Business Processes Modelling

MPB (6 cfu, 295AA)

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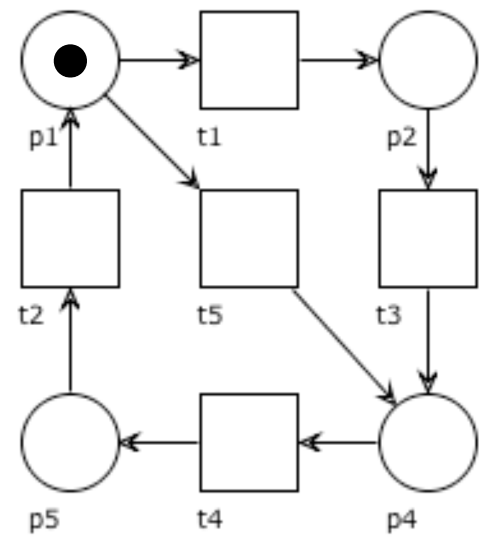
15 - S-systems and T-systems

Object



We study some “good” properties of S-systems

Free Choice Nets (book, optional reading)
https://www7.in.tum.de/~esparza/bookfc.html

Petri nets: structural properties

Structural properties

Structural (or **static**) properties
do not depend on the initial marking
(e.g., S-invariants, T-invariants, connectedness)

It is sometimes interesting to relate them to
behavioural properties
(i.e. that depend on the token game,
like boundedness and liveness)

This way we can give **structural characterization** of
behavioural properties for a whole class of nets
(computationally less expensive to check)

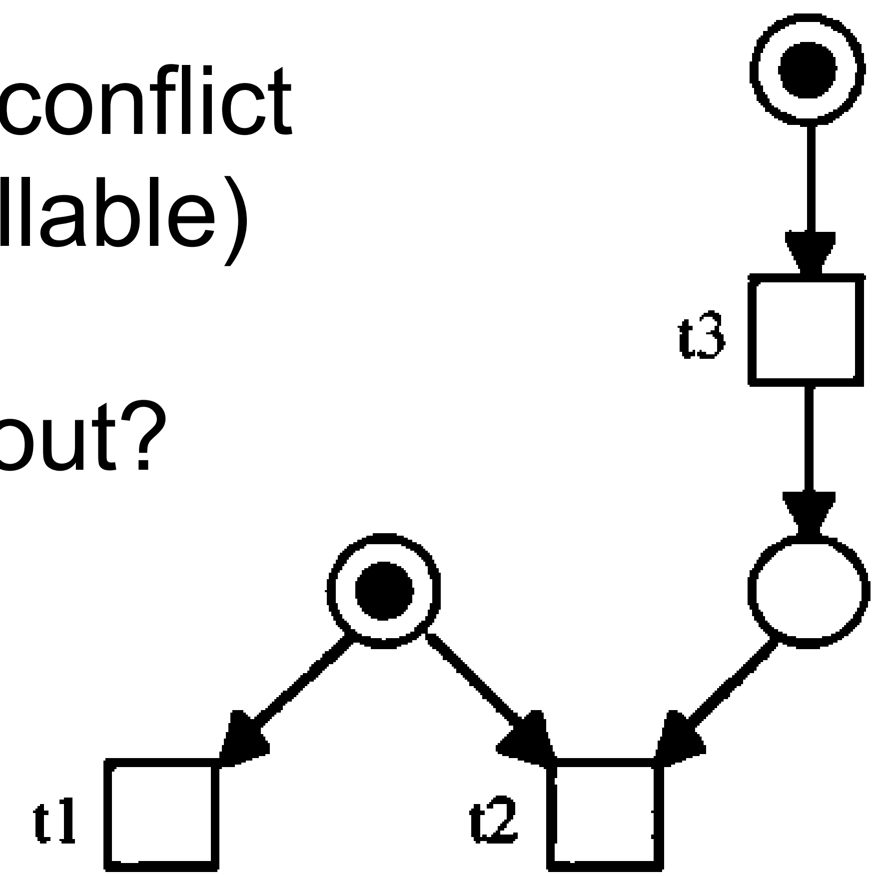
Interference of conflicts and synch

Typical situation:

initially t1 and t2 are not in conflict

but when t3 fires they are in conflict
(the firing of t3 is not controllable)

How to rule this situation out?

