

Master Program in *Data Science and Business Informatics*

Statistics for Data Science

Lesson 04 - Discrete random variables

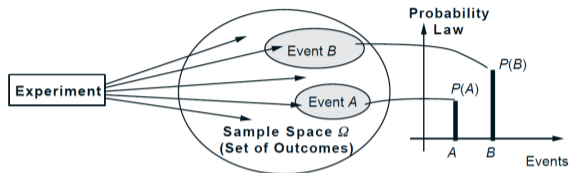
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Experiments



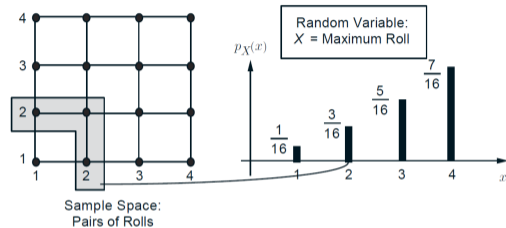
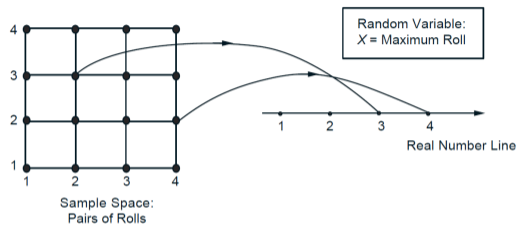
Experiment: roll two independent 4 sided die.

We are interested in probability of the *maximum of the two rolls*.

Modeling so far

- | $\Omega = f1;2;3;4g \quad f1;2;3;4g = f(1;1);(1;2);(1;3);(1;4);(2;1);:::(4;4)g$
- | $A = f\text{maximum roll is } 2g = f(1;2);(2;1);(2;2)g$
- | $P(A) = P(f(1;2);(2;1);(2;2)g) = 3/16$

Random variables

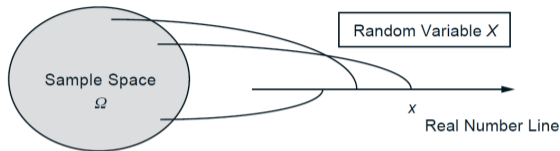


Modeling $X : \Omega \rightarrow \mathbb{R}$

- | $X((a; b)) = \max(a; b)$
- | $A = \{ \omega \mid \text{maximum roll is } 2 \} = \{ (a; b) \in \Omega \mid X((a; b)) = 2 \}$
- | $P(A) = P(X^{-1}(2)) = \frac{3}{16}$
- | We write $P_X(X = 2) \stackrel{\text{def}}{=} P(X^{-1}(2))$

[Induced probability]

(Discrete) Random variables



A random variable is a function $X : \Omega \rightarrow \mathbb{R}$

- it transforms Ω into a more tangible sample space \mathbb{R}
from $(a; b)$ to $\min(a; b)$
- it decouples the details of a specific Ω from the probability of events of interest
from $\{H, T\}$ or $\{good, bad\}$ or $\{::\}$ to $\{0; 1\}$
- it is not 'random' nor 'variable'

DEFINITION. Let Ω be a sample space. A *discrete random variable* is a function $X : \Omega \rightarrow \mathbb{R}$ that takes on a finite number of values a_1, a_2, \dots, a_n or an infinite number of values a_1, a_2, \dots

Probability Mass Function (PMF)

DEFINITION. The *probability mass function* p of a discrete random variable X is the function $p : \mathbb{R} \rightarrow [0, 1]$, defined by

$$p(a) = P(X = a) \quad \text{for } -\infty < a < \infty.$$

Support of X is $\{a \in \mathbb{R} \mid P(X = a) > 0\} = \{a_1; a_2; \dots; a_g\}$

- | $p(a_i) > 0$ for $i = 1; 2; \dots; g$
- | $p(a_1) + p(a_2) + \dots = 1$
- | $p(a) = 0$ if $a \notin \{a_1; a_2; \dots; a_g\}$

