\sigma^3$ *[(Pearson's moment) coefficient of skewness]*
- Skewness indicates direction and magnitude of a distribution's deviation from symmetry

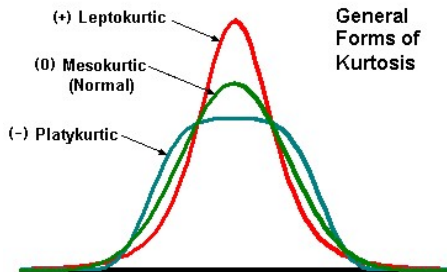


- E.g., for $X \sim \text{Exp}(\lambda)$, $\tilde{\mu}_3 = 2$

Prove it!

Kurtosis

- $\tilde{\mu}_4 = E[(\frac{X-\mu}{\sigma})^4]$ [(Pearson's moment) coefficient of kurtosis]
- For $X \sim \mathcal{N}(\mu, \sigma)$, $\tilde{\mu}_4 = 3$ $\tilde{\mu}_4 - 3$ is called *kurtosis in excess*
- Kurtosis is a measure of the dispersion of X around the two values $\mu \pm \sigma$



- $\tilde{\mu}_4 > 3$ *Leptokurtic* (slender) distribution has *fatter* tails. May have outlier problems.
- $\tilde{\mu}_4 < 3$ *Platykurtic* (broad) distribution has *thinner* tails

See R script

Functions of two or more random variables: expectation

- $V = \pi HR^2$ be the volume of a vase of height H and radius R
- $g(H, R) = \pi HR^2$ is a random variable (function of random variables)
- $P_V(V = 3) = P_{HR}(\pi HR^2 = 3)$
- How to calculate $E[V]$?

TWO-DIMENSIONAL CHANGE-OF-VARIABLE FORMULA. Let X and Y be random variables, and let $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function. If X and Y are *discrete* random variables with values a_1, a_2, \dots and b_1, b_2, \dots , respectively, then

$$E[g(X, Y)] = \sum_i \sum_j g(a_i, b_j) P(X = a_i, Y = b_j).$$

If X and Y are *continuous* random variables with joint probability density function f , then

$$E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y) dx dy.$$

If $H \perp\!\!\!\perp R$:

$$E[V] = E[\pi HR^2] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \pi hr^2 f_H(h) f_R(r) dh dr$$

Linearity of expectations

Theorem. For X and Y random variables, and $s, t \in \mathbb{R}$:

$$E[rX + sY + t] = rE[X] + sE[Y] + t$$

Proof. (discrete case)

$$\begin{aligned} E[rX + Ys + t] &= \sum_a \sum_b (ra + sb + t)P(X = a, Y = b) \\ &= \left(r \sum_a \sum_b aP(X = a, Y = b) \right) + \left(s \sum_a \sum_b bP(X = a, Y = b) \right) + \left(t \sum_a \sum_b P(X = a, Y = b) \right) \\ &= \left(r \sum_a aP(X = a) \right) + \left(s \sum_b bP(Y = b) \right) + t = rE[X] + sE[Y] + t \end{aligned}$$

Corollary. $E[a_0 + \sum_{i=1}^n a_i X_i] = a_0 + \sum_{i=1}^n a_i E[X_i]$

Corollary. $X \leq Y$ implies $E[X] \leq E[Y]$

Proof. $Z = Y - X \geq 0$ implies $E[Z] = E[Y] - E[X] \geq 0$, i.e., $E[Y] \geq E[X]$.

Applications

- Expectation of some discrete distributions
 - ▶ $X \sim \text{Ber}(p)$ $E[X] = p$
 - ▶ $X \sim \text{Bin}(n, p)$ $E[X] = n \cdot p$
 - Because $X = \sum_{i=1}^n X_i$ for $X_1, \dots, X_n \sim \text{Ber}(p)$
 - ▶ $X \sim \text{Geo}(p)$ $E[X] = \frac{1}{p}$
 - ▶ $X \sim \text{NBin}(n, p)$ $E[X] = \frac{n \cdot (1-p)}{p}$
 - Because $X = \sum_{i=1}^n X_i - n$ for $X_1, \dots, X_n \sim \text{Geo}(p)$
- Expectation of some continuous distributions
 - ▶ $X \sim \text{Exp}(\lambda)$ $E[X] = 1/\lambda$
 - ▶ $X \sim \text{Erl}(n, \lambda)$ $E[X] = \frac{n}{\lambda}$
 - Because $X = \sum_{i=1}^n X_i$ for $X_1, \dots, X_n \sim \text{Exp}(\lambda)$