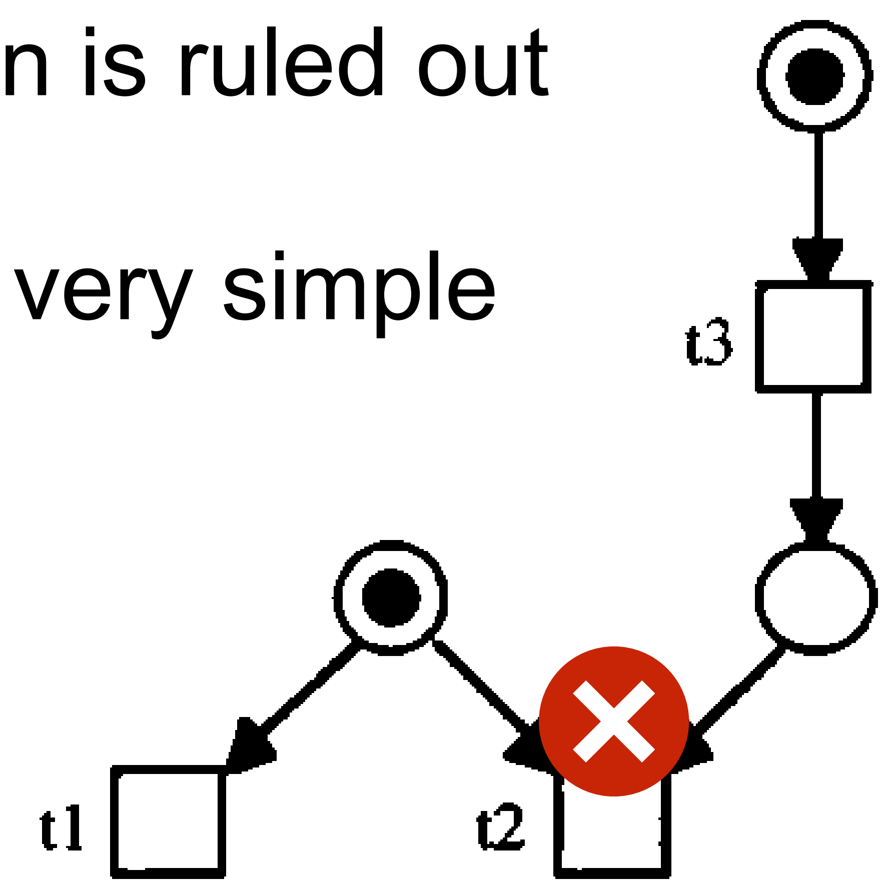


S-systems / S-nets

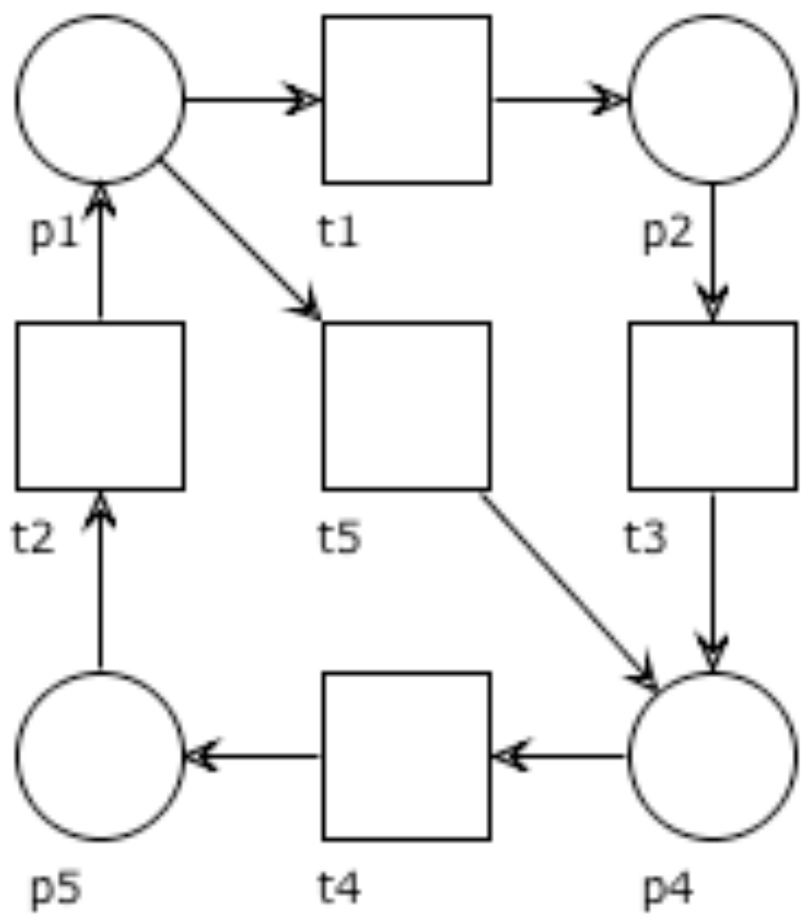
A Petri net is called **S-system** if every transition has one input place and one output place
(S comes from *Stellen*, the German word for place)

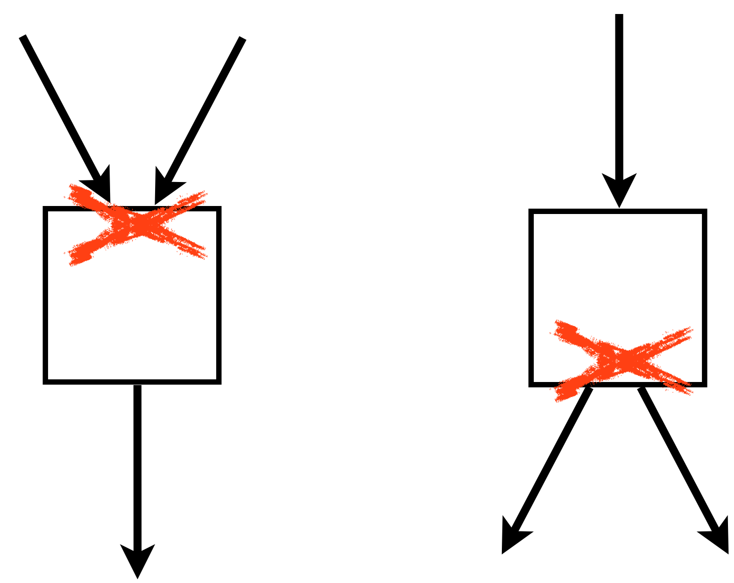
This way any synchronization is ruled out