Cumulative Distribution Function (CDF) and CCDF

DEFINITION. The *distribution function* F of a random variable X is the function $F : \mathbb{R} \rightarrow [0, 1]$, defined by

$$F(a) = P(X \leq a) \quad \text{for } -\infty < a < \infty.$$

$$F(a) = P(X \leq a) = \sum_{a_i \leq a} p(a_i)$$

if $a < b$ then $F(a) < F(b)$

$$P(a < X \leq b) = F(b) - F(a) = \sum_{a < a_i \leq b} p(a_i)$$

[Non-decreasing]

Complementary cumulative distribution function (CCDF)

$$\bar{F}(a) = P(X > a) = 1 - P(X \leq a) = 1 - F(a)$$

$$\bar{F}(a) = P(X > a) = \sum_{a_i > a} p(a_i)$$

See R script

Uniform discrete distribution

A discrete random variable X has the *uniform distribution* with parameters $m; M \in \mathbb{Z}$ such that $m < M$, if its pmf is given by

$$p(a) = \frac{1}{M - m + 1} \quad \text{for } a = m; m + 1; \dots; M$$

We denote this distribution by $U(m; M)$.

Intuition: all integers in $[m; M]$ have equal chances of being observed.

$$F(a) = \frac{a - m + 1}{M - m + 1} \quad \text{for } m \leq a \leq M$$

See R script

Benford's law

A discrete random variable X has the *Benford's distribution*, if its pmf is given by

$$p(a) = \log_{10} \left(1 + \frac{1}{a} \right) \quad \text{for } a = 1; 2; \dots; 9$$

We denote this distribution by *Ben*.

Plausible and empirically adequate model for to the frequency distribution of leading digits in many real-life numerical datasets.

See [Wikipedia](#) for its interesting history and applications!

See R script

X $Ber(p)$

DEFINITION. A discrete random variable X has a **Bernoulli distribution** with parameter p , where $0 \leq p \leq 1$, if its probability mass function is given by

$$p_X(1) = P(X = 1) = p \quad \text{and} \quad p_X(0) = P(X = 0) = 1 - p.$$

We denote this distribution by $Ber(p)$.

X models success/failure in tossing a coin (H, T), testing for a disease (infected, not infected), membership in a set (member, non-member), etc.

p_X is the *pmf* (to distinguish from parameter p)

Alternative definition: $p_X(a) = p^a (1 - p)^{1 - a}$ for $a \in \{0, 1\}$

See R script

Identically distributed random variables

Two random variables X and Y are said *identically distributed* (in symbols, $X \stackrel{d}{=} Y$), if $F_X = F_Y$, i.e.,

$$F_X(a) = F_Y(a) \quad \text{for } a \in \mathbb{R}$$

Identically distributed does **not** mean equal

Toss a fair coin

- | let X be 1 for H and 0 for T
- | let Y be $1 - X$

$X \sim \text{Ber}(0.5)$ and $Y \sim \text{Ber}(0.5)$

Thus, $X \stackrel{d}{=} Y$ but are clearly always different.

Joint p.m.f.

For a same Ω , several random variables can be defined

- Random variables related to the same experiment often influence one another

- $\Omega = \{(i; j) \mid i, j \in \{1, \dots, 6\}\}$ rolls of two dice

$$X((i; j)) = i + j \text{ and } Y((i; j)) = \max(i; j)$$

$$P(X = 4; Y = 3) = P(X^{-1}(4) \cap Y^{-1}(3)) = P(\{(3; 1); (1; 3)\}) = \frac{2}{36}$$

- $\Omega = \{(f, g) \mid f, g \in \{+, -\}\}$ (testing for Covid-19 - multivariate)

$$G((g; a; c)) = 0 \text{ if } g = f \text{ and } 1 \text{ otherwise} \quad A((g; a; c)) = a$$

$$Y((g; a; c)) = 0 \text{ if } c = \text{and } 1 \text{ otherwise}$$

In general:

$$P_{XY}(X = a; Y = b) = P(\{(! \in \Omega \mid X(!) = a \text{ and } Y(!) = b\}) = P(X^{-1}(a) \cap Y^{-1}(b))$$

DEFINITION. The **joint probability mass function** p of two discrete random variables X and Y is the function $p : \mathbb{R}^2 \rightarrow [0, 1]$, defined by

$$p(a, b) = P(X = a, Y = b) \quad \text{for } -\infty < a, b < \infty.$$

Joint and marginal p.m.f.

