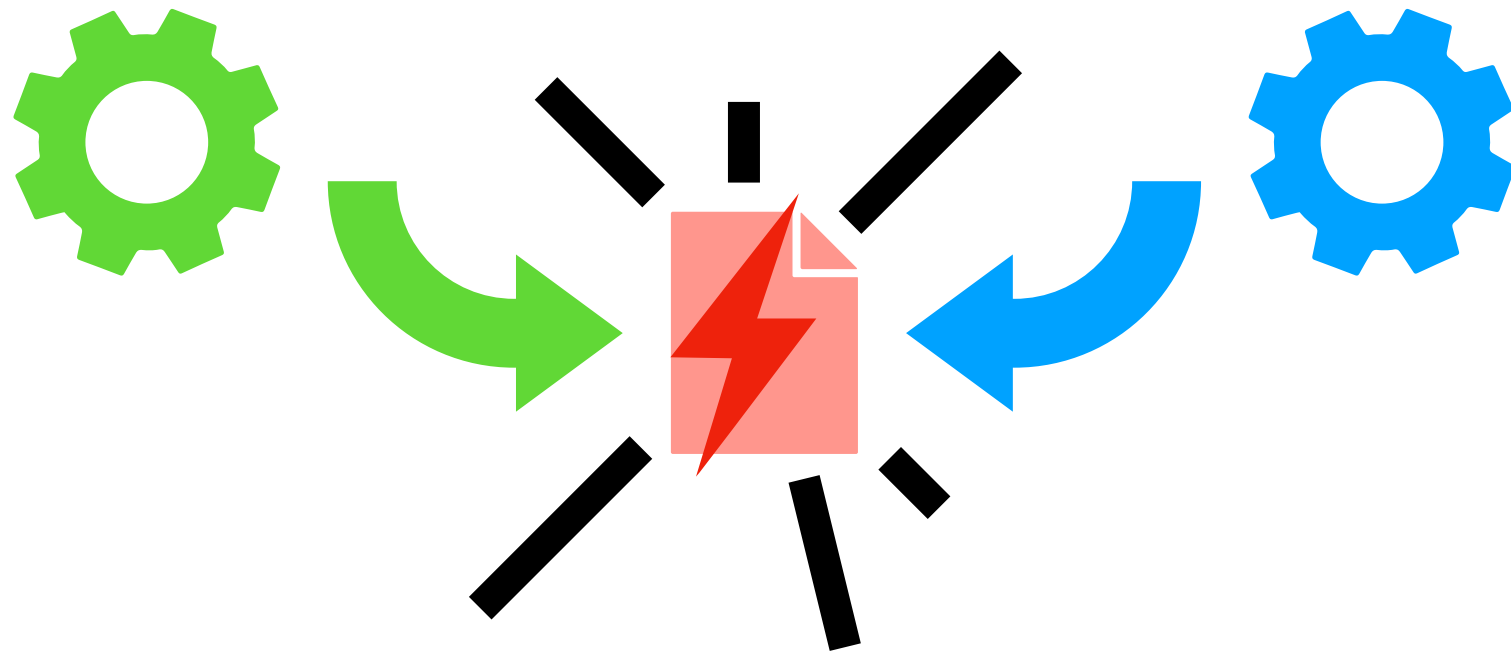


Linguaggi di Programmazione



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Semantica Denotazionale HOF-9

Domini di interpretazioni

L'idea

Per ogni tipo introduciamo il corrispondente dominio $(V_\tau)_\perp$ definito induttivamente sulla struttura di τ in modo da poter assegnare un elemento del dominio $(V_\tau)_\perp$ ad ogni termine t chiuso e tipabile con tipo τ

Domini di interpretazioni

$$D_{int} \triangleq \mathbb{Z}_{\perp}$$

$$D_{\tau_1 * \tau_2} \triangleq (D_{\tau_1} \times D_{\tau_2})_{\perp}$$

distinguere:

coppia di termini divergenti
da coppia divergente

$$D_{\tau_1 \rightarrow \tau_2} \triangleq [D_{\tau_1} \rightarrow D_{\tau_2}]_{\perp}$$

distinguere:

prende arg e diverge

dalla divergenza senza prendere arg

Esempio

$$D_{int * int} \triangleq (\mathbb{Z}_{\perp} \times \mathbb{Z}_{\perp})_{\perp}$$

$$\mathbf{rec} \, p. \, p \quad (\mathbf{rec} \, x. \, x, \mathbf{rec} \, y. \, y)$$

$$\perp_{D_{int * int}} \quad (\perp_{D_{int}}, \perp_{D_{int}})$$

Esempio

$$D_{int \rightarrow int} \triangleq [\mathbb{Z}_{\perp} \rightarrow \mathbb{Z}_{\perp}]_{\perp}$$

$$\mathbf{rec} \, f. \, f \quad \lambda x. \, \mathbf{rec} \, y. \, y$$

$$\perp_{D_{int \rightarrow int}} \quad \lambda d. \, \perp_{D_{int}}$$

Domini di interpretazioni

$$D_{int} \triangleq \mathbb{Z}_{\perp}$$

$$D_{\tau_1 * \tau_2} \triangleq (D_{\tau_1} \times D_{\tau_2})_{\perp}$$

$$D_{\tau_1 \rightarrow \tau_2} \triangleq [D_{\tau_1} \rightarrow D_{\tau_2}]_{\perp}$$

