# Business Processes Modelling

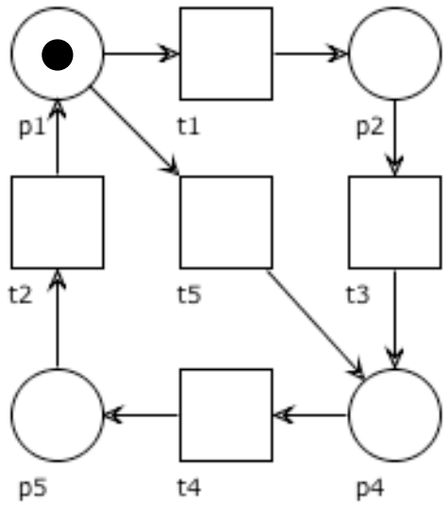
## MPB (6 cfu, 295AA)

Roberto Bruni

http://www.di.unipi.it/~bruni

## 16 - S-systems

# Object




We study some “good” properties of S-systems

Free Choice Nets (book, optional reading)
https://www7.in.tum.de/~esparza/bookfc.html

# S-systems

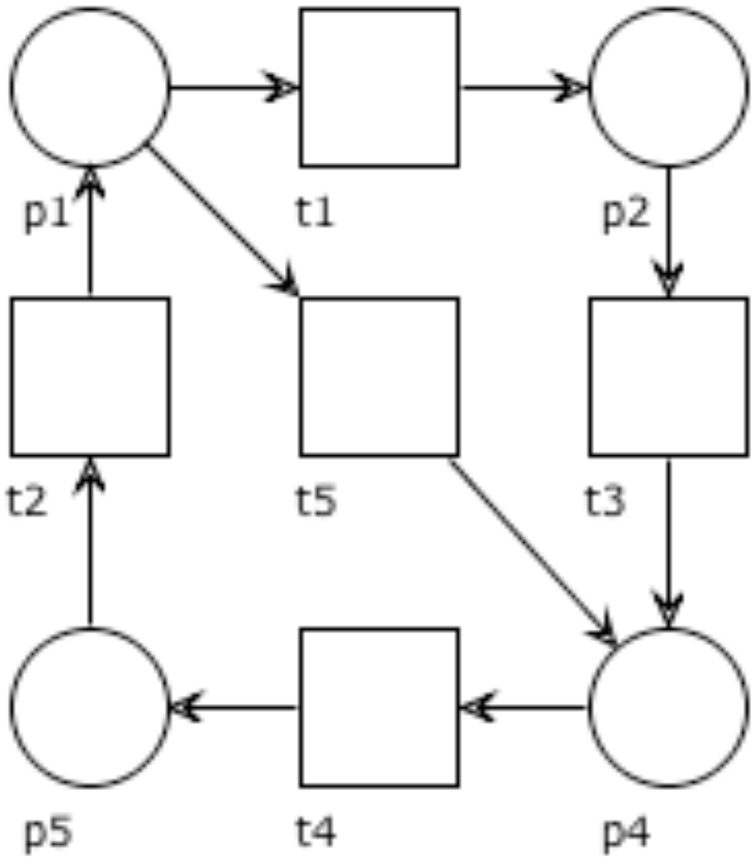
# S-system

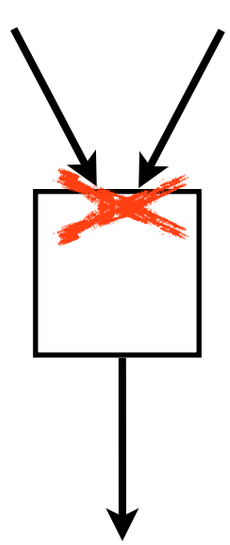
**Definition**: We recall that a net N is an **S-net** if each transition has exactly one input place and exactly one output place

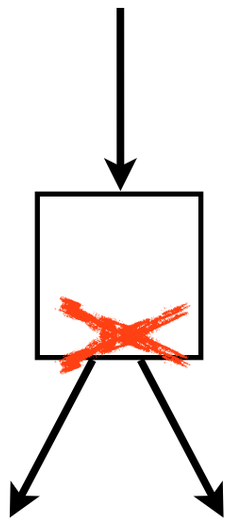
$$\forall t \in T, \qquad |\bullet t| = 1 = |t \bullet|$$

A system $(N,M_0)$ is an **S-system** if N is an S-net

# S-net: example





# S-net N*

**Proposition**: A workflow net N is an S-net
iff
N* is an S-net

N and N* differ only for the reset transition,
that has exactly one incoming arc
and exactly one outgoing arc

# S-systems: an observation

**Observation**: each transition t that fires removes exactly one token from some place p and inserts exactly one token in some place q (p and q can also coincide)

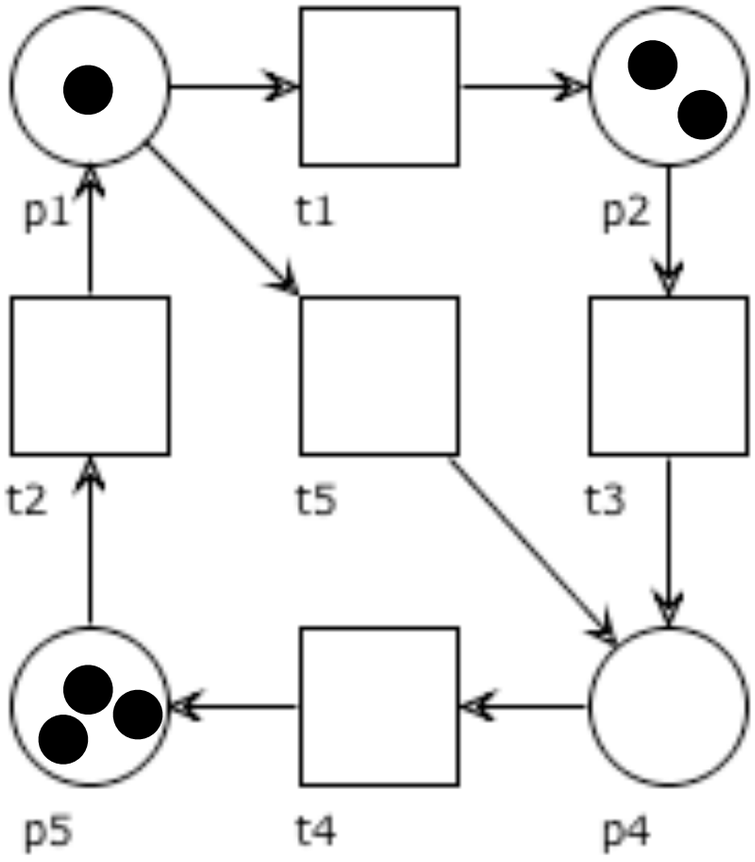
# Notation: token count

$$M(P) = \sum_{p \in P} M(p)$$

## Example

$P = \{p_1, p_2, p_3\}$ $M = 2p_1 + 3p_2$ $M(P) = 2 + 3 + 0 = 5$

# Question time




$M_0(P) = ?$

$M_0 = p_1 + 2p_2 + 3p_5$

$M_0(P) = 6$

# Fundamental property of S-systems

**The overall number of tokens in the S-net is an invariant** under any firing.

# Fundamental property of S-systems

**Proposition**: Let (P,T,F,$M_0$) be an S-system.
If M is a reachable marking, then M(P) = $M_0$(P)

We show that for any $M \xrightarrow{\sigma} M'$ we have $M'(P) = M(P)$

**base** ($\sigma = \epsilon$)**:** trivial ($M' = M$)

**induction** ($\sigma = \sigma' t$ **for some** $\sigma' \in T^*$ **and** $t \in T$)**:**

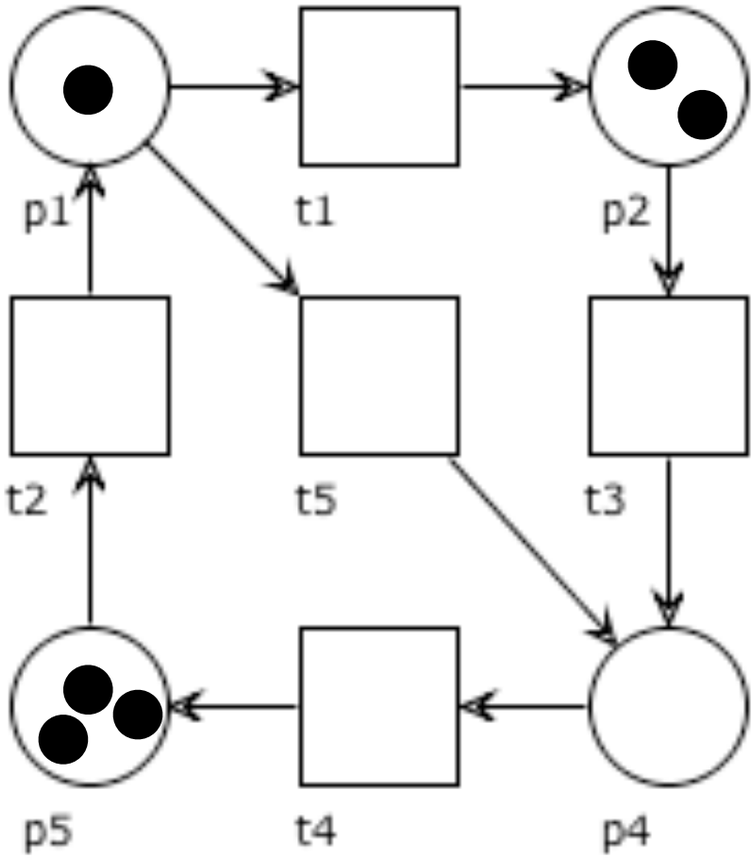
Let $M \xrightarrow{\sigma'} M'' \xrightarrow{t} M'$.

By inductive hypothesis: $M''(P) = M(P)$

By definition of S-system: $|\bullet t| = |t \bullet| = 1$

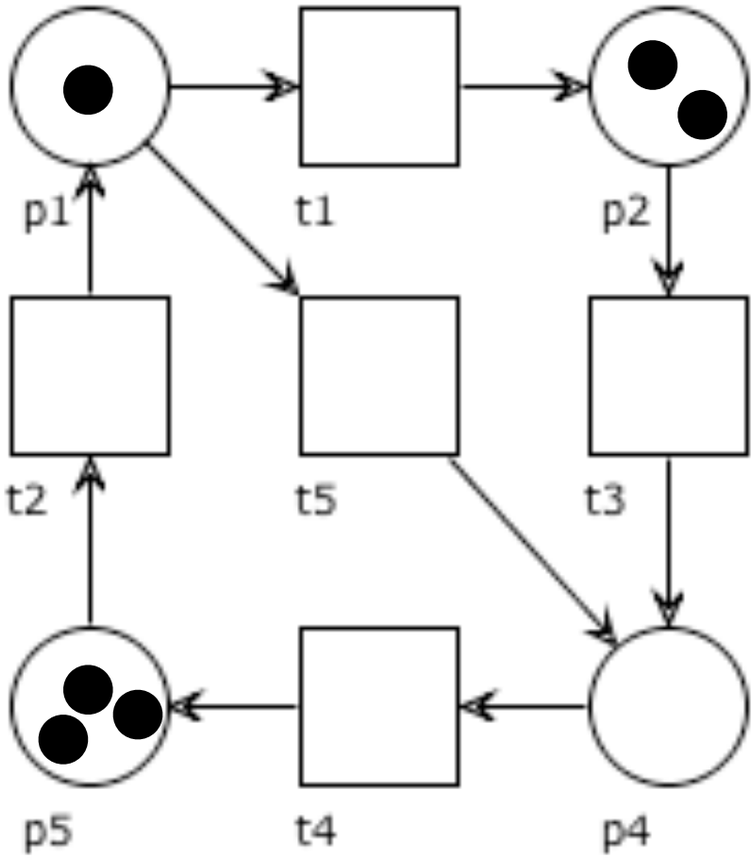
Thus, $M'(P) = M''(P) - |\bullet t| + |t \bullet| = M(P) - 1 + 1 = M(P)$

# Question time




Is the marking
$M = p_2 + 4p_4 + 2p_5$
reachable?