Joint distribution function $F : \mathbb{R} \times \mathbb{R} \rightarrow [0;1]$:

$$F_{XY}(a; b) = P(X \leq a; Y \leq b) = \sum_{a_i \leq a; b_i \leq b} p(a_i; b_i)$$

By generalized additivity, the **marginal p.m.f.**'s can be derived: *[Tabular method]*

$$p_X(a) = P_X(X = a) = \sum_b P_{XY}(X = a; Y = b) \quad p_Y(b) = P_Y(Y = b) = \sum_a P_{XY}(X = a; Y = b)$$

and the marginal distribution function of X as:

$$F_X(a) = P_X(X \leq a) = \lim_{b \uparrow \infty} F_{XY}(a; b) \quad F_Y(b) = P_Y(Y \leq b) = \lim_{a \uparrow \infty} F_{XY}(a; b)$$

Deriving the joint p.m.f. from marginal p.m.f.'s is not always possible!

- | **Exercise at home.** Prove it (hint: find two joint p.m.f.'s with the same marginals)

Deriving the joint p.m.f. from marginal p.m.f.'s is possible for independent events!

- | $\Omega = \{1; 2; 3; 4\} \times \{1; 2; 3; 4\}$, $X((a; b)) = a$, $Y((a; b)) = b$
- | $P(X = 1; Y = 2) = \frac{1}{16} = \frac{1}{4} \cdot \frac{1}{4} = P(X = 1) \cdot P(Y = 2)$

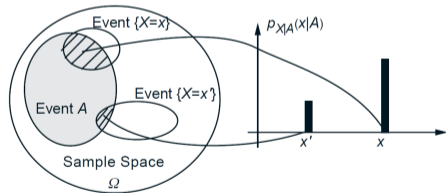
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Conditional distribution

Conditional distribution

Consider the joint distribution P_{XY} of X and Y . The conditional distribution of X given $Y \in B$ with $P_Y(Y \in B) > 0$, is the function $F_{X|Y \in B} : \mathbb{R} \rightarrow [0; 1]$:

$$F_{X|Y \in B}(a) = P_{X|Y}(X \leq a | Y \in B) = \frac{P_{XY}(X \leq a; Y \in B)}{P_Y(Y \in B)} \quad \text{for } 1 < a < 1$$



Distribution of X after knowing $Y \in B$.

Chain rule: $P_{XY}(X \leq a; Y \in B) = P_{X|Y}(X \leq a | Y \in B)P_Y(Y \in B)$

What if the distribution does not change w.r.t. the prior P_X ?

Independence of two random variables

Independence $X \perp\!\!\!\perp Y$

A random variable X is independent from a random variable Y , if for all $P_Y(Y = b) > 0$:

$$P_{X|Y}(X = a|Y = b) = P_X(X = a) \quad \text{for } 1 < a < 1$$

Properties

- | $X \perp\!\!\!\perp Y$ iff $P_{XY}(X = a; Y = b) = P_X(X = a) P_Y(Y = b)$ for $1 < a; b < 1$
- | $X \perp\!\!\!\perp Y$ iff $Y \perp\!\!\!\perp X$ [Symmetry]

For $X; Y$ **discrete** random variables:

- | $X \perp\!\!\!\perp Y$ iff $P_{XY}(X = a; Y = b) = P_X(X = a) P_Y(Y = b)$ for $1 < a; b < 1$
- | **Exercise at home.** Prove it!
- | $X \perp\!\!\!\perp Y$ iff $P_{XY}(X \in A; Y \in B) = P_X(X \in A) P_Y(Y \in B)$ for $A; B \subseteq \mathbb{R}$
- | **Exercise at home.** Prove it!