The theory of S-systems is very simple



S-net: example



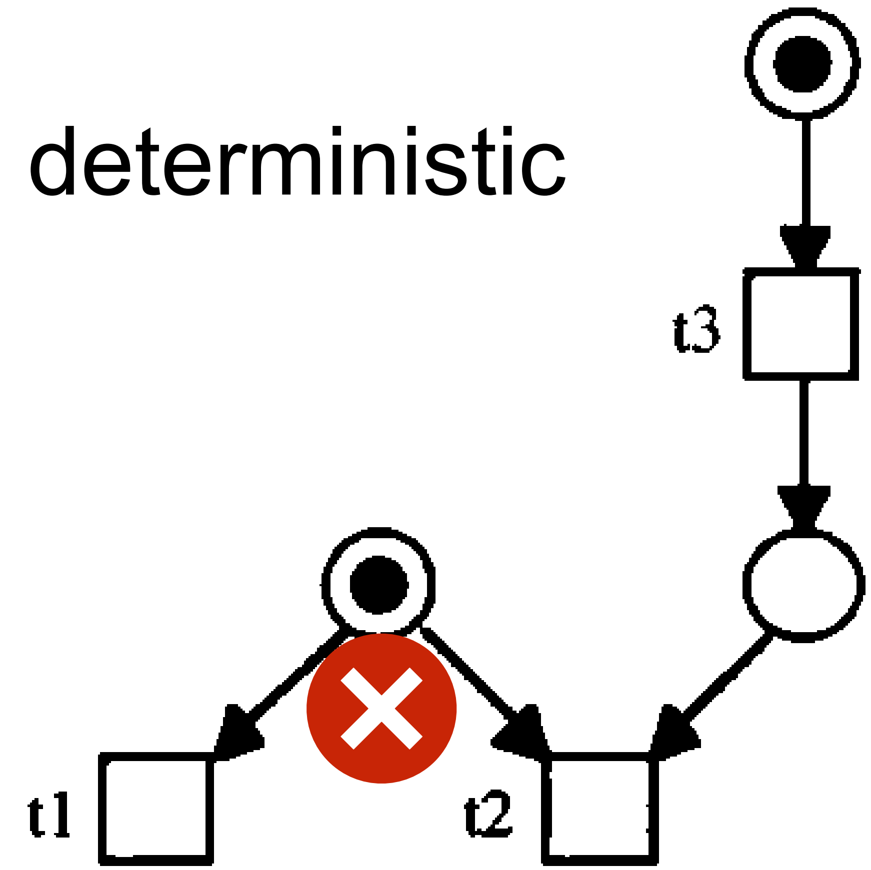


T-systems / T-nets

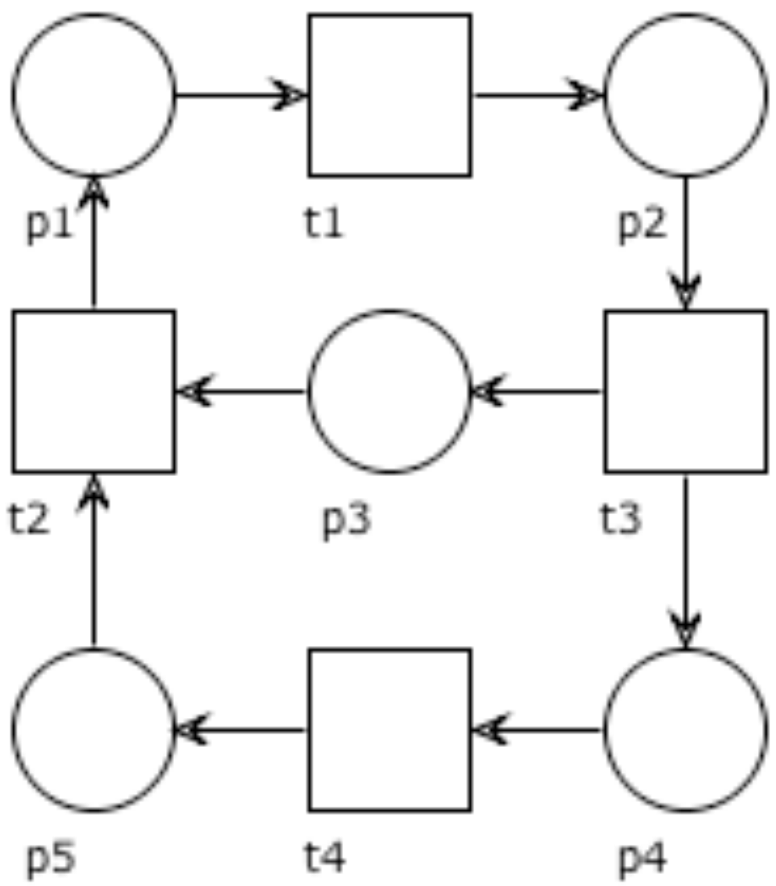
A Petri net is called **T-system** if every place has one input transition and one output transition

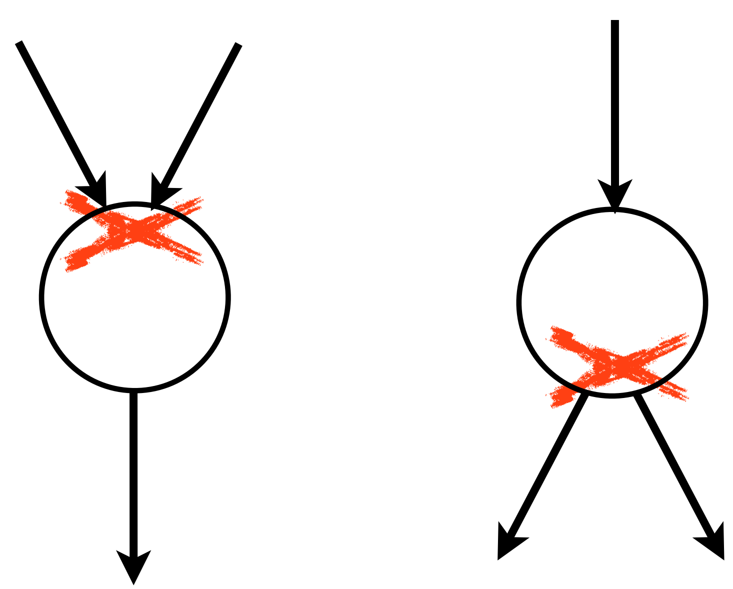
This way all choices/conflicts are ruled out

T-systems are concurrent but deterministic



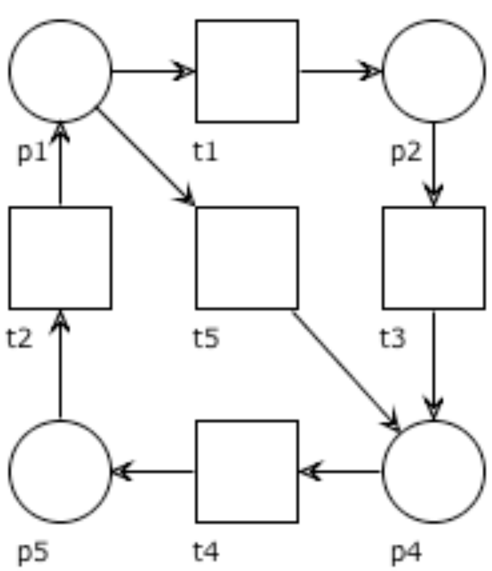
T-net: example

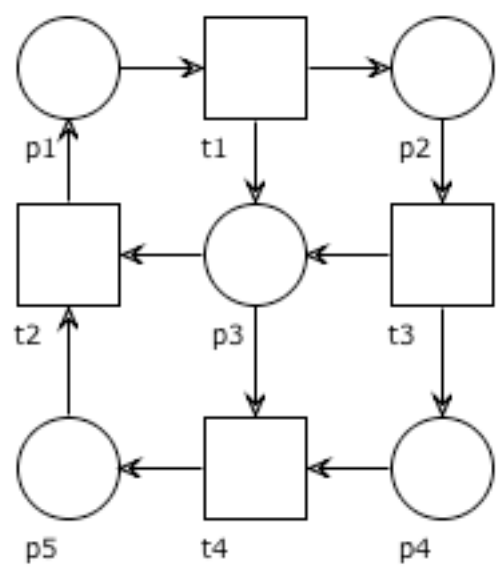


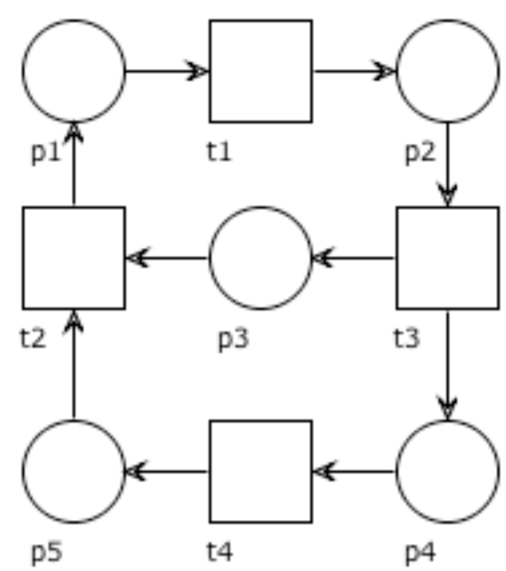


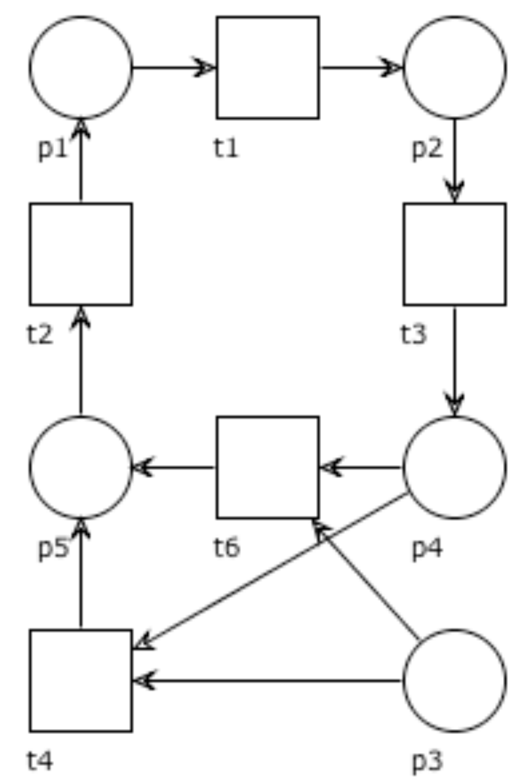
Question time

Is the net an S-net, a T-net?