Expectation of product and quotients

Theorem. For $X \perp\!\!\!\perp Y$, we have: $E[XY] = E[X]E[Y]$

Prove it!

PROPAGATION OF INDEPENDENCE. Let X_1, X_2, \dots, X_n be independent random variables. For each i , let $h_i : \mathbb{R} \rightarrow \mathbb{R}$ be a function and define the random variable

$$Y_i = h_i(X_i).$$

Then Y_1, Y_2, \dots, Y_n are also independent.

Corollary. For $X \perp\!\!\!\perp Y$ and $Y > 0$, we have: $E[X/Y] \geq E[X]/E[Y]$

Proof. $X \perp\!\!\!\perp Y$ implies $X \perp\!\!\!\perp 1/Y$. By theorem above:

$$E[X/Y] = E[X \cdot 1/Y] = E[X]E[1/Y] \geq E[X]/E[Y]$$

because by Jensen's inequality $E[1/Y] \geq 1/E[Y]$ since $1/y$ is convex for $y > 0$. □

Exercise at home. Show that $E[X/Y] = E[X]/E[Y]$ is a false claim.

Law of iterated/total expectation

Conditional expectation

$$E[X|Y = b] = \sum_i a_i p(a_i|b) \qquad E[X|Y = y] = \int_{-\infty}^{\infty} xf(x|y)dx$$

Theorem. (Law of iterated/total expectation)

$$E_Y[E[X|Y]] = E[X]$$

Proof. (for X, Y discrete random variables)

$$E_Y[E[X|Y]] = \sum_j \sum_i a_i p_{X|Y}(a_i|b_j) p_Y(b_j) = \sum_j \sum_i a_i p_{XY}(a_i, b_j) = \sum_i a_i p_X(a_i) = E[X]$$

Example (cfr the example from Lesson 1 on the Law of total probability)

- Factory 1's light bulbs working hours $\sim \text{Exp}(1/1000)$
- Factory 2's light bulbs working hours $\sim \text{Exp}(1/2000)$
- Factory 1 supplies 60% of the total bulbs on the market and Factory 2 supplies 40% of it.
- *What is the average work hour of a light bulb on the market?*

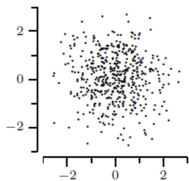
Variance of the sum and covariance

$$\begin{aligned} \text{Var}(X + Y) &= E[(X + Y - E[X + Y])^2] = E[((X - E[X]) + (Y - E[Y]))^2] \\ &= E[(X - E[X])^2] + E[(Y - E[Y])^2] + 2E[(X - E[X])(Y - E[Y])] \\ &= \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y) \end{aligned}$$

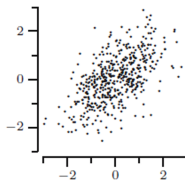
Covariance

The *covariance* $\text{Cov}(X, Y)$ of two random variables X and Y is the number:

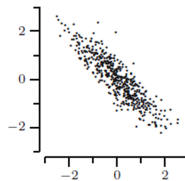
$$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])]$$



Uncorrelated



Positively correlated



Negatively correlated

Covariance

Theorem. $\text{Cov}(X, Y) = E[XY] - E[X]E[Y]$

Prove it!