# Question time




Is the marking
$M = p_2 + 4p_4 + 2p_5$
reachable?

No:
S-system
$M_0(P) = 6 \neq 7 = M(P)$

# A consequence of the fundamental property

**Corollary**: Any S-system is bounded

Let $M \in [\, M_0 \,\rangle$.

By the fundamental property of S-systems: $M(P) = M_0(P)$.

Then, for any $p \in P$ we have $M(p) \leq M(P) = M_0(P)$.

Thus the S-system is $k$-bounded for any $k \geq M_0(P)$.

$$M(P) = \sum_{p \in P} M(p)$$

# S-invariants of S-nets

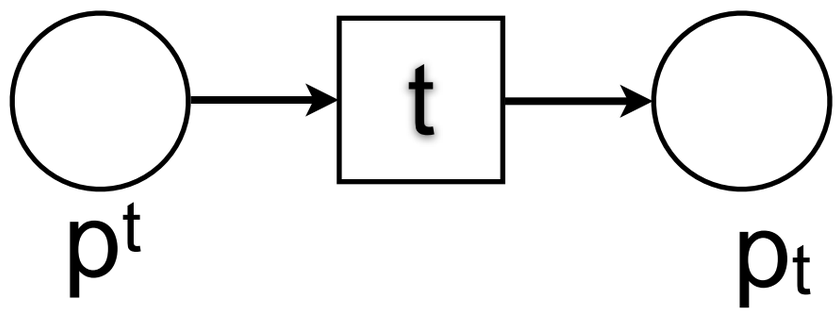
**Proposition**: Let N=(P,T,F) be a (connected) S-net.
**I** is an S-invariant of N **iff I**=[ k ... k ] for some value k

S-invariance $\forall t \in T, \sum_{p \in \bullet t} \mathbf{I}(p) = \sum_{p \in t \bullet} \mathbf{I}(p)$

S-nets $\forall t \in T, |\bullet t| = |t \bullet| = 1$

Let $\bullet t = \{p^t\}$ and $t \bullet = \{p_t\}$

$\forall t \in T, \mathbf{I}(p^t) = \mathbf{I}(p_t)$



# S-invariants of S-nets

**Proposition**: Let N=(P,T,F) be a (connected) S-net.
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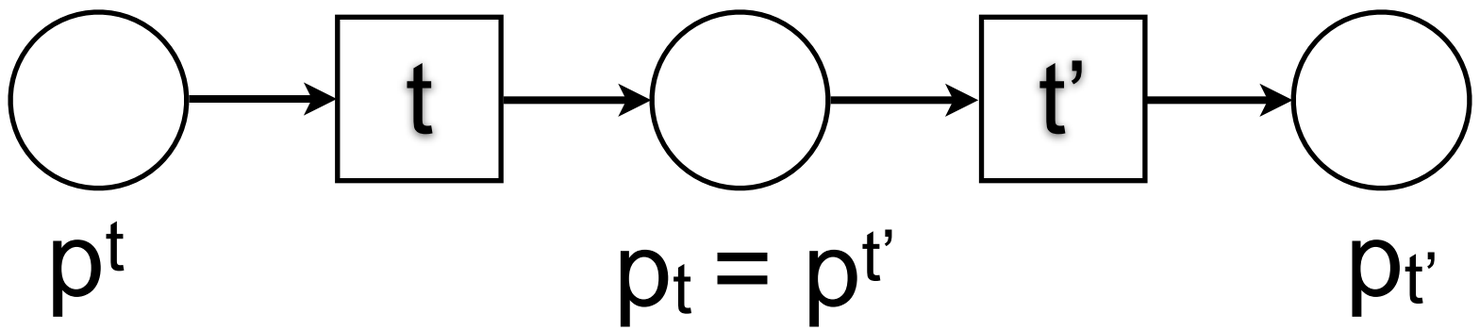


$$\mathbf{I}(p^t) = \mathbf{I}(p_t) = \mathbf{I}(p^{t'}) = \mathbf{I}(p_{t'})$$

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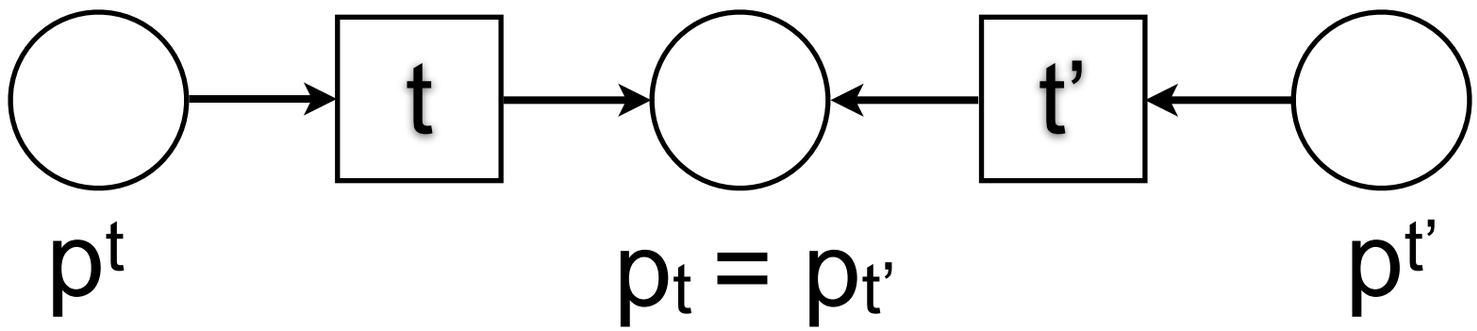


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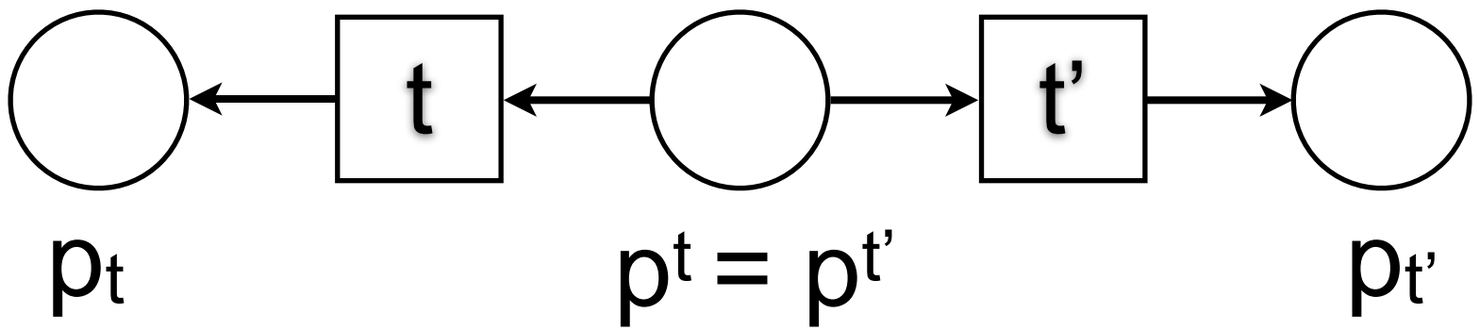


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# S-invariants of S-nets

**Proposition**: Let N=(P,T,F) be a (connected) S-net.
**I** is an S-invariant of N **iff I**=[ k ... k ] for some value k

$$\forall p_0, p_n \in P,\ \mathbf{I}(p_0) = \mathbf{I}(p_n)$$

# Quick Recap

Let N=(P,T,F,$M_0$) be an S-system

**Prop.** M reachable implies M(P) = $M_0$(P)

**Corollary.** N is bounded

**Prop. I** is an S-invariant iff **I**=[ k ... k ] for some k

# A note on S-invariants and S-nets

S-invariance

$$\forall M \in [\, M_0 \,\rangle, \quad \mathbf{I} \cdot M = \mathbf{I} \cdot M_0$$

S-invariant of S-nets

$$\mathbf{I} = [\, 1 \; 1 \; ... \; 1 \,]$$

consequence

$$\forall M, \quad \mathbf{I} \cdot M = \sum_{p \in P} 1 \cdot M(p) = \sum_{p \in P} M(p) = M(P)$$

We recover the Fundamental property of S-nets

$$\forall M \in [\, M_0 \,\rangle, \quad M(P) = \mathbf{I} \cdot M = \mathbf{I} \cdot M_0 = M_0(P)$$

# Liveness theorem for S-systems

**Theorem**: An S-system $(N,M_0)$ is live **iff**
N is strongly connected and $M_0$ marks at least one place

$\Rightarrow$) (quite obvious)
$(N, M_0)$ is live by hypothesis and bounded (because S-system).
By the strong connectedness theorem, $N$ is strongly connected.

Since $(N, M_0)$ is live, then $M_0 \xrightarrow{t}$ for some $t$.

Assume $\bullet t = \{p\}$. Thus, $M_0(p) \geq 1$.

# Liveness theorem for S-systems

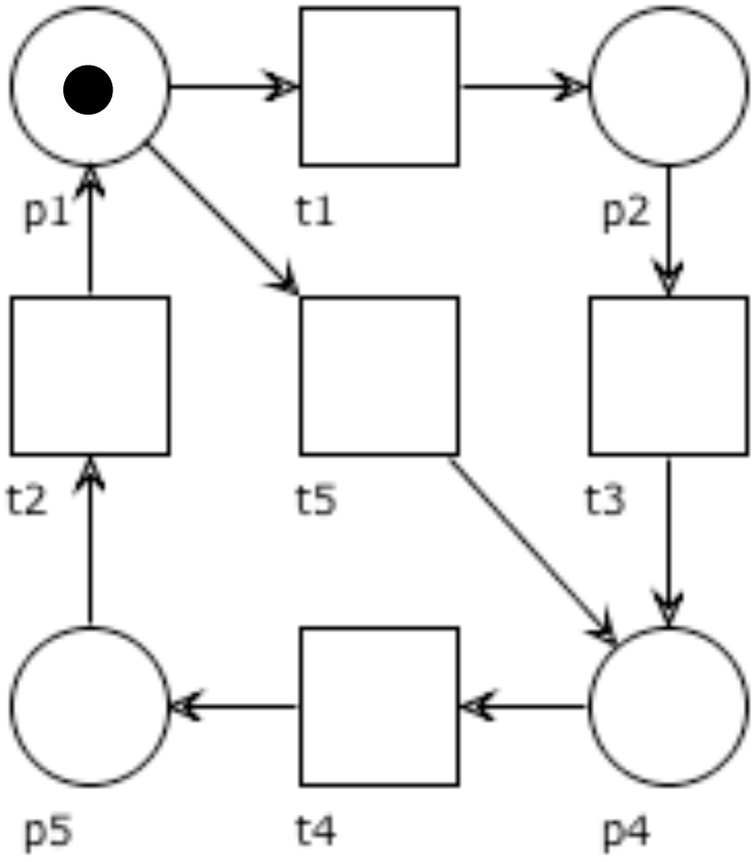
**Theorem**: An S-system $(N,M_0)$ is live **iff**
N is strongly connected and $M_0$ marks at least one place

$\Leftarrow$) (more interesting)
Take any $M \in [\, M_0 \,\rangle$ and $t \in T$.
We want to find $M' \in [\, M \,\rangle$ such that $M' \xrightarrow{t}$.