Domini di interpretazioni

$$t : \tau \quad \frac{\llbracket t \rrbracket \rho \in D_\tau}{\text{ambiente}}$$

tipo coerente con
assegnazione di
valori alle variabili

$$\rho : Var \rightarrow \bigcup_{\tau \in \mathcal{T}} D_\tau$$

$$x : \tau \Rightarrow \rho(x) \in D_\tau$$

definiamo la funzione di interpretazione per ricorsione
strutturale

Semantica Denotazionale

Constanti

$$t : \tau \quad \llbracket t \rrbracket \rho \in D_\tau$$

$$\rho : Var \rightarrow \bigcup_{\tau \in \mathcal{T}} D_\tau$$

$$x : \tau \Rightarrow \rho(x) \in D_\tau$$

$$\underbrace{\llbracket n \rrbracket \rho}_{\substack{int \\ D_{int} = \mathbb{Z}_\perp}} \triangleq \underbrace{\lfloor n \rfloor}_{\substack{\mathbb{Z} \\ \mathbb{Z}_\perp}}$$

Variabili

$$t : \tau \quad \llbracket t \rrbracket \rho \in D_\tau$$

$$\rho : Var \rightarrow \bigcup_{\tau \in \mathcal{T}} D_\tau$$

$$x : \tau \Rightarrow \rho(x) \in D_\tau$$

$$\underbrace{\llbracket x \rrbracket \rho}_{\tau} \stackrel{\Delta}{=} \underbrace{\rho(x)}_{D_\tau}$$

Operazioni aritmetiche

da dimostrare: $\underline{\text{op}}_{\perp}$ e' monotona e continua

$$\text{op} \in \{+, -, \times\}$$
$$\underbrace{\underbrace{\llbracket t_1 \rrbracket \rho}_{\text{int}} \text{ op } \underbrace{\llbracket t_2 \rrbracket \rho}_{\text{int}}}_{D_{\text{int}} = \mathbb{Z}_{\perp}} \triangleq \underbrace{\underbrace{\llbracket t_1 \rrbracket \rho}_{D_{\text{int}} = \mathbb{Z}_{\perp}} \underline{\text{op}}_{\perp} \underbrace{\llbracket t_2 \rrbracket \rho}_{D_{\text{int}} = \mathbb{Z}_{\perp}}}_{D_{\text{int}} = \mathbb{Z}_{\perp}}$$

$$\underline{\text{op}}_{\perp} : \mathbb{Z}_{\perp} \times \mathbb{Z}_{\perp} \rightarrow \mathbb{Z}_{\perp}$$

$$v_1 \underline{\text{op}}_{\perp} v_2 \triangleq \begin{cases} \lfloor n_1 \text{ op } n_2 \rfloor & \text{se } v_1 = \lfloor n_1 \rfloor \text{ e } v_2 = \lfloor n_2 \rfloor \\ \perp_{\mathbb{Z}_{\perp}} & \text{altrimenti (} v_1 = \perp_{\mathbb{Z}_{\perp}} \text{ o } v_2 = \perp_{\mathbb{Z}_{\perp}} \text{)} \end{cases}$$

chiamata estensione strict

Condizionale

da dimostrare: Cond_τ is monotona e continua

$$\underbrace{\underbrace{\underbrace{\llbracket \text{if } t \text{ then } t_1 \text{ else } t_2 \rrbracket \rho}_{\tau}}_{\tau}}_{D_\tau} \triangleq \text{Cond}_\tau \left(\underbrace{\underbrace{\llbracket t \rrbracket \rho}_{\tau}}_{D_{int} = \mathbb{Z}_\perp}, \underbrace{\underbrace{\llbracket t_1 \rrbracket \rho}_{\tau}}_{D_\tau}, \underbrace{\underbrace{\llbracket t_2 \rrbracket \rho}_{\tau}}_{D_\tau} \right)$$

$$\text{Cond}_\tau : \mathbb{Z}_\perp \times D_\tau \times D_\tau \rightarrow D_\tau$$

$$\text{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \perp_{D_\tau} & \text{se } v = \perp_{\mathbb{Z}_\perp} \\ d_1 & \text{se } v = \lfloor 0 \rfloor \\ d_2 & \text{altrimenti } (v = \lfloor n \rfloor \text{ con } n \neq 0) \end{cases}$$