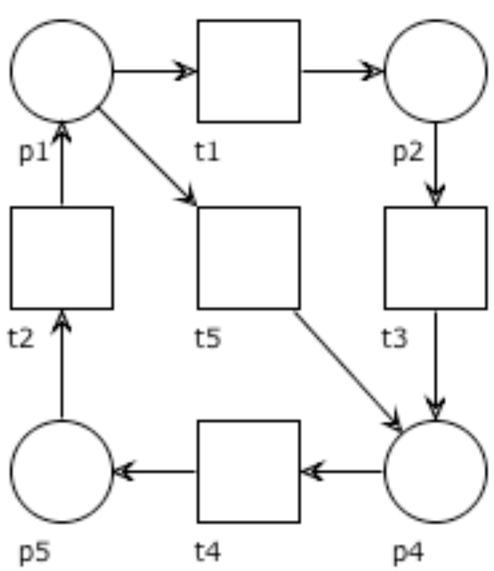


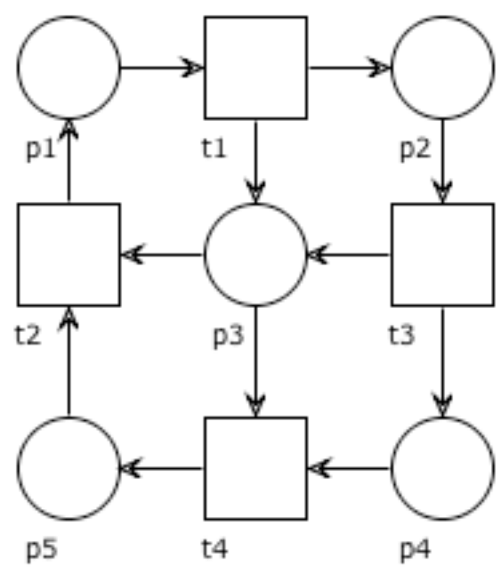
Question time

Is the net an S-net, a T-net?