Take $p_1 \in P$ such that $M(p_1) \geq 1$ (it exists, because $M(P) = M_0(P) \geq 1$).
By strong connectedness: there is a path from $p_1$ to $t_n = t$
$(p_1, t_1)(t_1, p_2)(p_2, t_2)...(p_n, t_n)$

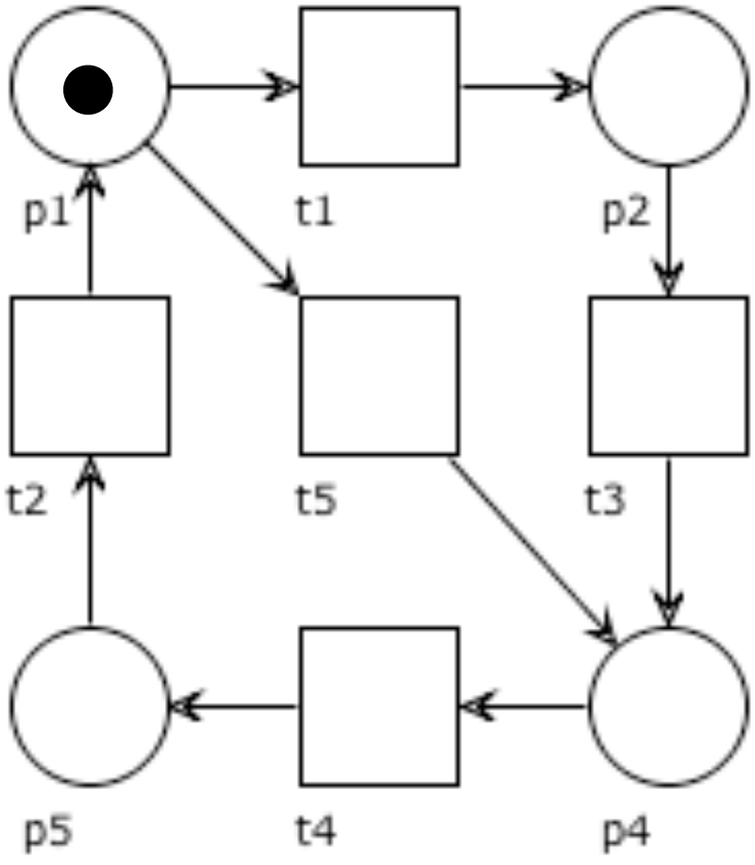
By definition of S-system: $\bullet t_i = \{p_i\}$ and $t_i \bullet = \{p_{i+1}\}$.
Thus, $M \xrightarrow{\sigma} M' \xrightarrow{t}$ for $\sigma = t_1 t_2 ... t_{n-1}$.

# Question time




Is this system live?

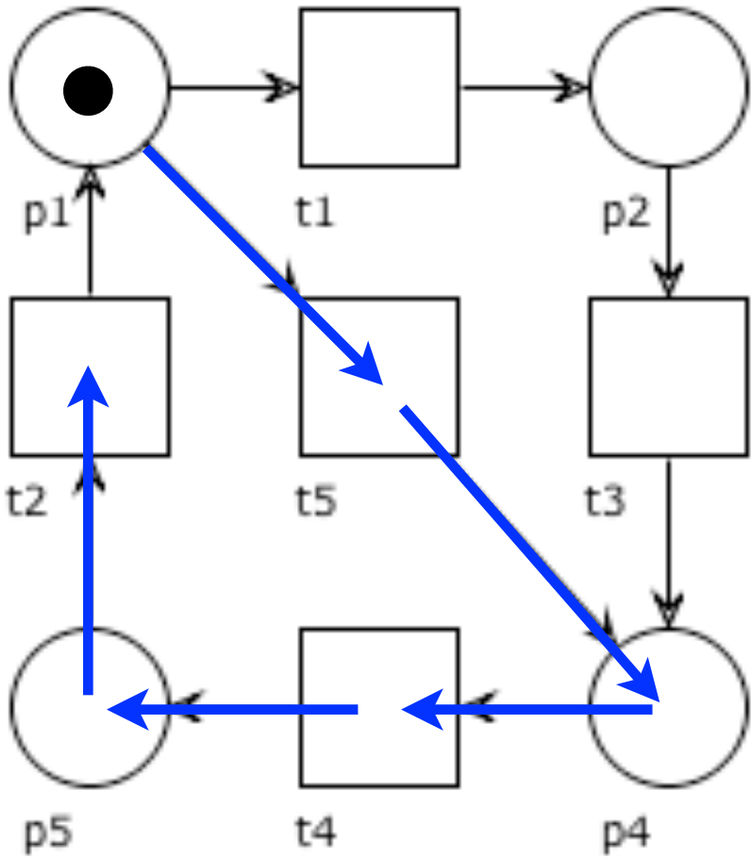
# Question time




Is this system live?

Yes:
S-system
strongly connected
$M_0(P) > 0$

# Question time

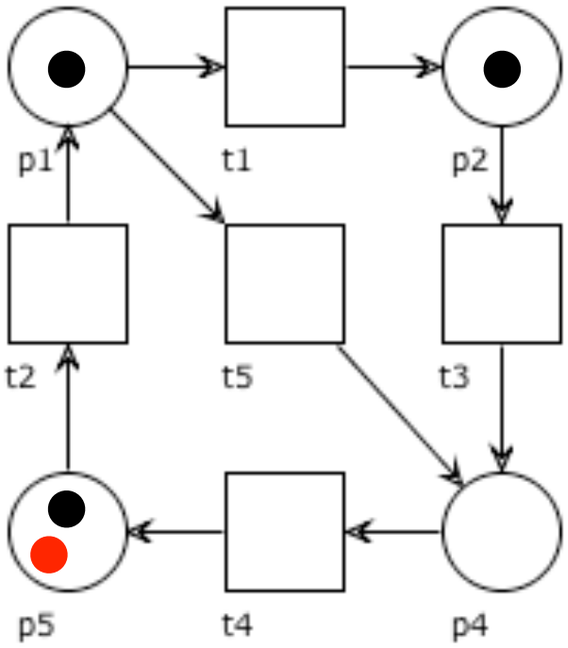


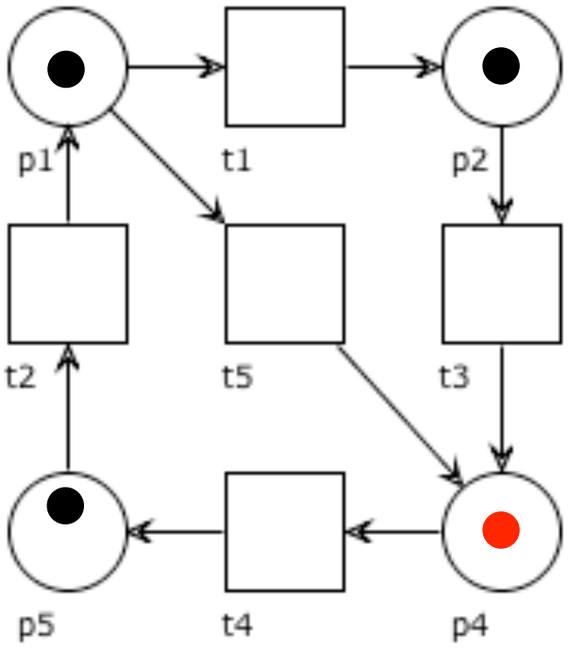

Is this system live?

How to enable t2?
just find a path
from p1 to t2

# Reachability lemma for S-nets

**Lemma**: Let (P,T,F) be a strongly connected S-net.
If M(P) = M'(P), then M' is reachable from M




**Intuitively:**

M and M' has the same number of tokens
but maybe they are in different places

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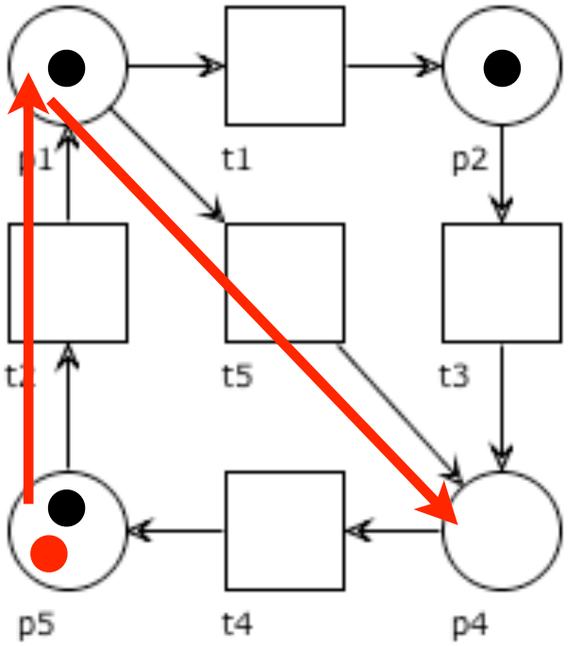
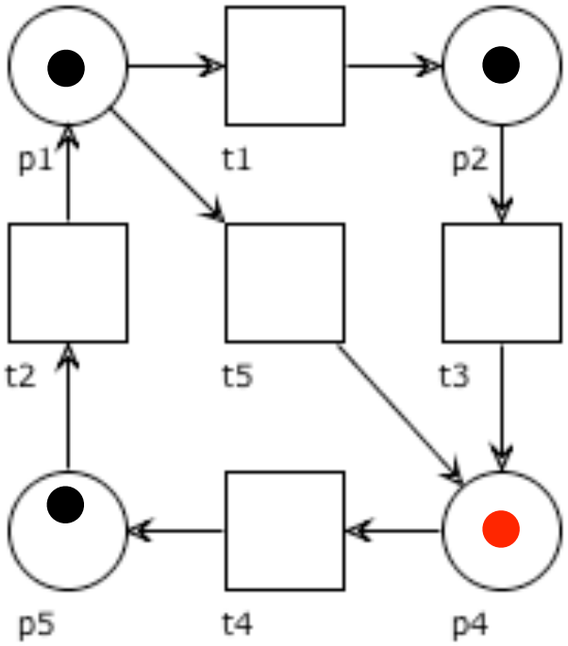



if there is only one token in the wrong place




**Intuitively:**

we need to move the token from p to q
by strong connectedness we have a path from p to q
by firing the transitions along the path we succeed