Coppie

$$D_{\tau_1 * \tau_2} \triangleq (D_{\tau_1} \times D_{\tau_2})_{\perp}$$

$$\begin{array}{c} \underbrace{\underbrace{\underbrace{t_1}_{\tau_1}}_{\tau_1 * \tau_2}}_{D_{\tau_1 * \tau_2} = (D_{\tau_1} \times D_{\tau_2})_{\perp}} \quad \underbrace{\underbrace{\underbrace{t_2}_{\tau_2}}_{D_{\tau_2}}}_{D_{\tau_1} \times D_{\tau_2}} \\ \underbrace{\underbrace{\underbrace{t_1}_{\tau_1}}_{D_{\tau_1}}}_{D_{\tau_1 * \tau_2} = (D_{\tau_1} \times D_{\tau_2})_{\perp}} \quad \underbrace{\underbrace{\underbrace{t_2}_{\tau_2}}_{D_{\tau_2}}}_{D_{\tau_1} \times D_{\tau_2}} \end{array}$$

Proiezioni

Equivalentemente: $\llbracket \mathbf{fst}(t) \rrbracket \rho \triangleq \mathbf{let} \ d \Leftarrow \llbracket t \rrbracket \rho. \ \pi_1(d)$

$$\llbracket \mathbf{fst}(t) \rrbracket \rho \triangleq \pi_1^* \left(\llbracket t \rrbracket \rho \right)$$

$$\begin{array}{c} \underbrace{\underbrace{\tau_1 * \tau_2}_{\tau_1}}_{D_{\tau_1}} \quad \underbrace{D_{\tau_1} \times D_{\tau_2} \rightarrow D_{\tau_1} \quad \underbrace{\tau_1 * \tau_2}_{D_{\tau_1 * \tau_2} = (D_{\tau_1} \times D_{\tau_2})^\perp}}_{(D_{\tau_1} \times D_{\tau_2})^\perp \rightarrow D_{\tau_1}}_{D_{\tau_1}} \end{array}$$

$$\llbracket \mathbf{snd}(t) \rrbracket \rho \triangleq \pi_2^* \left(\llbracket t \rrbracket \rho \right)$$

Astrazione

$$D_{\tau_1 \rightarrow \tau_2} \triangleq [D_{\tau_1} \rightarrow D_{\tau_2}]_{\perp}$$

$$\begin{array}{c} \underbrace{\underbrace{\underbrace{\lambda x. t}_{\tau_1}}_{\tau_1 \rightarrow \tau_2}}_{D_{\tau_1 \rightarrow \tau_2} = [D_{\tau_1} \rightarrow D_{\tau_2}]_{\perp}} \triangleq \underbrace{\underbrace{\underbrace{\lambda d. \underbrace{[t]_{\rho[d/x]}}_{\tau_2}}_{D_{\tau_1}}}_{[D_{\tau_1} \rightarrow D_{\tau_2}]}}_{[D_{\tau_1} \rightarrow D_{\tau_2}]_{\perp}} \end{array}$$

Applicazione (lazy)

Equivalentemente:

$$\llbracket t \ t_0 \rrbracket \rho \triangleq (\lambda \varphi. \varphi(\llbracket t_0 \rrbracket \rho))^* (\llbracket t \rrbracket \rho)$$

$$\llbracket t \ t_0 \rrbracket \rho \triangleq \mathbf{let} \ \varphi \Leftarrow \llbracket t \rrbracket \rho. \ \varphi(\llbracket t_0 \rrbracket \rho)$$

$$\underbrace{\underbrace{\underbrace{\tau_0 \rightarrow \tau}_{\tau_0}}_{\tau}}_{D_\tau}$$

$$\underbrace{\underbrace{V_{\tau_0 \rightarrow \tau} = [D_{\tau_0} \rightarrow D_\tau]}_{\tau_0 \rightarrow \tau} \underbrace{[D_{\tau_0} \rightarrow D_\tau]}_{\tau_0}}_{D_{\tau_0 \rightarrow \tau} = (V_{\tau_0 \rightarrow \tau})_\perp} \underbrace{\quad}_{D_\tau}$$

$$D_\tau$$

Ricorsione

$$\begin{array}{c}
 \underbrace{\underbrace{\underbrace{\mathbf{rec} \ x. \ t}_{\tau}}_{\tau}}_{\tau} \rho \triangleq \underbrace{\underbrace{\underbrace{[t] \rho}_{\tau}}_{\tau} \underbrace{[\underbrace{\mathbf{rec} \ x. \ t}_{D_\tau} \rho]_{D_\tau}}_{D_\tau}}_{D_\tau} / x]
 \end{array}$$