S-net

T-net



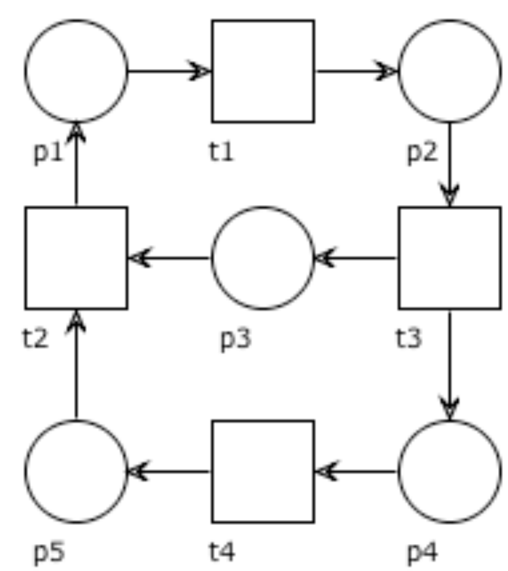


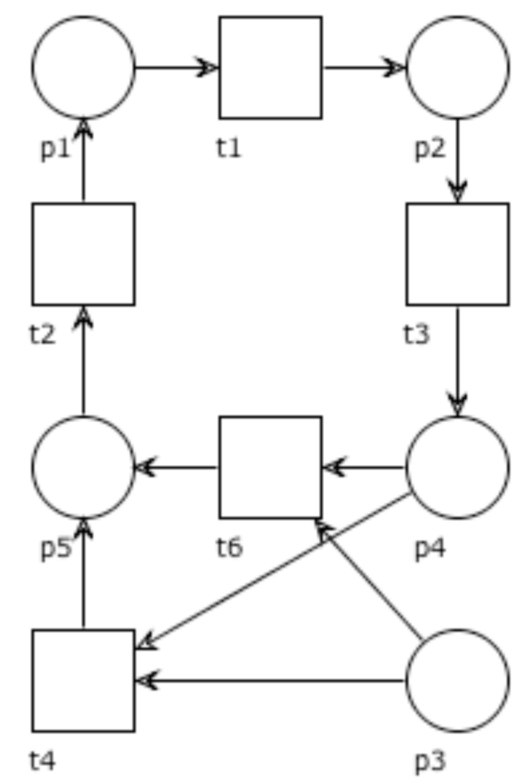
S-net

T-net

S-net

T-net



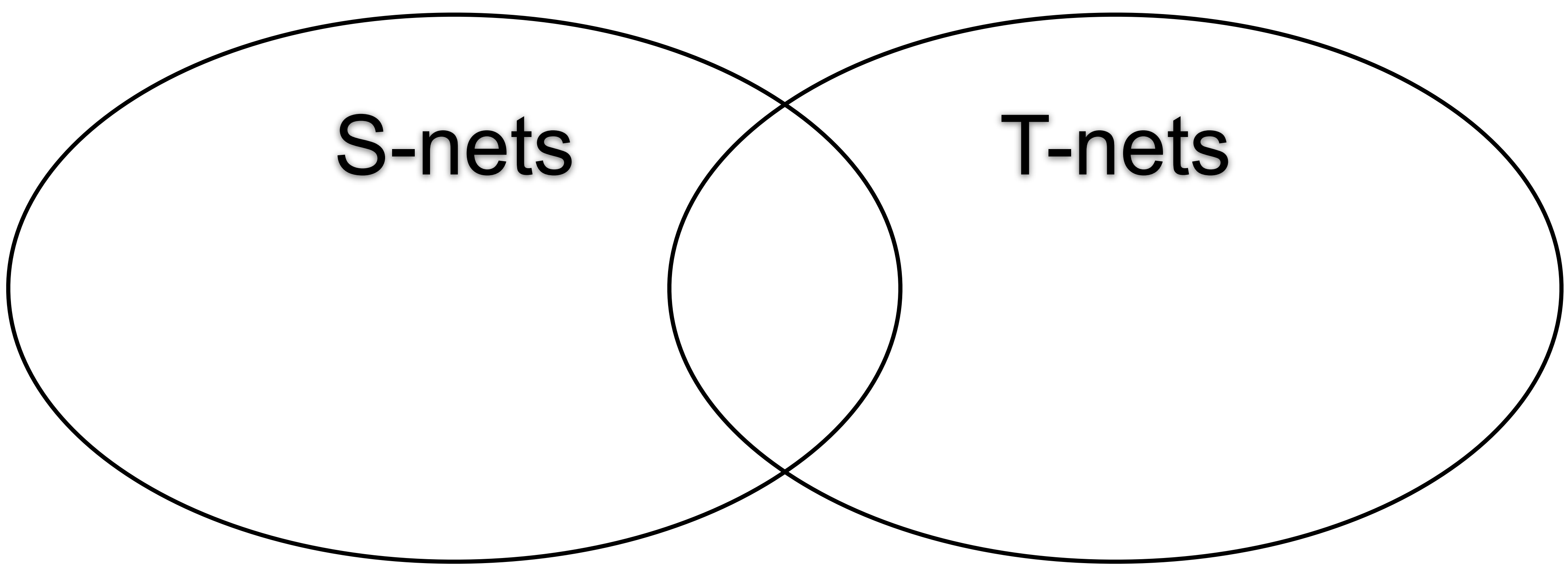


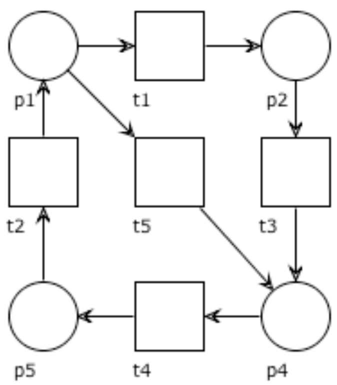
S-net

T-net

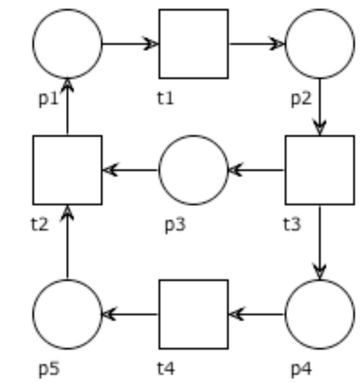
Exercises

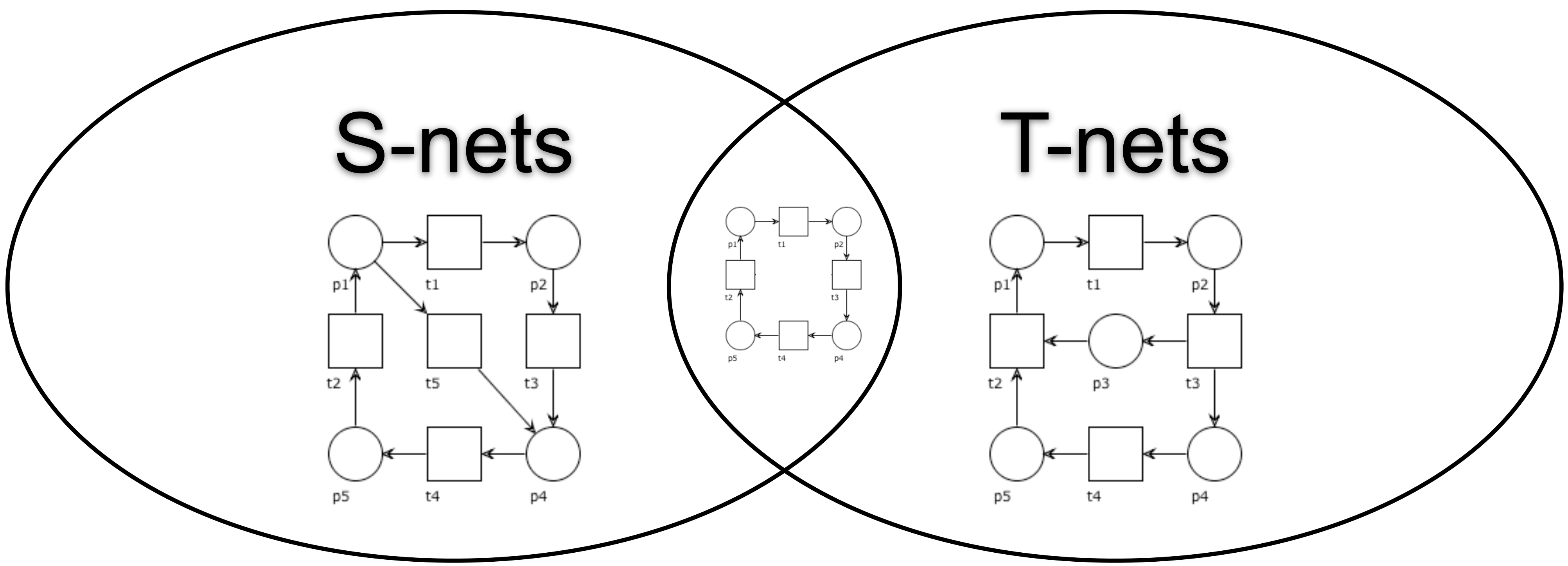
Show a net for each area of the Eulero-Venn diagram below

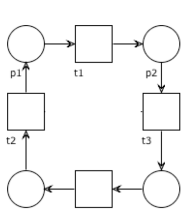



p1
t1
p2
t2
t5
t3
p5
t4
p4

Exercises

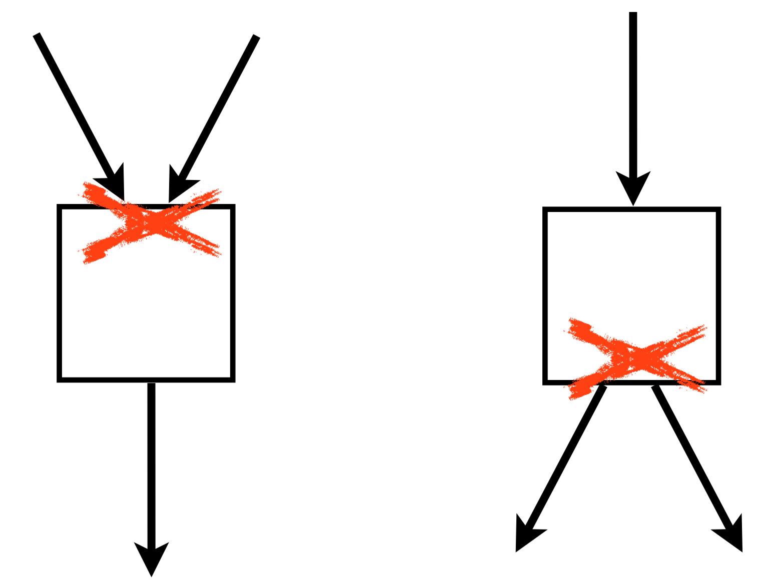

p1
t1
p2
t2
p3
t3
p5
t4
p4


S-nets
T-nets
p1
t1
p2
t2
t5
t3
p5
t4
p4
p1
t1
p2
t2
p3
t3
p5
t4
p4


p1
t1
p2
t2
t3
p5
t4
p4

S-systems

S-system



Definition: We recall that a net N is an **S-net** if each transition has exactly one input place and exactly one output place

$$\forall t \in T, \qquad |\bullet t| = 1 = |t \bullet|$$

A system (N,M_0) is an **S-system** if N is an S-net

Take-home message

Any workflow net N that is an S-net
is **safe and sound**

Fundamental property of S-systems

Observation: each transition t that fires removes exactly one token from some place p and inserts exactly one token in some place q (p and q can also coincide)

The overall number of tokens in the S-net is an invariant under any firing.