# Reachability lemma for S-nets

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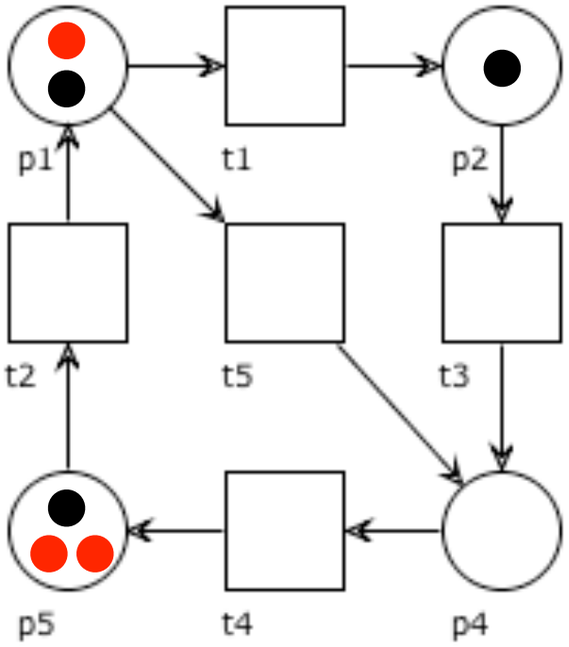
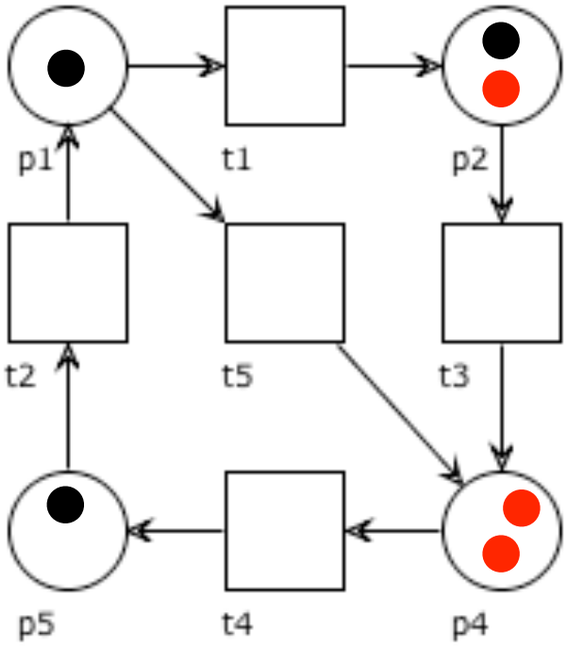



if there are many (n) tokens in the wrong places



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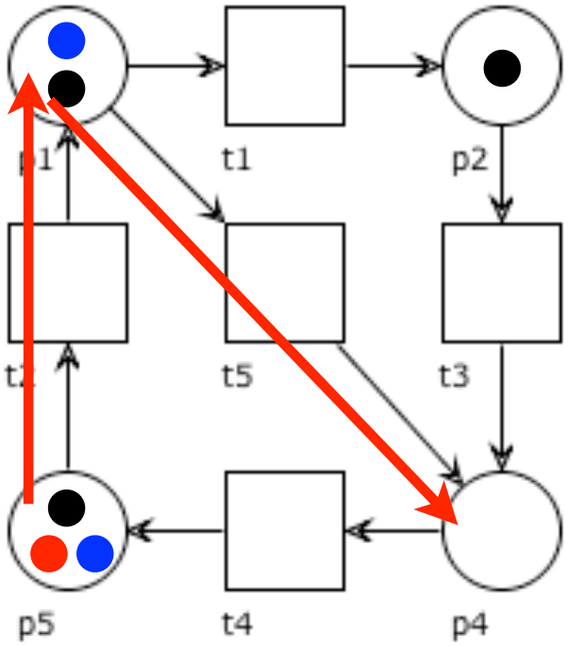


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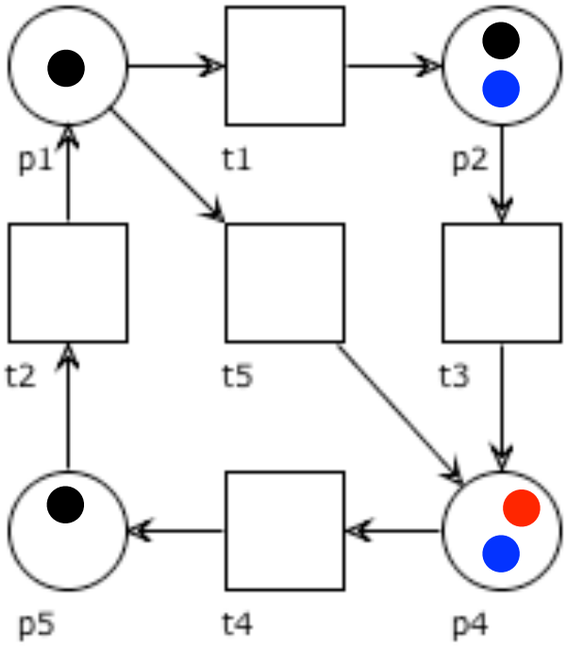



**Intuitively:**
we move one of them…
and inductively the remaining n-1

# Reachability lemma for S-nets

**Lemma**: Let (P,T,F) be a strongly connected S-net. If M(P) = M'(P), then M' is reachable from M

We proceed by induction on $M(P)$

**base** $(M(P) = M'(P) = 0)$**:** trivial $(M' = M)$

**induction** $(M(P) = M'(P) > 0)$**:**

Let $p, p' \in P$ be such that $M(p) > 0$ and $M'(p') > 0$.

Let $K = M - p$ and $K' = M' - p'$.

Clearly $K'(P) = K(P) < M(P) = M'(P)$.

By inductive hypothesis: $\exists \sigma,\ K \xrightarrow{\sigma} K'$

By strong connectedness: there is a path from $p_0 = p$ to $p_n = p'$

$(p_0, t_1)(t_1, p_1)(p_1, t_2)...(t_n, p_n)$

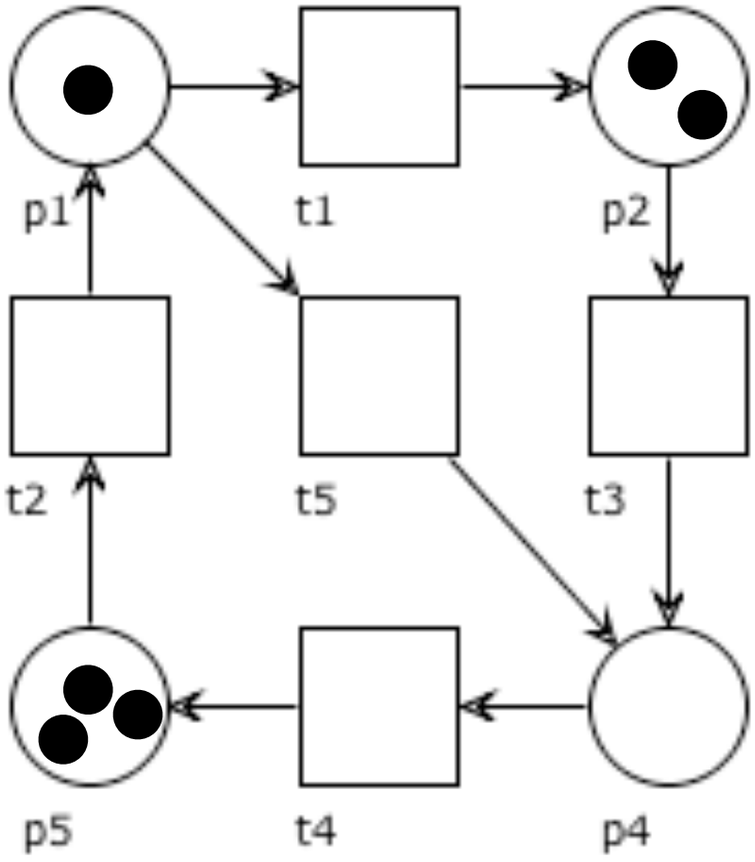
By definition of S-system: $\bullet t_i = \{p_{i-1}\}$ and $t_i \bullet = \{p_i\}$.

Thus, $p = p_0 \xrightarrow{\sigma'} p_n = p'$ for $\sigma' = t_1 t_2 ... t_n$.

By the monotonicity lemma: $M = K + p \xrightarrow{\sigma} K' + p \xrightarrow{\sigma'} K' + p' = M'$

Only for the interested reader!

# Question time

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$M = p_2 + 4p_4 + p_5$
reachable?

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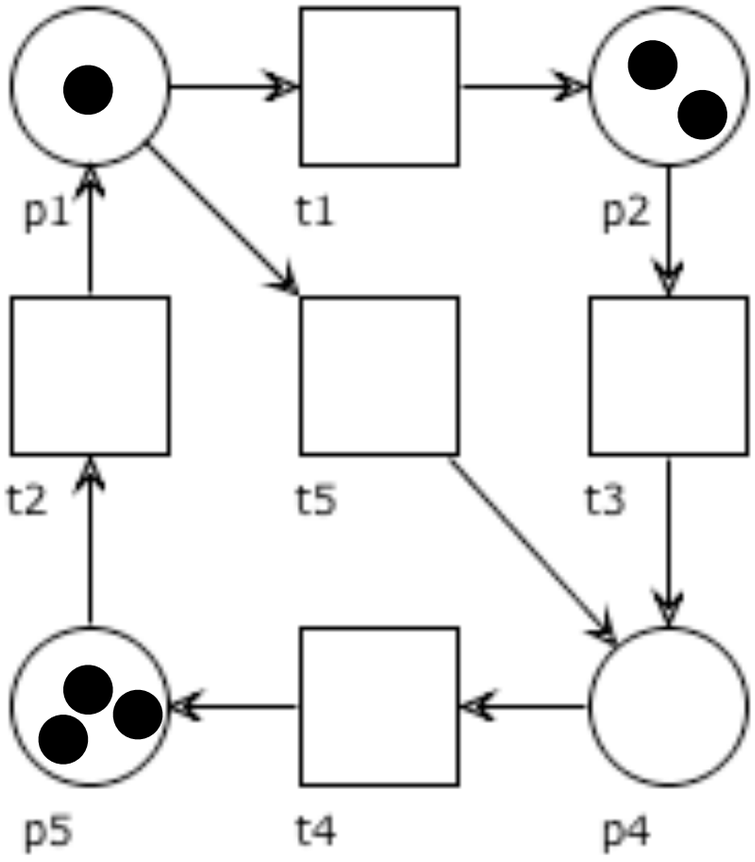



Is the marking
$M = p_2 + 4p_4 + p_5$
reachable?

Yes:
S-system
strongly connected
$M_0(P) = 6 = M(P)$

# Reachability Theorem for S-systems

**Theorem**: Let $(P,T,F,M_0)$ be a live S-system.
A marking M is reachable **iff** $M(P)=M_0(P)$

=>) Follows from the fundamental property of S-systems

<=) By the previous liveness theorem, the S-net is strongly connected.
We conclude by applying the reachability lemma for S-systems.