Ricorsione

$$\frac{\frac{\frac{\text{rec } x. t}{\tau}}{\tau}}{D_\tau} \triangleq \frac{\frac{\text{fix } \lambda d. \frac{\frac{t}{\tau}}{D_\tau}}{D_\tau}}{D_\tau} \frac{d}{x}$$

Recap

$$\llbracket n \rrbracket \rho \triangleq \lfloor n \rfloor$$

$$\llbracket x \rrbracket \rho \triangleq \rho(x)$$

$$\llbracket t_1 \text{ op } t_2 \rrbracket \rho \triangleq \llbracket t_1 \rrbracket \rho \text{ \underline{op} }_\perp \llbracket t_2 \rrbracket \rho$$

$$\llbracket \textbf{if } t \textbf{ then } t_1 \textbf{ else } t_2 \rrbracket \rho \triangleq \text{Cond}_\tau(\llbracket t \rrbracket \rho , \llbracket t_1 \rrbracket \rho , \llbracket t_2 \rrbracket \rho)$$

$$\llbracket (t_1 , t_2) \rrbracket \rho \triangleq \lfloor (\llbracket t_1 \rrbracket \rho , \llbracket t_2 \rrbracket \rho) \rfloor$$

$$\llbracket \textbf{fst}(t) \rrbracket \rho \triangleq \pi_1^* (\llbracket t \rrbracket \rho)$$

$$\llbracket \textbf{snd}(t) \rrbracket \rho \triangleq \pi_2^* (\llbracket t \rrbracket \rho)$$

$$\llbracket \lambda x. t \rrbracket \rho \triangleq \lfloor \lambda d. \llbracket t \rrbracket \rho^{[d/x]} \rfloor$$

$$\llbracket t \ t_0 \rrbracket \rho \triangleq \textbf{let } \varphi \leftarrow \llbracket t \rrbracket \rho. \varphi(\llbracket t_0 \rrbracket \rho)$$

$$\llbracket \textbf{rec } x. t \rrbracket \rho \triangleq \textit{fix } \lambda d. \llbracket t \rrbracket \rho^{[d/x]}$$

Esempi

I seguenti termini hanno la stessa semantica?

$$f \stackrel{\text{def}}{=} \lambda x : \textit{int}. 3$$

$$g \stackrel{\text{def}}{=} \lambda x : \textit{int}. \mathbf{if} \ x \ \mathbf{then} \ 3 \ \mathbf{else} \ 3$$

$$h \stackrel{\text{def}}{=} \mathbf{rec} \ y : \textit{int} \rightarrow \textit{int}. \lambda x : \textit{int}. 3$$

Esempio

$$f \stackrel{\text{def}}{=} \lambda x : \text{int}. 3$$

$$\llbracket \lambda x. t \rrbracket \rho \triangleq \llbracket \lambda d. \llbracket t \rrbracket \rho[d/x] \rrbracket \quad \llbracket n \rrbracket \rho \triangleq \llbracket n \rrbracket$$

$$\llbracket f \rrbracket \rho = \llbracket \lambda x. 3 \rrbracket \rho = \llbracket \lambda d. \llbracket 3 \rrbracket \rho[d/x] \rrbracket = \llbracket \lambda d. \llbracket 3 \rrbracket \rrbracket$$

Esempio

$$g \stackrel{\text{def}}{=} \lambda x : \text{int}. \text{ if } x \text{ then } 3 \text{ else } 3$$

$$\llbracket \lambda x. t \rrbracket \rho \triangleq \llbracket \lambda d. \llbracket t \rrbracket \rho^{[d/x]} \rrbracket$$

$$\llbracket g \rrbracket \rho = \llbracket \lambda x. \text{ if } x \text{ then } 3 \text{ else } 3 \rrbracket \rho$$

$$= \llbracket \lambda d. \llbracket \text{if } x \text{ then } 3 \text{ else } 3 \rrbracket \rho^{[d/x]} \rrbracket$$

$$\llbracket \text{if } t \text{ then } t_1 \text{ else } t_2 \rrbracket \rho \triangleq \text{Cond}_\tau(\llbracket t \rrbracket \rho, \llbracket t_1 \rrbracket \rho, \llbracket t_2 \rrbracket \rho)$$

$$= \llbracket \lambda d. \text{Cond}(d, [3], [3]) \rrbracket$$

$$= \llbracket \lambda d. \text{let } x \Leftarrow d. [3] \rrbracket$$

$$\text{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \perp_{D_\tau} & \text{se } v = \perp_{\mathbb{Z}_\perp} \\ d_1 & \text{se } v = [0] \\ d_2 & \text{altrimenti } (v = [n] \text{ con } n \neq 0) \end{cases}$$

$$\llbracket f \rrbracket \rho \quad \llbracket g \rrbracket \rho$$

$$\llbracket \lambda d. [3] \rrbracket \quad f \stackrel{\text{def}}{=} \lambda x : \text{int}. 3$$