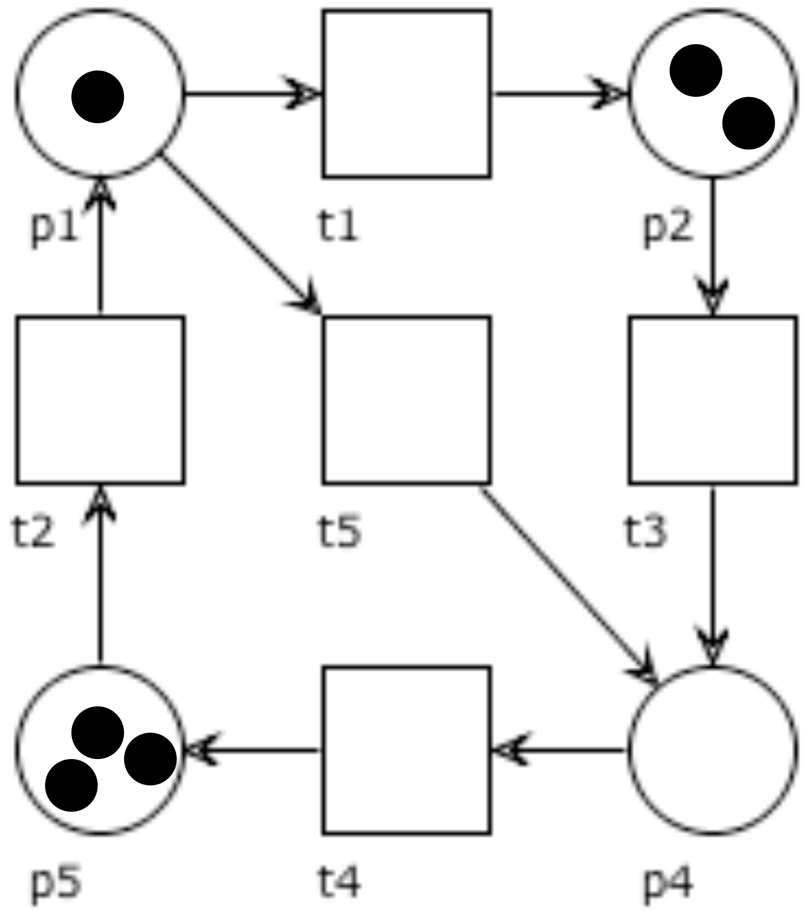
Notation: token count

$$M(P) = \sum_{p \in P} M(p)$$

Example

$$P = \{p_1, p_2, p_3\} \qquad M = 2p_1 + 3p_2 \qquad M(P) = 2 + 3 + 0 = 5$$

Question time



$M_0(P) = ?$

$M_0 = p_1 + 2p_2 + 3p_5$

$M_0(P) = M_0(p_1) + M_0(p_2)$
$+ M_0(p_4) + M_0(p_5)$

$M_0(P) = 6$

Fundamental property of S-systems

Proposition: Let (P,T,F,M_0) be an S-system. If M is a reachable marking, then M(P) = M_0(P)

We show that for any $M \xrightarrow{\sigma} M'$ we have $M'(P) = M(P)$

$\mathbf{base}$ $(\sigma = \epsilon)$**:** trivial $(M' = M)$

induction $(\sigma = \sigma' t$ **for some** $\sigma' \in T^*$ **and** $t \in T)$**:**

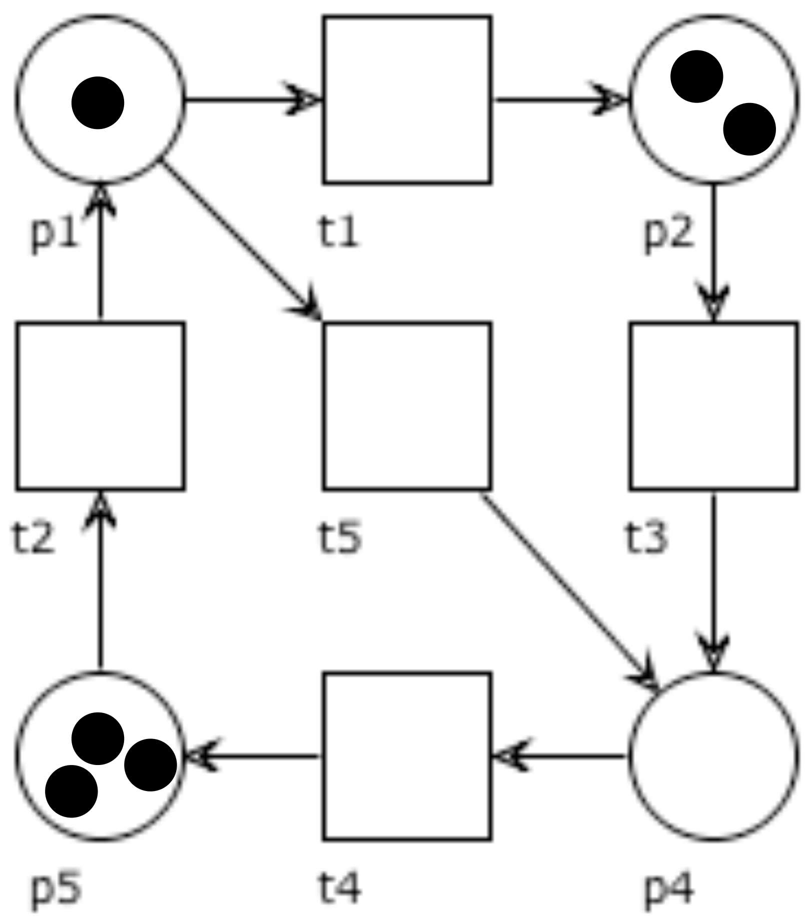
Let $M \xrightarrow{\sigma'} M'' \xrightarrow{t} M'$.

By inductive hypothesis: $M''(P) = M(P)$

By definition of S-system: $|\bullet t| = |t \bullet| = 1$

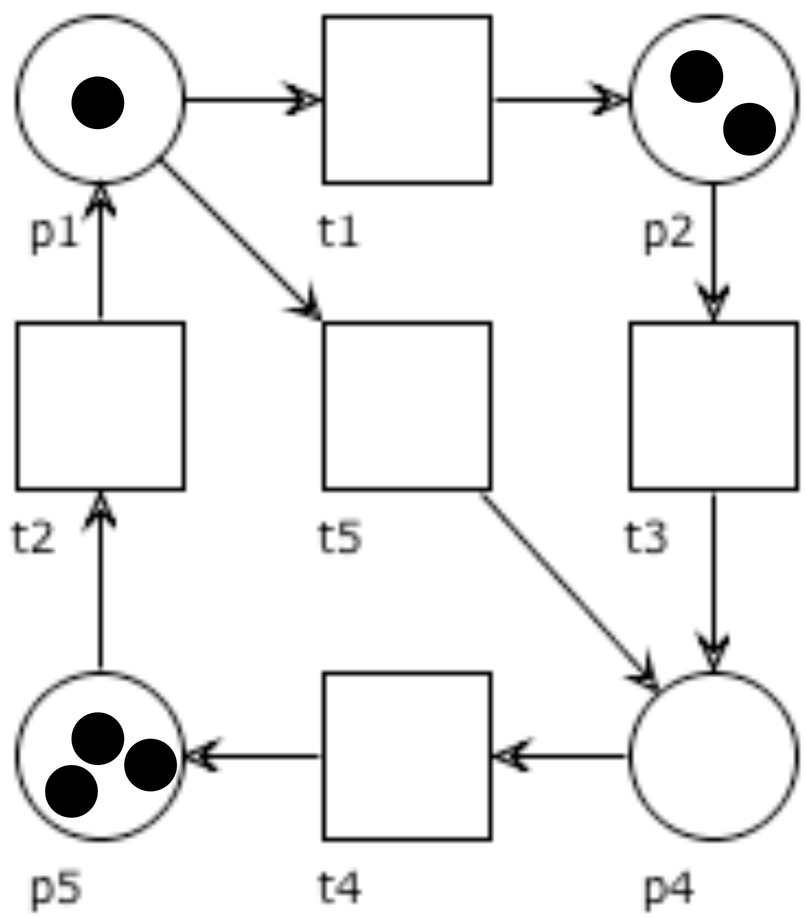
Thus, $M'(P) = M''(P) - |\bullet t| + |t \bullet| = M(P) - 1 + 1 = M(P)$

Question time



Is the marking
$M = p_2 + 4p_4 + 2p_5$
reachable?

