Optimality, relaxations and bounds

(Wolsey : Chapter 2)

Given an optimization problem of form

$$z = \max_{x \in X} c(x)$$

how can we prove that a feasible x^* is optimal? If we find a lower bound (l.b.) $\underline{z} \leq z$ and an upper bound (u.b.) $\overline{z} \geq z$ such that $\underline{z} = c(x^*) = \overline{z}$, then x^* is optimal.

Algorithmic consequence:

- find a decreasing sequence of u.b.:

$$\overline{z}_1 > \overline{z}_2 > \dots > \overline{z}_s$$

- find an increasing sequence of l.b.:

$$\underline{z}_1 < \underline{z}_2 < \dots < \underline{z}_t$$

- stop when

$$\bar{z}_s - \underline{z}_t \leq \varepsilon$$

for a given $\varepsilon \geq 0$.

So : <u>how to compute l.b. and u.b.?</u>

<u>Primal bounds</u> ($\equiv$ l.b.)

each feasible solution $\bar{x} \in X$ gives a l.b. (for a maximization problem); in fact

$$c(\bar{x}) \leq z$$

* this is the way to compute l.b: *

<u>heuristics</u>

<u>Dual bounds</u> ($\equiv$ u.b.)

finding u.b. for a maximization problem (l.b. for a minimization problem) is not immediate:

* the most important approach is *

"<u>by relaxation</u>"

<u>Idea</u>: replace a "difficult" max (min) optimization problem by a "simpler" optimization problem s.t:

<u>Def</u>: a problem (RP):

$$(RP) \qquad z^R = \max_{x \in T} f(x)$$

is a relaxation of problem (P):

$$(P) \qquad z = \max_{x \in X} c(x)$$

if:

(i) $X \subseteq T$

(ii) $f(x) \geq c(x) \quad \forall x \in X$

<u>Proposition</u>: if (RP) is a relaxation of (P), then $\boxed{z^R \geq z}$ (u.b.!)

<u>Proof</u>:

Assume x^* optimal solution to (P)