Esempio

$$h \stackrel{\text{def}}{=} \mathbf{rec} \, y : \mathit{int} \rightarrow \mathit{int}. \lambda x : \mathit{int}. 3$$

$$\begin{aligned} \llbracket h \rrbracket \rho &= \llbracket \mathbf{rec} \, y. \lambda x. 3 \rrbracket \rho && \llbracket \mathbf{rec} \, x. t \rrbracket \rho \triangleq \mathit{fix} \, \lambda d. \llbracket t \rrbracket \rho[d/x] \\ &= \mathit{fix} \, \lambda d_y. \llbracket \lambda x. 3 \rrbracket \rho[d_y/y] && \llbracket \lambda x. t \rrbracket \rho \triangleq \llbracket \lambda d. \llbracket t \rrbracket \rho[d/x] \rrbracket \\ &= \mathit{fix} \, \lambda d_y. \llbracket \lambda d_x. \llbracket 3 \rrbracket \rho[d_y/y, d_x/x] \rrbracket \\ &= \mathit{fix} \, \lambda d_y. \llbracket \lambda d_x. \llbracket 3 \rrbracket \rrbracket && \Gamma_h = \lambda d_y. \llbracket \lambda d_x. \llbracket 3 \rrbracket \rrbracket \end{aligned}$$

$$d_0 = \Gamma_h^0(\perp_{[\mathbb{Z}_\perp \rightarrow \mathbb{Z}_\perp]_\perp}) = \perp_{[\mathbb{Z}_\perp \rightarrow \mathbb{Z}_\perp]_\perp}$$

$$d_1 = \Gamma_h(d_0) = (\lambda d_y. \llbracket \lambda d_x. \llbracket 3 \rrbracket \rrbracket) \perp = \llbracket \lambda d_x. \llbracket 3 \rrbracket \rrbracket$$

$$d_2 = \Gamma_h(d_1) = (\lambda d_y. \llbracket \lambda d_x. \llbracket 3 \rrbracket \rrbracket) \llbracket \lambda d_x. \llbracket 3 \rrbracket \rrbracket = \llbracket \lambda d_x. \llbracket 3 \rrbracket \rrbracket = d_1$$

Esempio

$$h \stackrel{\text{def}}{=} \mathbf{rec} \, y : \mathit{int} \rightarrow \mathit{int}. \lambda x : \mathit{int}. 3$$

$$\Gamma_h = \lambda d_y. [\lambda d_x. [3]]$$

$$d_0 = \Gamma_h^0(\perp_{[\mathbb{Z}_\perp \rightarrow \mathbb{Z}_\perp]_\perp}) = \perp_{[\mathbb{Z}_\perp \rightarrow \mathbb{Z}_\perp]_\perp}$$

$$d_1 = \Gamma_h(d_0) = (\lambda d_y. [\lambda d_x. [3]])\perp = [\lambda d_x. [3]]$$

Elemento massimale in $[\mathbb{Z}_\perp \rightarrow \mathbb{Z}_\perp]_\perp$
potremmo già fermarci

Esempio

$$h \stackrel{\text{def}}{=} \mathbf{rec} \, y : \mathit{int} \rightarrow \mathit{int}. \lambda x : \mathit{int}. 3$$

$$\llbracket h \rrbracket \rho = \llbracket \lambda d_x. \llbracket 3 \rrbracket \rrbracket = \llbracket f \rrbracket \rho$$

$$\llbracket f \rrbracket \rho = \llbracket \lambda x. 3 \rrbracket \rho = \llbracket \lambda d. \llbracket 3 \rrbracket \rho^{[d/x]} \rrbracket = \llbracket \lambda d. \llbracket 3 \rrbracket \rrbracket$$

$$f \stackrel{\text{def}}{=} \lambda x : \mathit{int}. 3$$

Esempio

$$x : \tau \qquad \llbracket \mathbf{rec} \ x. \ t \rrbracket \rho \triangleq \mathit{fix} \ \lambda d. \llbracket t \rrbracket \rho [d/x]$$

$$\begin{aligned} \llbracket \mathbf{rec} \ x. \ x \rrbracket \rho &= \mathit{fix} \ \lambda d_x. \llbracket x \rrbracket \rho [d_x/x] \\ &= \mathit{fix} \ \lambda d_x. \ d_x \end{aligned}$$