Question time



Is the marking
$M = p_2 + 4p_4 + 2p_5$
reachable?

No:
S-system
$M_0(P) = 6 \neq 7 = M(P)$

A consequence of the fundamental property

Corollary: Any S-system is bounded

Let $M \in [\, M_0 \,\rangle$.

By the fundamental property of S-systems: $M(P) = M_0(P)$.

Then, for any $p \in P$ we have $M(p) \leq M(P) = M_0(P)$.

Thus the S-system is k-bounded for any $k \geq M_0(P)$.

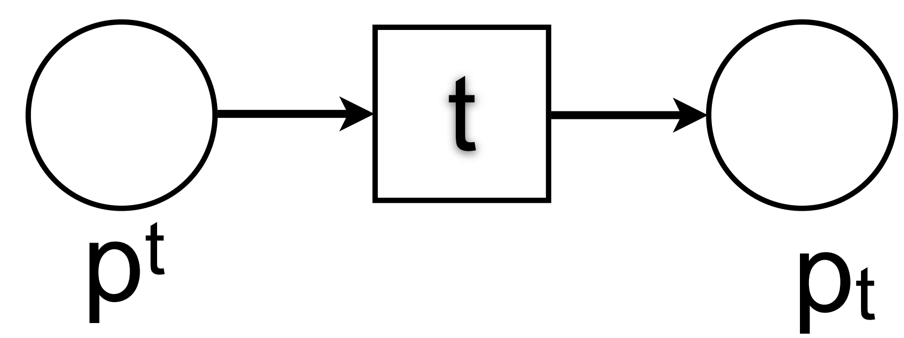
$$M(P) = \sum_{p \in P} M(p)$$

S-invariants of S-nets

Proposition: Let N=(P,T,F) be a (connected) S-net.
I is an S-invariant of N **iff I**=[k ... k] for some value k

S-invariance $$\forall t \in T,\ \sum_{p \in \bullet t} \mathbf{I}(p) = \sum_{p \in t \bullet} \mathbf{I}(p)$$

S-nets $$\forall t \in T,\ |\bullet t| = |t \bullet| = 1$$

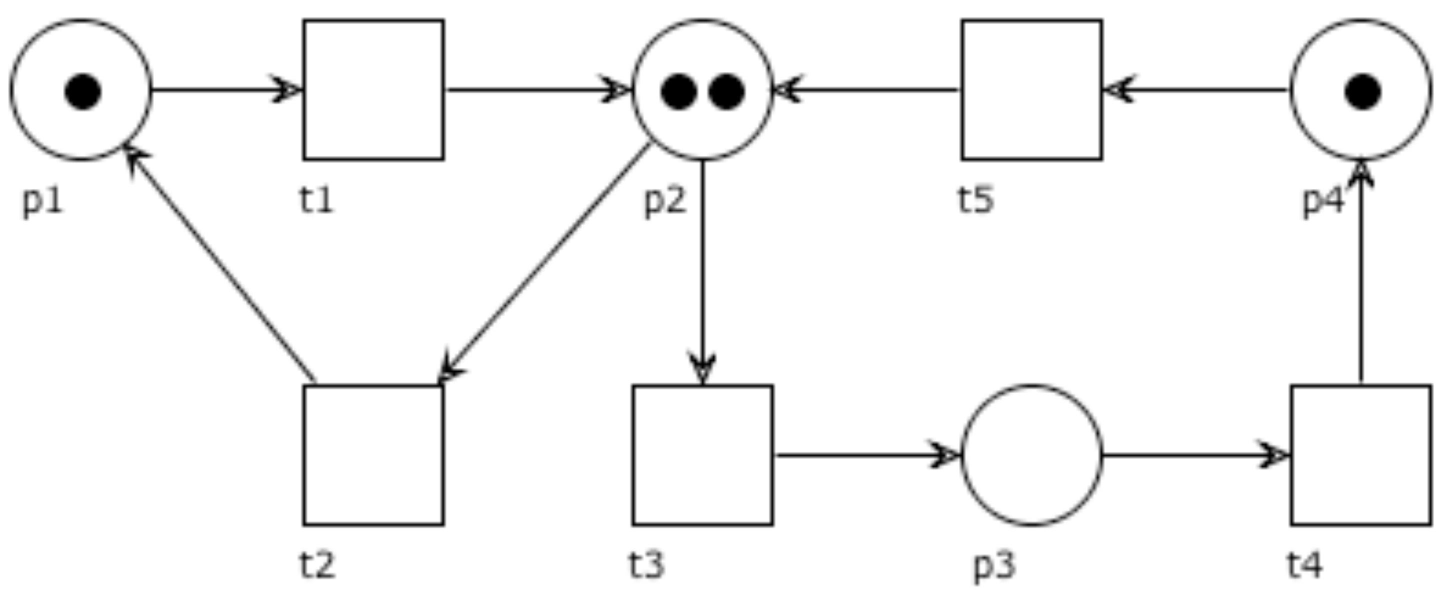


Let $\bullet t = \{p^t\}$ and $t\bullet = \{p_t\}$

$$\forall t \in T,\ \mathbf{I}(p^t) = \mathbf{I}(p_t)$$

Question time

Which of the following are S-invariants? (why?)



[0 0 2 2]
[1 1 1 1]
[2 2 1 1]
[2 2 2 2]