# S-systems: recap

S-system => bounded
S-system: strong conn. + $M_0(P)>0$ <=> live

S-system + M reachable => $M(P) = M_0(P)$
S-system + str. conn.: $M(P)=M_0(P)$ <=> M reachable
S-system + live: $M(P)=M_0(P)$ <=> M reachable

S-system: S-invariant **I** <=> **I** = [ k k ... k ]

# Workflow S-nets

**Theorem**: If a workflow net N is an S-system then it is sound

N is S-system <=> N* is S-system

N and N* S-systems => N and N* bounded

N workflow net => N* strong connected
$M_0(P)=1$ (initially one token in place i)
N* strong connected + $M_0(P) > 0$ <=> N* live

N* bounded and live <=> N sound

# Workflow S-nets

**Theorem**: If a workflow net N is an S-system
then it is **safe** and sound

N is S-system <=> N* is S-system

$M_0(P)=1$ (initially one token in place i)

**N and N* S-systems + $M_0(P)=1$ => N and N* safe**

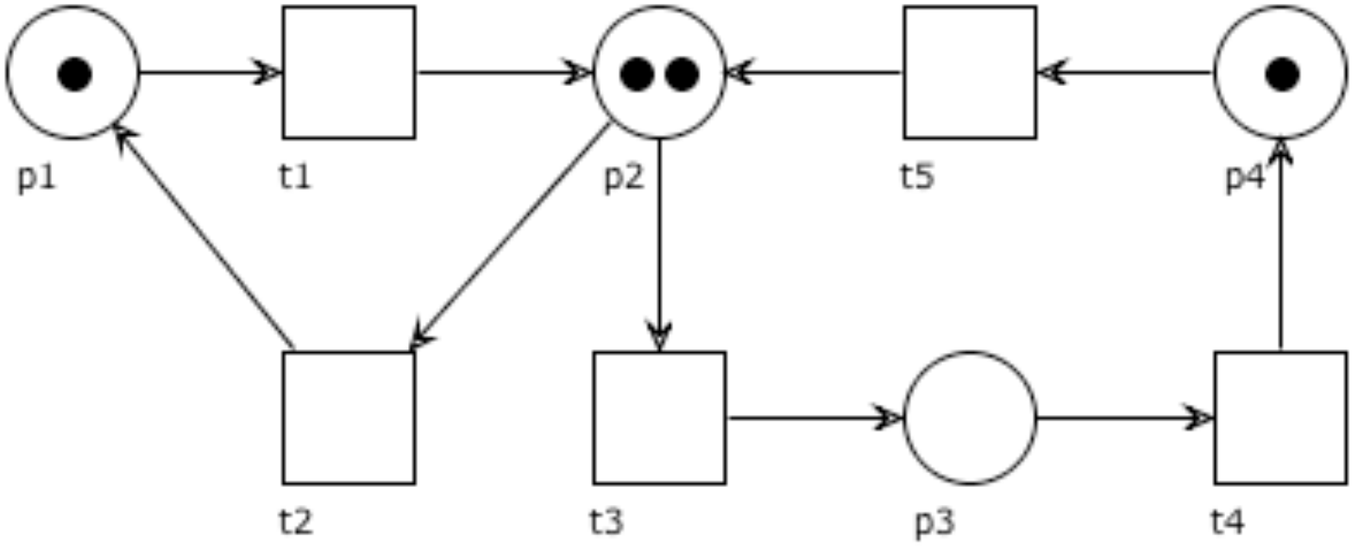
N workflow net => N* strong connected
N* strong connected + $M_0(P) = 1$ <=> N* live

N* bounded and live <=> N sound

# Question time

Which of the following markings are reachable? (why?)

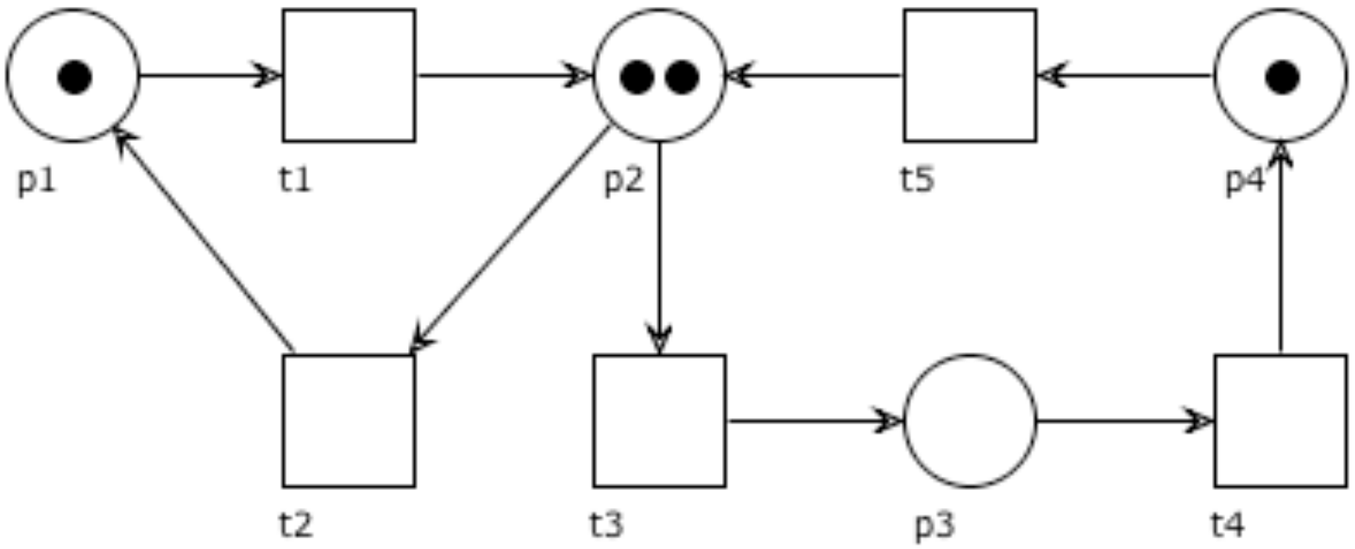
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 0 \\ 1 & 2 & 1 & 2 \\ 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 4 \end{bmatrix}$$



# Question time

Which of the following markings are reachable? (why?)

strong connected




[ 1 1 1 1 ]
[ 2 0 2 0 ]
[ 1 2 1 2 ]
[ 4 0 0 0 ]
[ 0 4 0 4 ]

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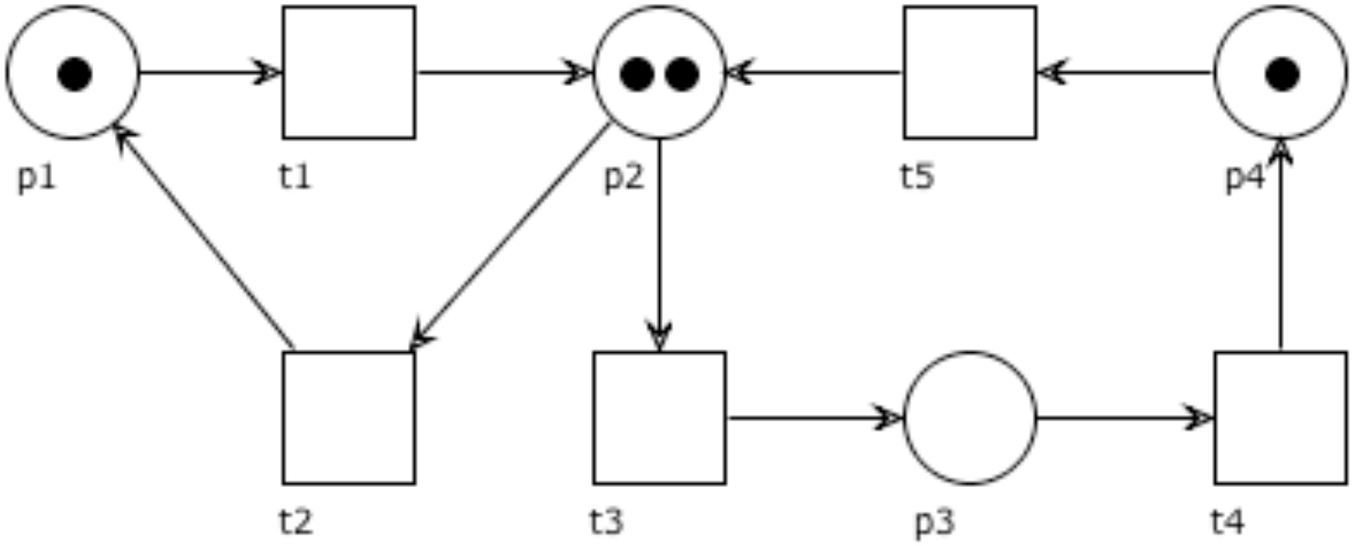
Which of the following markings are reachable? (why?)

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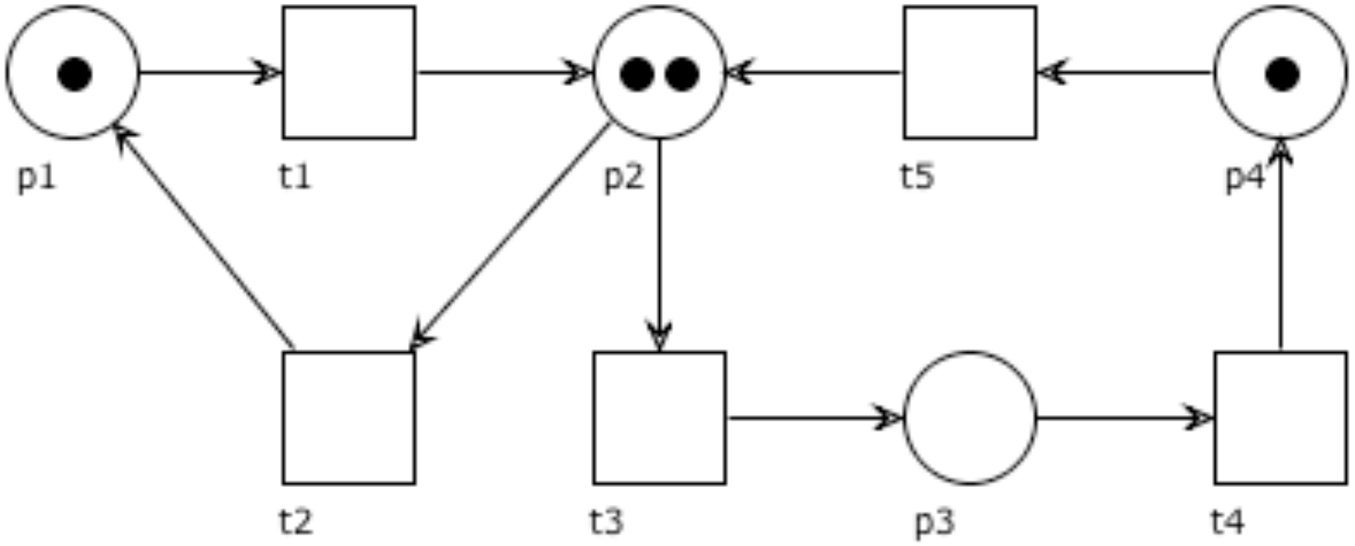