See R script

Sum of independent discrete random variables

ADDING TWO INDEPENDENT DISCRETE RANDOM VARIABLES. Let X and Y be two independent discrete random variables, with probability mass functions p_X and p_Y . Then the probability mass function p_Z of $Z = X + Y$ satisfies

$$p_Z(c) = \sum_j p_X(c - b_j) p_Y(b_j),$$

where the sum runs over all possible values b_j of Y .

Proof (sketch). $P(Z = c) = \sum_j P(Z = c | Y = b_j) P(Y = b_j) = \sum_j P(X = c - b_j | Y = b_j) P(Y = b_j) = \sum_j P(X = c - b_j) P(Y = b_j)$

Independence of multiple random variables

Independence (factorization formula)

Random variables X_1, \dots, X_n are independent, if:

$$P_{X_1, \dots, X_n}(X_1 = a_1, \dots, X_n = a_n) = \prod_{i=1}^n P_{X_i}(X_i = a_i) \quad \text{for } 1 < a_1, \dots, a_n < 1$$

X_1, \dots, X_n **discrete** random variables are independent iff:

$$P_{X_1, \dots, X_n}(X_1 = a_1, \dots, X_n = a_n) = \prod_{i=1}^n P_{X_i}(X_i = a_i) \quad \text{for } 1 < a_1, \dots, a_n < 1$$

Definition: X_1, \dots, X_n are **i.i.d.** (independent and identically distributed) if X_1, \dots, X_n are independent and $X_i \sim F$ for $i = 1, \dots, n$ for some distribution F

$X \sim \text{Bin}(n; p)$

X models the number of successes in n Bernoulli trials (How many H's when tossing n coins?)

Intuition: for $X_1; X_2; \dots; X_n$ such that $X_i \sim \text{Ber}(p)$ and independent (**i.i.d.**):

$$X = \sum_{i=1}^n X_i \sim \text{Bin}(n; p)$$

$p^k (1-p)^{n-k}$ is the probability of observing first k H's and then $n-k$ T's

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$ number of ways to choose the first k variables *[Binomial coefficient]*

$p_X(k)$ computationally expensive to calculate (no closed formula, but approximation/bounds)

Exercise at home. Prove $X_1 + X_2 \sim \text{Bin}(2; p)$ using the sum of independent random variables.

See R script

Exercise: there are c bikes shared among n persons. Assuming that each person needs a bike with probability p , what is the probability that all bikes will be in use?

$$P(X = c) = \binom{n}{c} p^c (1 - p)^{n - c} = \text{dbinom}(c-1, n, p)$$

X $Geo(p)$

DEFINITION. A discrete random variable X has a *geometric distribution* with parameter p , where $0 < p \leq 1$, if its probability mass function is given by

$$p_X(k) = P(X = k) = (1 - p)^{k-1} p \quad \text{for } k = 1, 2, \dots$$

We denote this distribution by $Geo(p)$.

X models the number of Bernoulli trials before a success (how many tosses to have a H?)

Intuition: for $X_1; X_2; \dots$ such that $X_i \sim Ber(p)$ i.i.d.:

$$X = \min_i (X_i = 1) \sim Geo(p)$$

$$\bar{F}(a) = P(X > a) = (1 - p)^{a-1}$$

$$F(a) = P(X \leq a) = 1 - \bar{F}(a) = 1 - (1 - p)^{a-1}$$

See R script

You cannot always loose

H is 1, T is 0, $0 < p < 1$

$B_n = \text{first } n \text{ coin tosses are T}$

$P(\bigcap_{n=1}^{\infty} B_n) = ?$

$X \sim \text{Geom}(p)$

$P(B_n) = P(X > n) = (1 - p)^n$

$P(\bigcap_{n=1}^{\infty} B_n) = \lim_{n \rightarrow \infty} P(B_n) = \lim_{n \rightarrow \infty} (1 - p)^n = 0$

$P(\bigcap_{n=1}^{\infty} B_n) = \lim_{n \rightarrow \infty} P(B_n)$ for B_n non-increasing