$$d_0 = \perp_{D_\tau}$$

$$d_1 = (\lambda d_x. d_x) \ d_0 = d_0 = \perp_{D_\tau}$$

$$\llbracket \mathbf{rec} \ x. \ x \rrbracket \rho = \perp_{D_\tau}$$

$$x : \mathit{int} \rightarrow \mathit{int} \qquad \llbracket \mathbf{rec} \ x. \ x \rrbracket \rho = \perp_{[\mathbb{Z}_\perp \rightarrow \mathbb{Z}_\perp]_\perp}$$

$$x : \mathit{int} * \mathit{int} \qquad \llbracket \mathbf{rec} \ x. \ x \rrbracket \rho = \perp_{(\mathbb{Z}_\perp \times \mathbb{Z}_\perp)_\perp}$$

Esempi

I seguenti termini hanno la stessa semantica?

rec $x. x$

$\lambda y. \text{rec } z. z$

Esempio

$$y : \tau_1 \quad z : \tau_2 \quad \llbracket \lambda x. t \rrbracket \rho \triangleq \lfloor \lambda d. \llbracket t \rrbracket \rho[d/x] \rfloor$$

Ricordiamoci che $\llbracket \mathbf{rec} \ x. x \rrbracket \rho = \perp_{D_\tau}$

$$\begin{aligned} \llbracket \lambda y. \mathbf{rec} \ z. z \rrbracket \rho &= \lfloor \lambda d_y. \llbracket \mathbf{rec} \ z. z \rrbracket \rho[d_y/y] \rfloor \\ &= \lfloor \lambda d_y. \perp_{D_{\tau_2}} \rfloor \\ &= \lfloor \perp_{[D_{\tau_1} \rightarrow D_{\tau_2}]} \rfloor \\ &\neq \perp_{D_{\tau_1 \rightarrow \tau_2}} \end{aligned}$$

$$x : int \rightarrow int \quad \llbracket \mathbf{rec} \ x. x \rrbracket \rho = \perp_{[\mathbb{Z}_\perp \rightarrow \mathbb{Z}_\perp]_\perp} \quad \text{diverge}$$

$$y : int, z : int \quad \llbracket \lambda y. \mathbf{rec} \ z. z \rrbracket \rho = \lfloor \perp_{[\mathbb{Z}_\perp \rightarrow \mathbb{Z}_\perp]} \rfloor \quad \text{aspetta arg e diverge}$$

Esercizio

$x : int * int , y : int , z : int$

$$\llbracket \mathbf{rec} \ x. \ x \rrbracket \rho \stackrel{?}{=} \llbracket (\mathbf{rec} \ y. \ y , \mathbf{rec} \ z. \ z) \rrbracket \rho$$



diverge

una coppia
di computazioni divergenti

$\perp_{D_{int * int}}$

$\llbracket (\perp_{D_{int}} , \perp_{D_{int}}) \rrbracket$

Lazy vs Eager

Applicazioni Eager

returns $\perp$ quando $\llbracket t \rrbracket \rho = \perp$

lazy $\llbracket t \ t_0 \rrbracket \rho \triangleq \mathbf{let} \ \varphi \Leftarrow \llbracket t \rrbracket \rho. \ \varphi(\llbracket t_0 \rrbracket \rho)$

eager $\llbracket t \ t_0 \rrbracket \rho \triangleq \mathbf{let} \ \varphi \Leftarrow \llbracket t \rrbracket \rho. \ \mathbf{let} \ d \Leftarrow \llbracket t_0 \rrbracket \rho. \ \varphi(\lfloor d \rfloor)$

returns $\perp$ quando $\llbracket t \rrbracket \rho = \perp$ o $\llbracket t_0 \rrbracket \rho = \perp$

Definizioni ben costruite

Ben definite

Dobbiamo garantire che tutte le funzioni che abbiamo usato siano monotone e continue,
in modo che la teoria dei punti fissi di Kleene sia applicabile

$\pi_1 \ \pi_2 \ (\cdot)^* \ \textit{apply} \ \textit{fix} \ \textit{let}$ già' analizzate

$\underline{\textit{op}}_{\perp} \ \textit{Cond}_{\tau} \ \lambda$ da controllare

TH. $\underline{\text{op}}_{\perp}$ e' monotono e continuo

$$\underline{\text{op}}_{\perp} : \mathbb{Z}_{\perp} \times \mathbb{Z}_{\perp} \rightarrow \mathbb{Z}_{\perp}$$

$$v_1 \underline{\text{op}}_{\perp} v_2 \triangleq \begin{cases} \lfloor n_1 \underline{\text{op}} n_2 \rfloor & \text{se } v_1 = \lfloor n_1 \rfloor \text{ e } v_2 = \lfloor n_2 \rfloor \\ \perp_{\mathbb{Z}_{\perp}} & \text{altrimenti } (v_1 = \perp_{\mathbb{Z}_{\perp}} \text{ o } v_2 = \perp_{\mathbb{Z}_{\perp}}) \end{cases}$$