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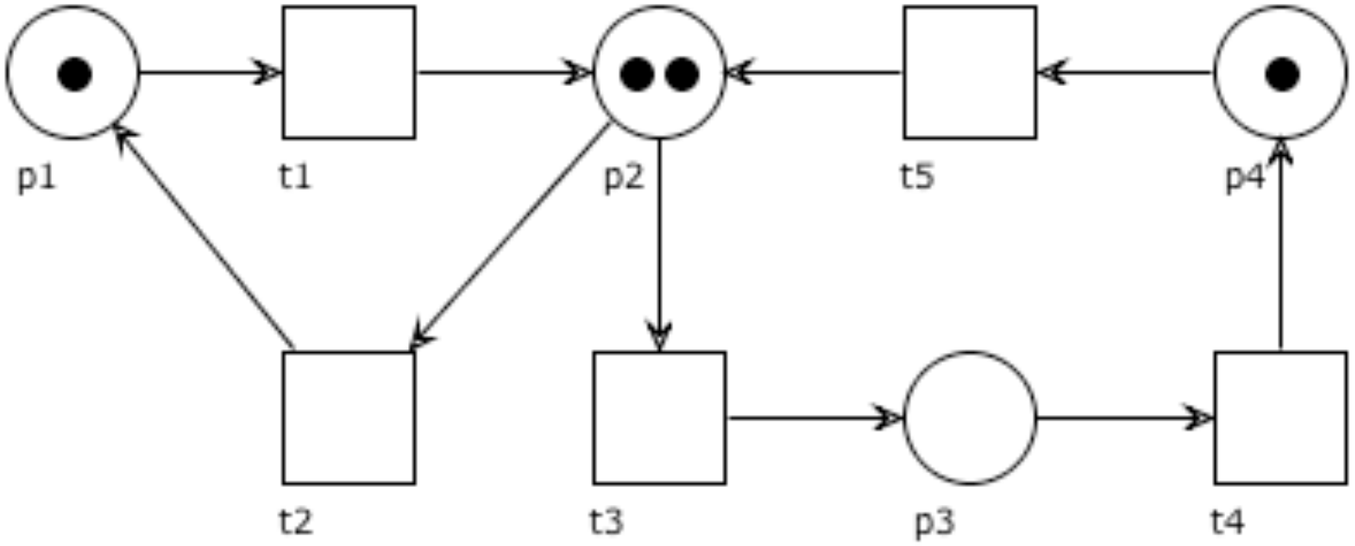


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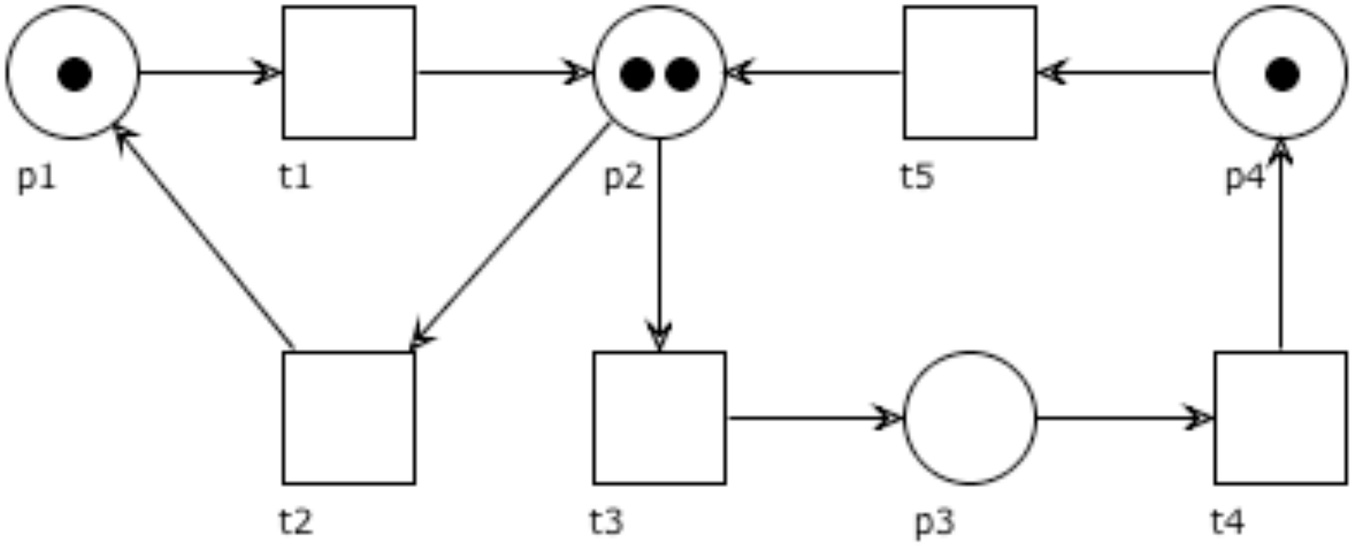
# Question time

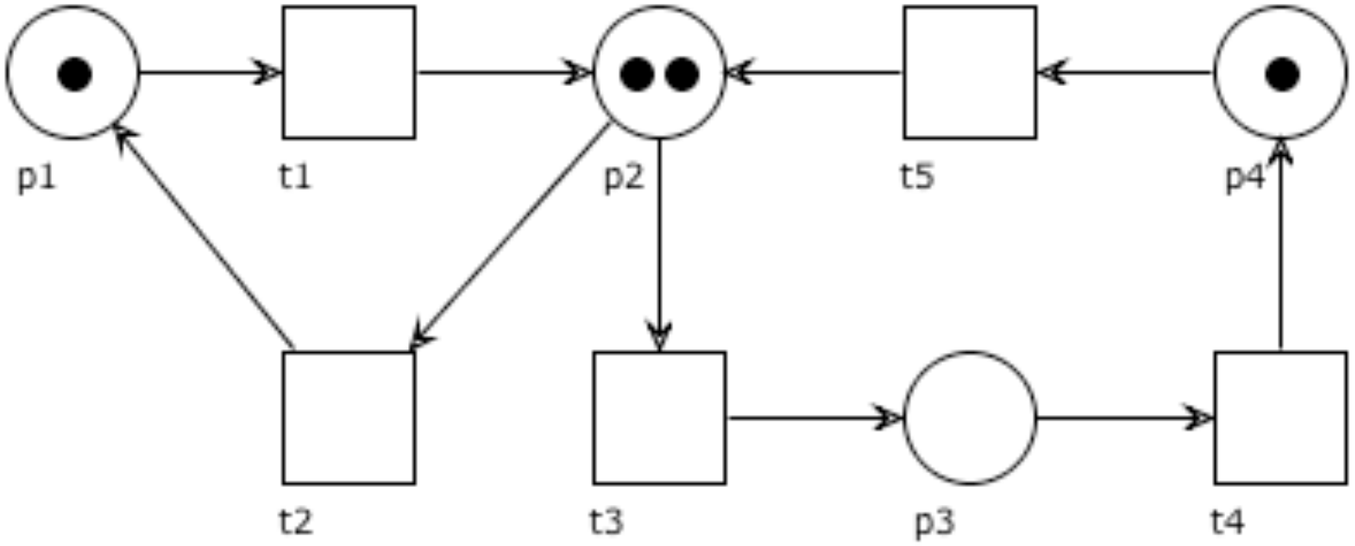
Which of the following markings are reachable? (why?)




[ 1 1 1 1 ]
[ 2 0 2 0 ]
[ 1 2 1 2 ]
[ 4 0 0 0 ]
[ 0 4 0 4 ]

# Question time

Which of the following are S-invariants? (why?)




$$\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 & 2 & 2 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 2 & 1 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 2 & 2 & 2 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 2 & 2 & 1 \end{bmatrix}$$

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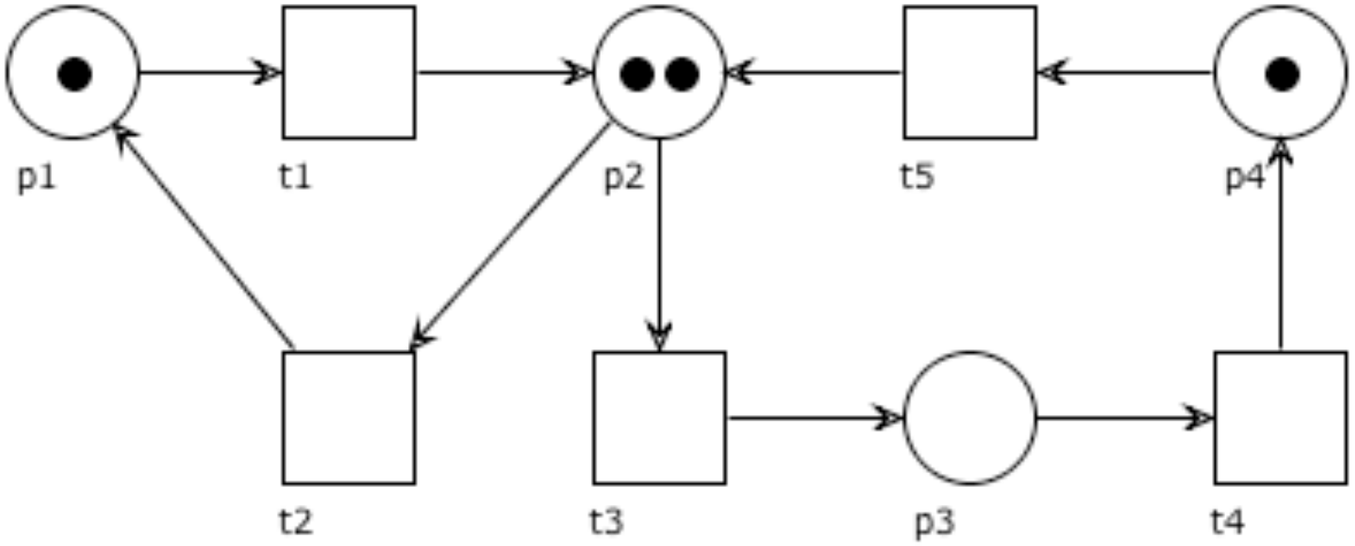