- If X and Y are independent ($X \perp\!\!\!\perp Y$):

$$\text{Cov}(X, Y) = 0 \qquad \text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$$

- But there are X and Y uncorrelated (ie., $\text{Cov}(X, Y) = 0$) that are dependent!
- Variances of some discrete distributions
 - ▶ $X \sim \text{Ber}(p)$ $\text{Var}(X) = p(1 - p)$
 - ▶ $X \sim \text{Bin}(n, p)$ $\text{Var}(X) = np(1 - p)$
 - Because $X = \sum_{i=1}^n X_i$ for $X_1, \dots, X_n \sim \text{Ber}(p)$ and independent
 - ▶ $X \sim \text{Geo}(p)$ $\text{Var}(X) = \frac{1-p}{p^2}$
 - ▶ $X \sim \text{NBin}(n, p)$ $\text{Var}(X) = n \frac{1-p}{p^2}$
 - Because $X = \sum_{i=1}^n X_i - n$ for $X_1, \dots, X_n \sim \text{Geo}(p)$ and independent
- Variances of some continuous distributions
 - ▶ $X \sim \text{Exp}(\lambda)$ $\text{Var}(X) = 1/\lambda^2$
 - ▶ $X \sim \text{Erl}(n, \lambda)$ $\text{Var}(X) = \frac{n}{\lambda^2}$
 - Because $X = \sum_{i=1}^n X_i$ for $X_1, \dots, X_n \sim \text{Exp}(\lambda)$ and independent

Covariance and covariance matrix

COVARIANCE UNDER CHANGE OF UNITS. Let X and Y be two random variables. Then

$$\text{Cov}(rX + s, tY + u) = rt \text{Cov}(X, Y)$$

for all numbers r, s, t , and u .

- Hence, $\text{Var}(rX + sY + t) = r^2 \text{Var}(X) + s^2 \text{Var}(Y) + 2rs \text{Cov}(X, Y)$
- **Bivariate** Normal/Gaussian distribution:

$$(X, Y) \sim \mathcal{N}((\mu_X, \mu_Y), \begin{pmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{pmatrix})$$

- ▶ where marginals are $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$, $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$, and $\text{Cov}(X, Y) = \sigma_{XY}$
 - ▶ **Covariance matrix** $\Sigma_{ij} = \text{Cov}(X_i, X_j)$ for a vector $\mathbf{X} = (X_1, \dots, X_n)$ of r.v.'s
- Covariance depends on the unit of measure!

See R script lesson 08

Correlation coefficient

DEFINITION. Let X and Y be two random variables. The **correlation coefficient** $\rho(X, Y)$ is defined to be 0 if $\text{Var}(X) = 0$ or $\text{Var}(Y) = 0$, and otherwise

$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}}.$$

- Correlation coefficient is *dimensionless* (not affected by change of units)
 - ▶ E.g., if X and Y are in Km, then $\text{Cov}(X, Y)$, $\text{Var}(X)$ and $\text{Var}(Y)$ are in Km^2
- Moreover: $-1 \leq \rho(X, Y) \leq 1$
 - ▶ The bounds are derived from the **Cauchy–Schwarz's inequality**:

$$E[|XY|] \leq \sqrt{E[X^2]} \sqrt{E[Y^2]}$$

Proof. For any $u, w \in \mathbb{R}$, we have $2|uw| \leq u^2 + w^2$. Therefore, $2|UW| \leq U^2 + W^2$ for r.v.'s U and V . By defining $U = X/\sqrt{E[X^2]}$ and $W = Y/\sqrt{E[Y^2]}$ (*), we have

$2 \cdot |XY|/\sqrt{E[X^2]}\sqrt{E[Y^2]} \leq X^2/E[X^2] + Y^2/E[Y^2]$. Taking the expectations, we conclude:

$$2 \cdot E[|XY|]/\sqrt{E[X^2]}\sqrt{E[Y^2]} \leq 2.$$

(*) The case $E[X^2] = 0$ or $E[Y^2] = 0$ is left as an exercise. □

Bivariate Normal/Gaussian distribution

$$(X, Y) \sim \mathcal{N}((\mu_X, \mu_Y), \begin{pmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{pmatrix})$$

where marginals are $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$, $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$, and $\text{Cov}(X, Y) = \sigma_{XY}$