Omettiamo il controllo di monotonicit 

Dal momento che il dominio ha solo catene finite, e' anche continuo

TH. Cond_τ e' monotona e continua

$$\text{Cond}_\tau : \mathbb{Z}_\perp \times D_\tau \times D_\tau \rightarrow D_\tau$$

$$\text{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \perp_{D_\tau} & \text{se } v = \perp_{\mathbb{Z}_\perp} \\ d_1 & \text{se } v = \lfloor 0 \rfloor \\ d_2 & \text{altrimenti } (v = \lfloor n \rfloor \text{ con } n \neq 0) \end{cases}$$

Omettiamo il controllo di monotonicità

Dimostriamo la continuità su ogni parametro separatamente

Il primo parametro e' in $\mathbb{Z}_\perp$

sono possibili solo catene finite, quindi la continuità è garantita

Dimostriamo la continuità sul secondo parametro

Per il terzo parametro la prova è analoga e viene omessa

(continua)

$$\text{Cond}_\tau : \mathbb{Z}_\perp \times D_\tau \times D_\tau \rightarrow D_\tau$$

$$\text{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \perp_{D_\tau} & \text{se } v = \perp_{\mathbb{Z}_\perp} \\ d_1 & \text{se } v = \lfloor 0 \rfloor \\ d_2 & \text{altrimenti } (v = \lfloor n \rfloor \text{ con } n \neq 0) \end{cases}$$

Continuità' sul secondo parametro

prendiamo $v \in \mathbb{Z}_\perp, d \in D_\tau, \{d_i\}_{i \in \mathbb{N}} \subseteq D_\tau$

vogliamo provare $\text{Cond}_\tau \left(v, \bigsqcup_{i \in \mathbb{N}} d_i, d \right) = \bigsqcup_{i \in \mathbb{N}} \text{Cond}_\tau(v, d_i, d)$

procediamo per analisi dei casi $v \begin{cases} \perp_{\mathbb{Z}_\perp} \\ \lfloor 0 \rfloor \\ \lfloor n \rfloor, n \neq 0 \end{cases}$

(continua)

$$\text{Cond}_\tau : \mathbb{Z}_\perp \times D_\tau \times D_\tau \rightarrow D_\tau$$

$$\text{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \perp_{D_\tau} & \text{se } v = \perp_{\mathbb{Z}_\perp} \\ d_1 & \text{se } v = \lfloor 0 \rfloor \\ d_2 & \text{altrimenti } (v = \lfloor n \rfloor \text{ con } n \neq 0) \end{cases}$$

$$v = \perp_{\mathbb{Z}_\perp}$$

$$\text{Cond}_\tau \left(\perp_{\mathbb{Z}_\perp}, \bigsqcup_{i \in \mathbb{N}} d_i, d \right) = \perp_{D_\tau} = \bigsqcup_{i \in \mathbb{N}} \perp_{D_\tau} = \bigsqcup_{i \in \mathbb{N}} \text{Cond}_\tau(\perp_{\mathbb{Z}_\perp}, d_i, d)$$

(continua)

$$\text{Cond}_\tau : \mathbb{Z}_\perp \times D_\tau \times D_\tau \rightarrow D_\tau$$

$$\text{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \perp_{D_\tau} & \text{se } v = \perp_{\mathbb{Z}_\perp} \\ d_1 & \text{se } v = \lfloor 0 \rfloor \\ d_2 & \text{altrimenti } (v = \lfloor n \rfloor \text{ con } n \neq 0) \end{cases}$$

$$v = \lfloor 0 \rfloor$$

$$\text{Cond}_\tau \left(\lfloor 0 \rfloor, \bigsqcup_{i \in \mathbb{N}} d_i, d \right) = \bigsqcup_{i \in \mathbb{N}} d_i = \bigsqcup_{i \in \mathbb{N}} \text{Cond}_\tau(\lfloor 0 \rfloor, d_i, d)$$

(continua)

$$\text{Cond}_\tau : \mathbb{Z}_\perp \times D_\tau \times D_\tau \rightarrow D_\tau$$

$$\text{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \perp_{D_\tau} & \text{se } v = \perp_{\mathbb{Z}_\perp} \\ d_1 & \text{se } v = \lfloor 0 \rfloor \\ d_2 & \text{altrimenti } (v = \lfloor n \rfloor \text{ con } n \neq 0) \end{cases}$$

$$v = \lfloor n \rfloor, n \neq 0$$

$$\text{Cond}_\tau \left(\lfloor n \rfloor, \bigsqcup_{i \in \mathbb{N}} d_i, d \right) = d = \bigsqcup_{i \in \mathbb{N}} d = \bigsqcup_{i \in \mathbb{N}} \text{Cond}_\tau(\lfloor n \rfloor, d_i, d)$$