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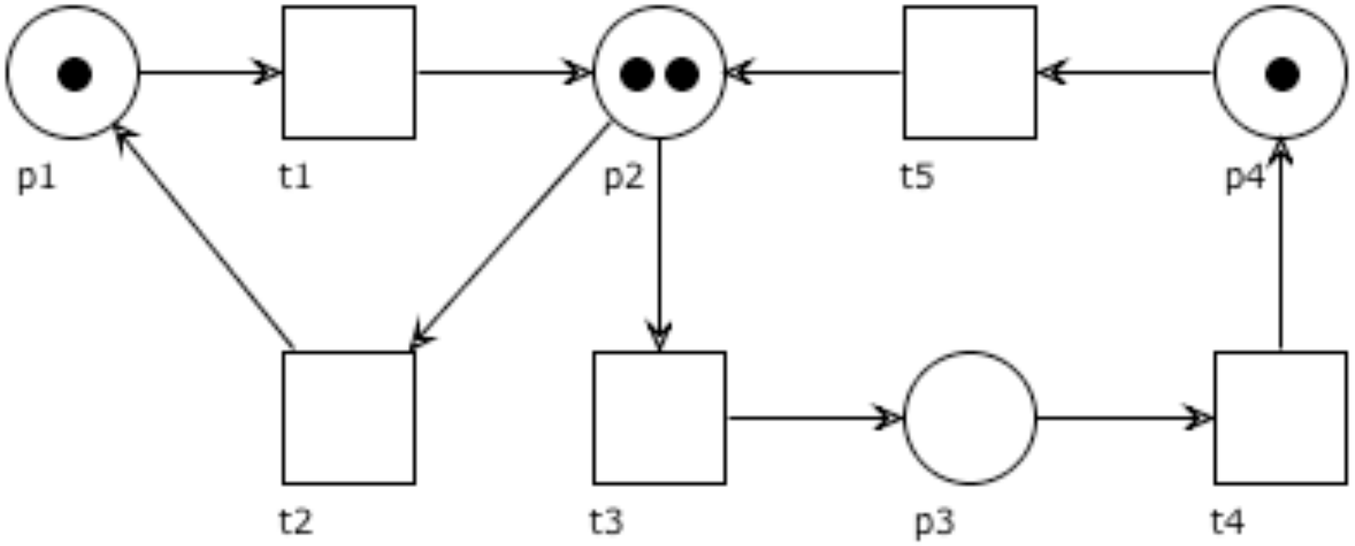


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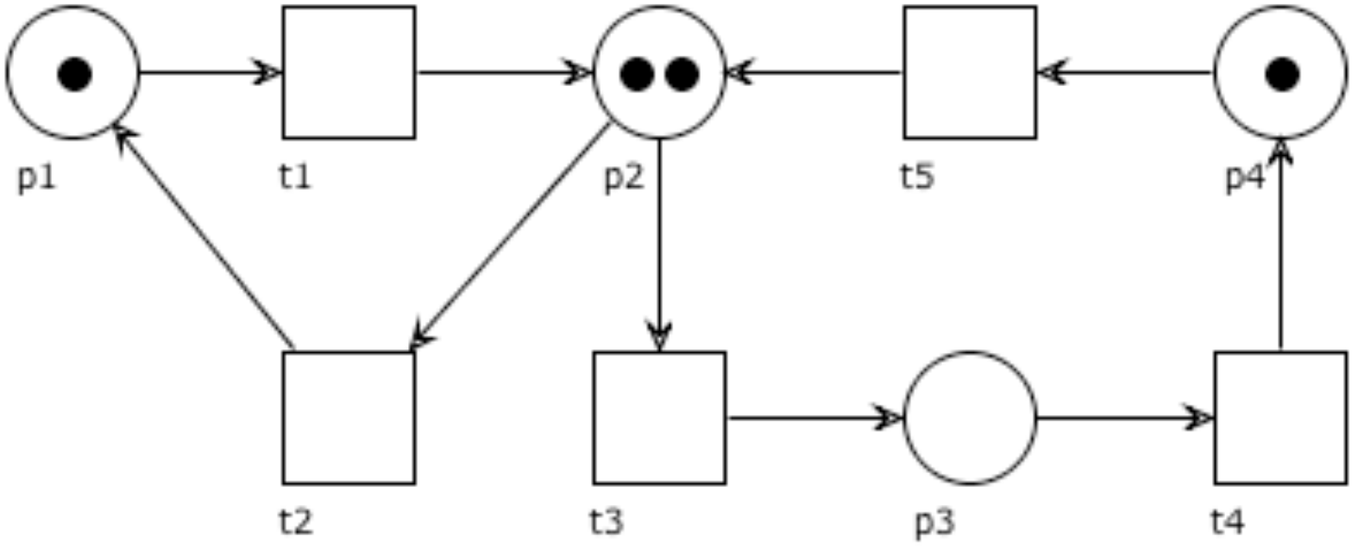


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$$\begin{bmatrix} 0 & 0 & 2 & 2 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$
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$$\begin{bmatrix} 1 & 2 & 2 & 1 \end{bmatrix}$$

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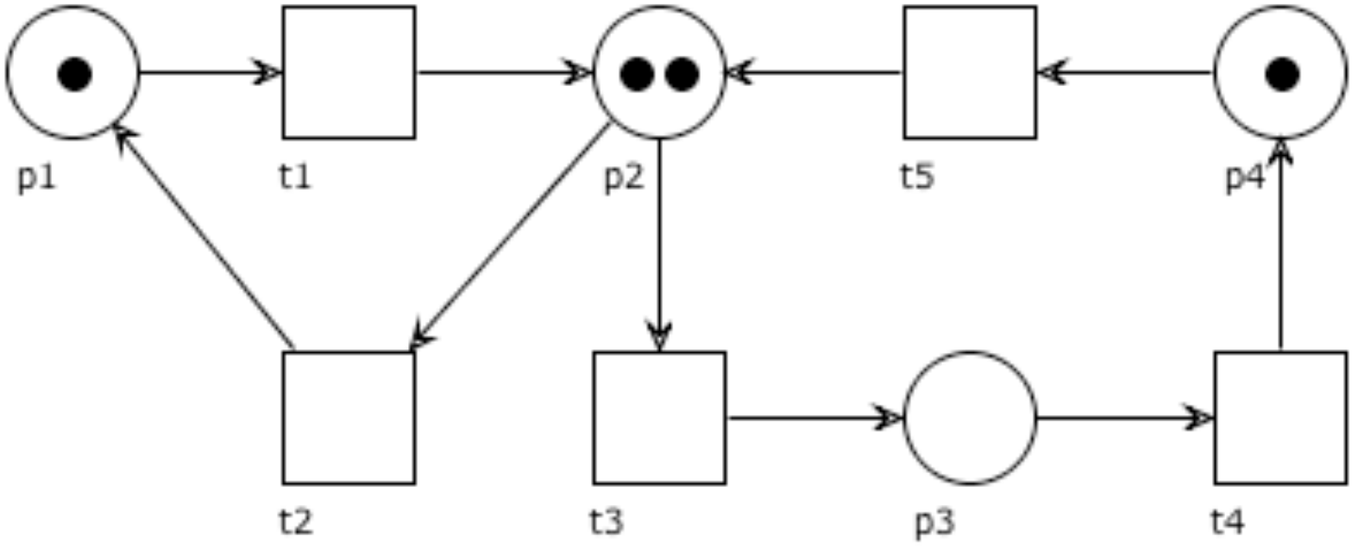


[ 1 1 0 0 ]
[ 0 0 2 2 ]
[ 1 1 1 1 ]
[ 2 2 1 1 ]
[ 2 2 2 2 ]
[ 1 2 2 1 ]

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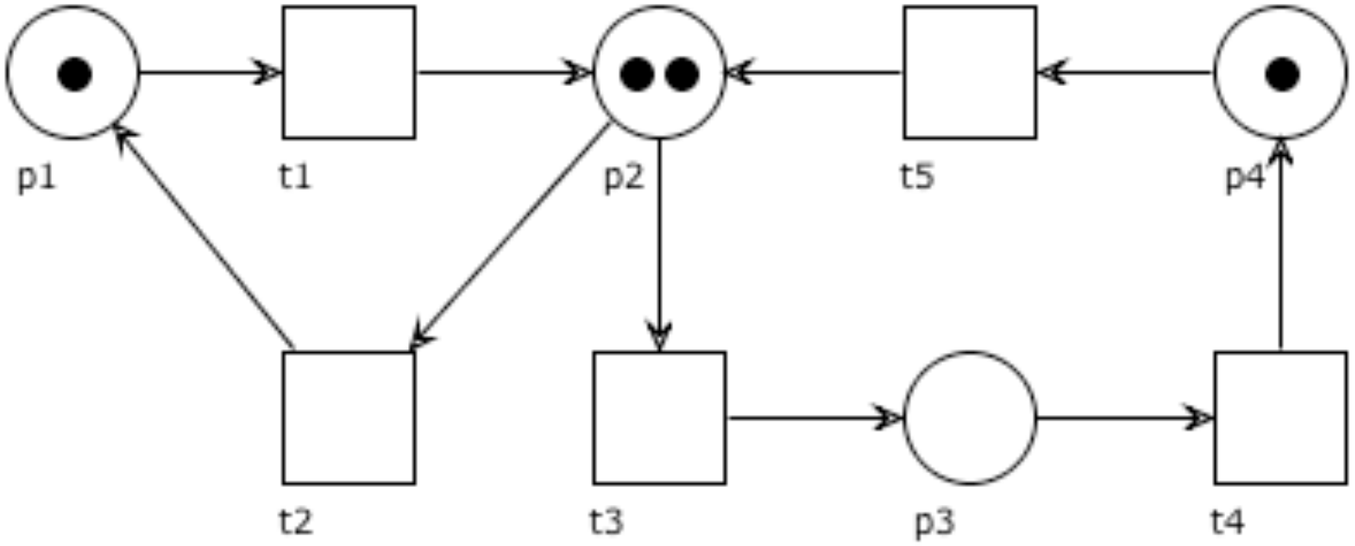


[ 1 1 0 0 ]
[ 0 0 2 2 ]
[ 1 1 1 1 ]
[ 2 2 1 1 ]
[ 2 2 2 2 ]
[ 1 2 2 1 ]