- Since $\sigma_{XY} = \rho(X, Y) \cdot \sigma_X \cdot \sigma_Y$:

$$(X, Y) \sim \mathcal{N}((\mu_X, \mu_Y), \begin{pmatrix} \sigma_X^2 & \rho(X, Y) \cdot \sigma_X \cdot \sigma_Y \\ \rho(X, Y) \cdot \sigma_X \cdot \sigma_Y & \sigma_Y^2 \end{pmatrix})$$

- Density of $\mathcal{N}((0, 0), (1, \sigma_{XY}, \sigma_{XY}, 1))$:

$$f(x, y) = \frac{1}{2\pi\sqrt{1 - \sigma_{XY}^2}} e^{-\frac{1}{2(1 - \sigma_{XY}^2)}(x^2 + y^2 - 2xy\sigma_{XY})}$$

- Useful facts for (X, Y) bivariate Normal:

- ▶ for (X, Y) bivariate Normal: $\rho(X, Y) = 0$ iff $X \perp\!\!\!\perp Y$, i.e., uncorrelation equals independence
- ▶ (X, Y) bivariate Normal iff $aX + bY$ is Normal for any $a, b \in \mathbb{R}$

Sum of independent Normal random variables

- See Lesson 04 and Lesson 08 for convolution formulas

ADDING TWO INDEPENDENT CONTINUOUS RANDOM VARIABLES. Let X and Y be two independent continuous random variables, with probability density functions f_X and f_Y . Then the probability density function f_Z of $Z = X + Y$ is given by

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y)f_Y(y) \, dy$$

for $-\infty < z < \infty$.

Theorem. If $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ and $X \perp\!\!\!\perp Y$, then:

$$Z = X + Y \sim \mathcal{N}(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$$

Proof. See [T, Sect. 11.2]

□

- In general: $Z = rX + sY + t \sim \mathcal{N}(r\mu_X + s\mu_Y + t, r^2\sigma_X^2 + s^2\sigma_Y^2)$
- The converse of the theorem also holds: [[Lévy-Cramér theorem](#)]
 - ▶ If $X \perp\!\!\!\perp Y$ and $Z = X + Y$ is normally distributed, then X and Y follow a normal distribution.

Extremes of independent random variables

THE DISTRIBUTION OF THE MAXIMUM. Let X_1, X_2, \dots, X_n be n independent random variables with the same distribution function F , and let $Z = \max\{X_1, X_2, \dots, X_n\}$. Then

$$F_Z(a) = (F(a))^n.$$

- $P(Z \leq a) = P(X_1 \leq a, \dots, X_n \leq a) = \prod_{i=1}^n P(X_i \leq a) = (F(a))^n$
- Example: maximum water level over 365 days assuming water level on a day is $U(0, 1)$
- Example: maximum of two rolls of **a die with 4 sides**

THE DISTRIBUTION OF THE MINIMUM. Let X_1, X_2, \dots, X_n be n independent random variables with the same distribution function F , and let $V = \min\{X_1, X_2, \dots, X_n\}$. Then

$$F_V(a) = 1 - (1 - F(a))^n.$$

- $P(V \leq a) = 1 - P(X_1 > a, \dots, X_n > a) = 1 - \prod_{i=1}^n (1 - P(X_i \leq a)) = 1 - ((1 - F(a))^n)$

Product and quotient of independent random variables

PRODUCT OF INDEPENDENT CONTINUOUS RANDOM VARIABLES. Let X and Y be two independent continuous random variables with probability densities f_X and f_Y . Then the probability density function f_Z of $Z = XY$ is given by

$$f_Z(z) = \int_{-\infty}^{\infty} f_Y\left(\frac{z}{x}\right) f_X(x) \frac{1}{|x|} dx$$

for $-\infty < z < \infty$.

QUOTIENT OF INDEPENDENT CONTINUOUS RANDOM VARIABLES. Let X and Y be two independent continuous random variables with probability densities f_X and f_Y . Then the probability density function f_Z of $Z = X/Y$ is given by

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(zx) f_Y(x) |x| dx$$

for $-\infty < z < \infty$.

- $X, Y \sim \mathcal{N}(0, 1)$ independent, $Z = X/Y \sim \text{Cau}(0, 1)$ where:

$$f_Z(x) = \frac{1}{\pi(1+x^2)}$$