TH. la lambda astrazione e' monotona e continua

$t : \tau$ $\lambda d. \llbracket t \rrbracket \rho[d/x]$ e' continua

ci concentriamo su una proprietà più forte

$\tilde{x}$ ennupla $x_1, \dots, x_n$ $\tilde{d}$ ennupla $d_1, \dots, d_n$

$\lambda \tilde{d}. \llbracket t \rrbracket \rho[\tilde{d}/\tilde{x}]$ e' continua

la prova e' per induzione strutturale su t

(provateci)

Corollary $t : \tau_0 \rightarrow \tau$ $\text{fix } \lambda d. \llbracket t \rrbracket \rho[d/x]$ e' continua

(il limite di funzioni continue è continuo)

Proprieta' principali

Lemma di sostituzione

$x, t_0 : \tau_0$
 $t : \tau$

$$\llbracket t[t_0 / x] \rrbracket \rho = \llbracket t \rrbracket \rho[\llbracket t_0 \rrbracket \rho / x]$$

update dell'ambiente

sostituzione sintattica

la prova e' per induzione strutturale su t
(provateci)

Composizionalità

Il lemma di sostituzione $\llbracket t^{t_0} / x \rrbracket \rho = \llbracket t \rrbracket \rho[\llbracket t_0 \rrbracket \rho / x]$ è importante perché garantisce la composizionalità della semantica denotazionale

TH. $\llbracket t_1 \rrbracket \rho = \llbracket t_2 \rrbracket \rho \quad \Rightarrow \quad \llbracket t^{t_1} / x \rrbracket \rho = \llbracket t^{t_2} / x \rrbracket \rho$

proof. assumiamo $\llbracket t_1 \rrbracket \rho = \llbracket t_2 \rrbracket \rho$

$$\begin{array}{ccccc} \llbracket t^{t_1} / x \rrbracket \rho = \llbracket t \rrbracket \rho[\llbracket t_1 \rrbracket \rho / x] & = & \llbracket t \rrbracket \rho[\llbracket t_2 \rrbracket \rho / x] & = & \llbracket t^{t_2} / x \rrbracket \rho \\ | & & | & & | \\ \text{lemma} & \llbracket t_1 \rrbracket \rho = \llbracket t_2 \rrbracket \rho & \text{lemma} \\ \text{di sostituzione} & & \text{di sostituzione} \end{array}$$

Solo le variabili free hanno importanza

TH. $t : \tau$
 $\forall x \in \text{fv}(t). \rho(x) = \rho'(x) \quad \Rightarrow \quad \llbracket t \rrbracket \rho = \llbracket t \rrbracket \rho'$

la prova e' per induzione strutturale su t
(provateci)

Corollary t chiuso $\Rightarrow \quad \forall \rho, \rho'. \llbracket t \rrbracket \rho = \llbracket t \rrbracket \rho'$

TH. I termini canonici non sono bottom

$$c \in C_\tau \Rightarrow \forall \rho. \llbracket c \rrbracket \rho \neq \perp_{D_\tau}$$

prova. per induzione sulle regole dei termini canonici

$$P(c \in C_\tau) \triangleq \forall \rho. \llbracket c \rrbracket \rho \neq \perp_{D_\tau}$$

$$\frac{}{n \in C_{int}}$$

$$\llbracket n \rrbracket \rho = \lfloor n \rfloor \neq \perp_{D_{int}}$$

$$\frac{t_0 : \tau_0 \quad t_1 : \tau_1 \quad t_0, t_1 \text{ closed}}{(t_0, t_1) \in C_{\tau_0 * \tau_1}}$$

$$\llbracket (t_0, t_1) \rrbracket \rho = \lfloor (\llbracket t_0 \rrbracket \rho, \llbracket t_1 \rrbracket \rho) \rfloor \neq \perp_{D_{\tau_0 * \tau_1}}$$

$$\frac{\lambda x. t : \tau_0 \rightarrow \tau_1 \quad \lambda x. t \text{ closed}}{\lambda x. t \in C_{\tau_0 \rightarrow \tau_1}}$$

$$\llbracket \lambda x. t \rrbracket \rho = \lfloor \lambda d. \llbracket t \rrbracket \rho [^d / x] \rfloor \neq \perp_{D_{\tau_0 \rightarrow \tau_1}}$$