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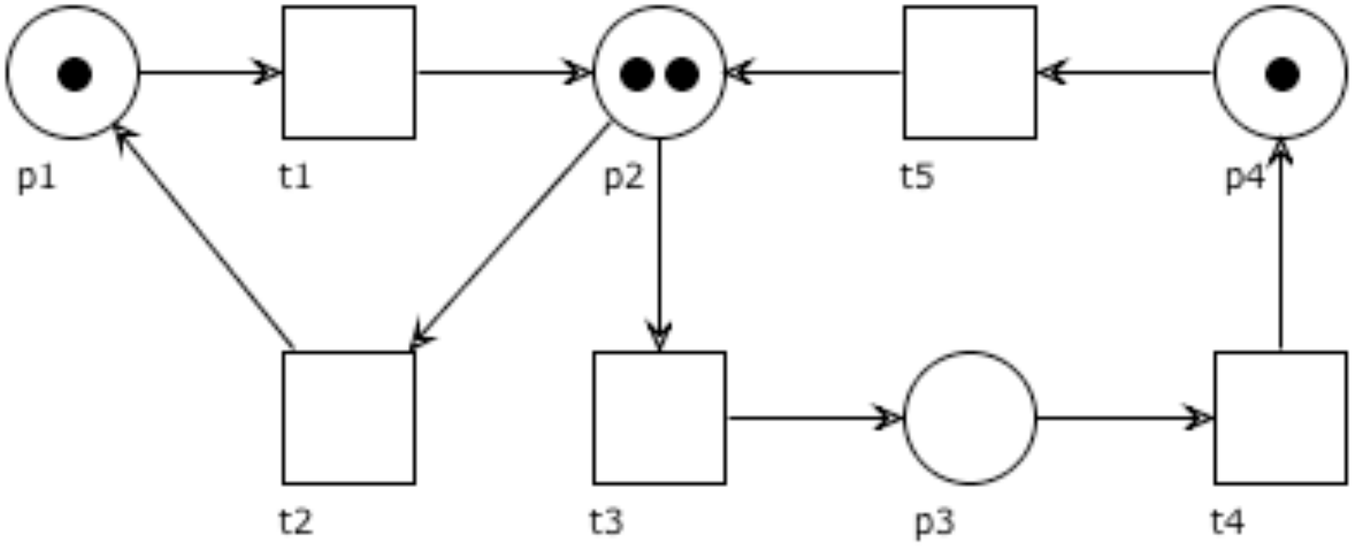
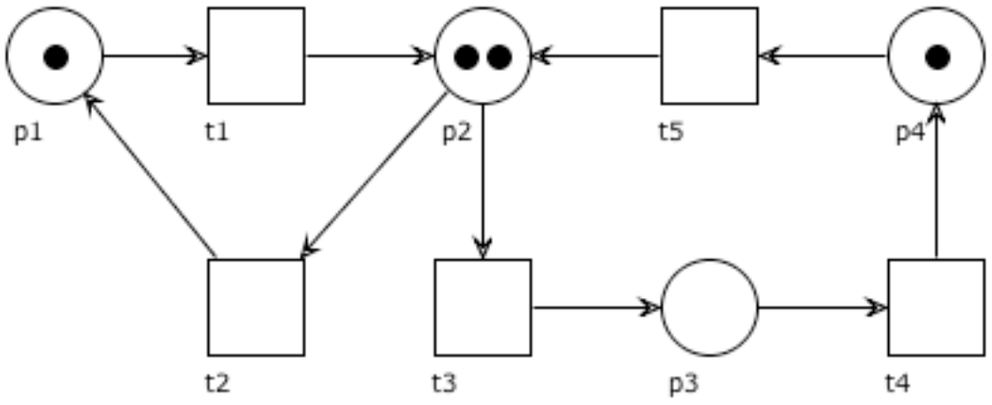
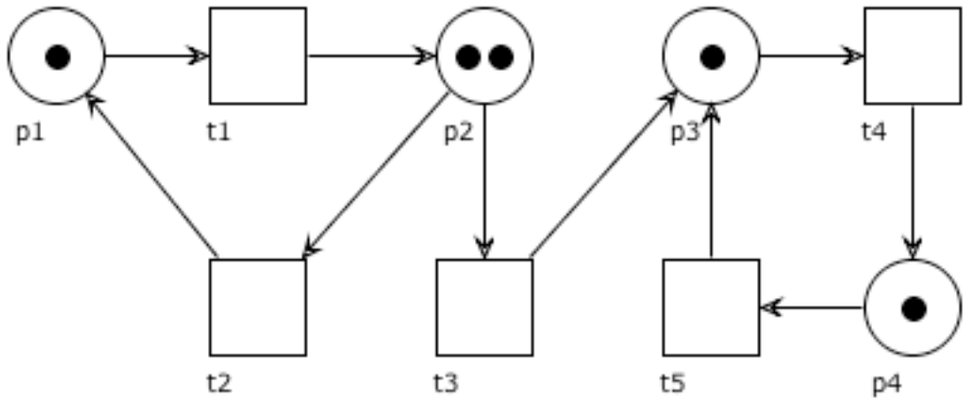


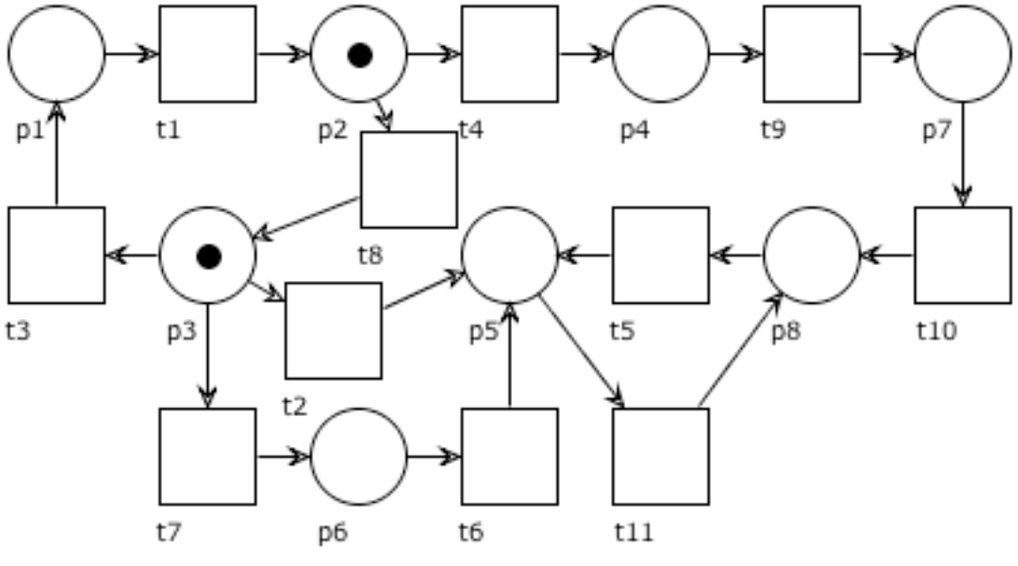

$$[\,1\;1\;0\;0\,]$$
$$[\,0\;0\;2\;2\,]$$
$$[\,1\;1\;1\;1\,]$$
$$[\,2\;2\;1\;1\,]$$
$$[\,2\;2\;2\;2\,]$$
$$[\,1\;2\;2\;1\,]$$

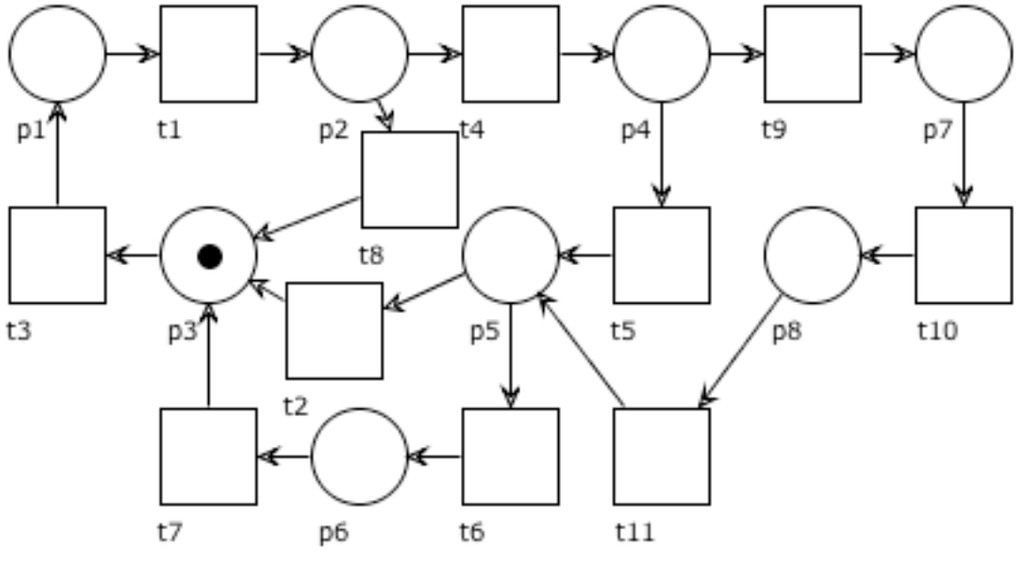
# Exercises

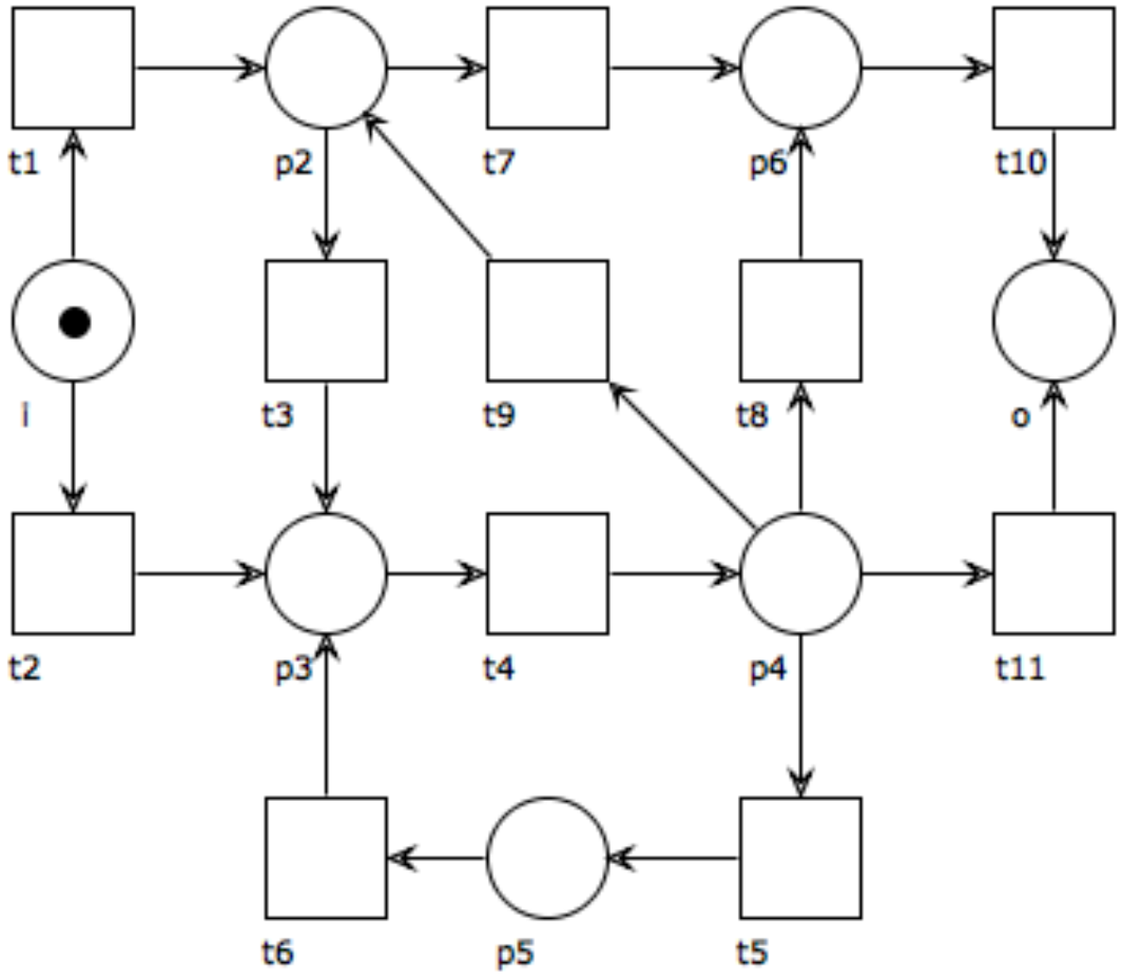
Which of the following S-systems are live? (why?)



# Exercise

Is the net below a workflow net?
Is it sound?