[Borel–Cantelli Lemma]

But if you lost so far, you can lose again

Memoryless property

For $X \sim \text{Geo}(p)$, and $n, k = 0; 1; 2; \dots$:

$$P(X > n + k | X > k) = P(X > n)$$

Proof

$$\begin{aligned} P(X > n + k | X > k) &= \frac{P(fX > n + kg \mid fX > kg)}{P(fX > kg)} \\ &= \frac{P(fX > n + kg)}{P(fX > kg)} \\ &= \frac{(1 - p)^{n+k}}{(1 - p)^k} \\ &= (1 - p)^n = P(X > n) \end{aligned}$$

$X \sim NBin(n; p)$

Negative binomial (or Pascal distribution)

A discrete random variable X has a negative binomial with parameters n and p , where $n = 0; 1; 2; \dots$ and $0 < p < 1$, if its probability mass function is given by

$$p_X(k) = P(X = k) = \binom{k+n-1}{k} (1-p)^k p^n \quad \text{for } k = 0; 1; 2; \dots$$

X models the number of failures before the n -th success in Bernoulli trials (how many T's to have n H's?)

Intuition: for $X_1; X_2; \dots; X_n$ such that $X_i \sim Geo(p)$ i.i.d.:

$$X = \sum_{i=1}^n X_i \sim n \cdot NBin(1; p)$$

$(1-p)^k p^n$ is the probability of observing first k T's and then n H's

$\binom{k+n-1}{k} = \frac{(k+n-1)!}{k!(n-1)!}$ number of ways to choose the first k variables among $k+n-1$ (the last one must be a success!)

See R script

$X \sim \text{Poi}(\mu)$

DEFINITION. A discrete random variable X has a *Poisson distribution* with parameter μ , where $\mu > 0$ if its probability mass function p is given by

$$p(k) = \text{P}(X = k) = \frac{\mu^k}{k!} e^{-\mu} \quad \text{for } k = 0, 1, 2, \dots$$

We denote this distribution by $\text{Pois}(\mu)$.

X models the number of events in a fixed interval if these events occur with a known constant mean rate and independently of the last event

- | telephone calls arriving in a system
- | number of patients arriving at an hospital
- | customers arriving at a counter

μ denotes the mean number of events

$\text{Bin}(n; p)$ is the number of successes in n trials, assuming $p = \mu/n$, i.e., $p \ll 1$

When $n \rightarrow \infty$ and $p \rightarrow 0$ such that $\mu = np$ is constant, $\text{Bin}(n; p) \rightarrow \text{Poi}(\mu)$ [Law of rare events]

- | Number of typos in a book, number of cars involved in accidents, etc.

See R script

The discrete Bayes' rule

BAYES' RULE. Suppose the events C_1, C_2, \dots, C_m are disjoint and $C_1 \cup C_2 \cup \dots \cup C_m = \Omega$. The conditional probability of C_i , given an arbitrary event A , can be expressed as:

$$P(C_i | A) = \frac{P(A | C_i) \cdot P(C_i)}{P(A | C_1)P(C_1) + P(A | C_2)P(C_2) + \dots + P(A | C_m)P(C_m)}.$$

Definition. Conditional p.m.f. of X given $Y = b$ with $P_Y(Y = b) > 0$

$$p_{X|Y}(a|b) = \frac{p_{XY}(a; b)}{p_Y(b)} \quad \text{i.e.,} \quad P_{X|Y}(X = a | Y = b) = \frac{P_{XY}(X = a; Y = b)}{P_Y(Y = b)}$$

Discrete Bayes' rule:

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x)p_X(x)}{p_Y(y)} = \frac{p_{Y|X}(y|x)p_X(x)}{\sum_{a \in \text{dom}(X)} p_{Y|X}(y|a)p_X(a)}$$

Exercise at home. A machine fails after n days with a p.m.f. $X \sim \text{Geo}(p)$. p is known to be either $p = 0.1$ or 0.05 with equal probability. What can we say about the distribution of p given n ? Code your solution in R.