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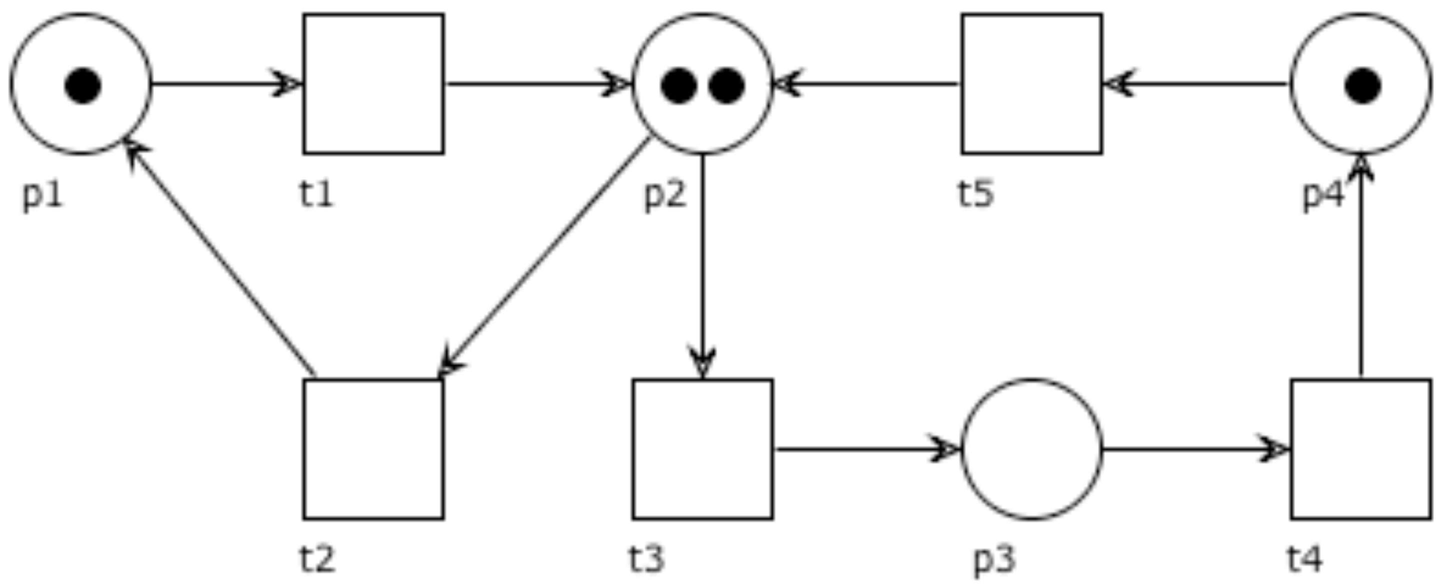

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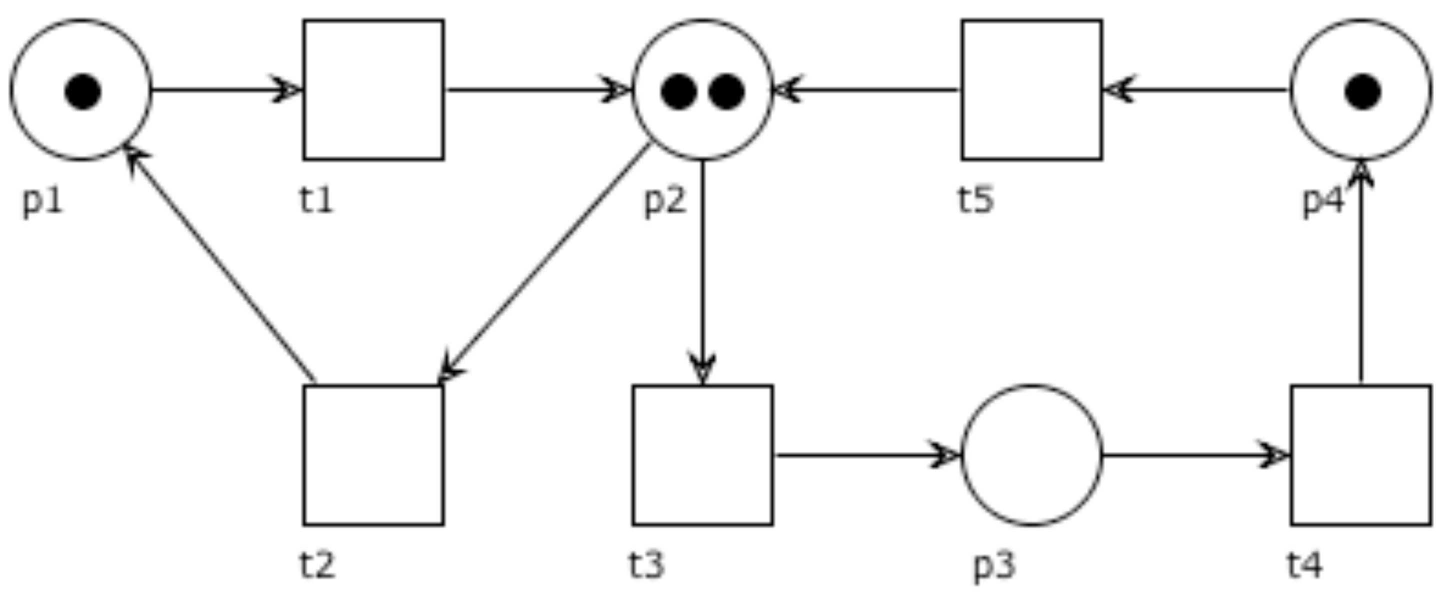

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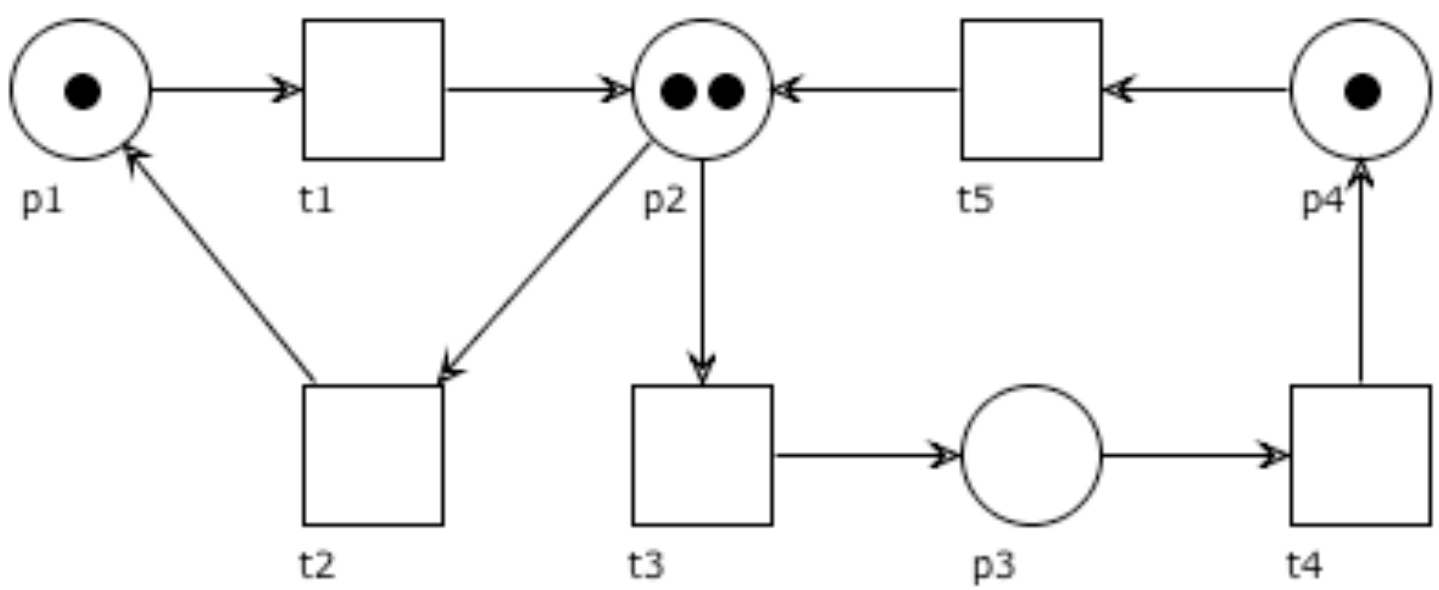

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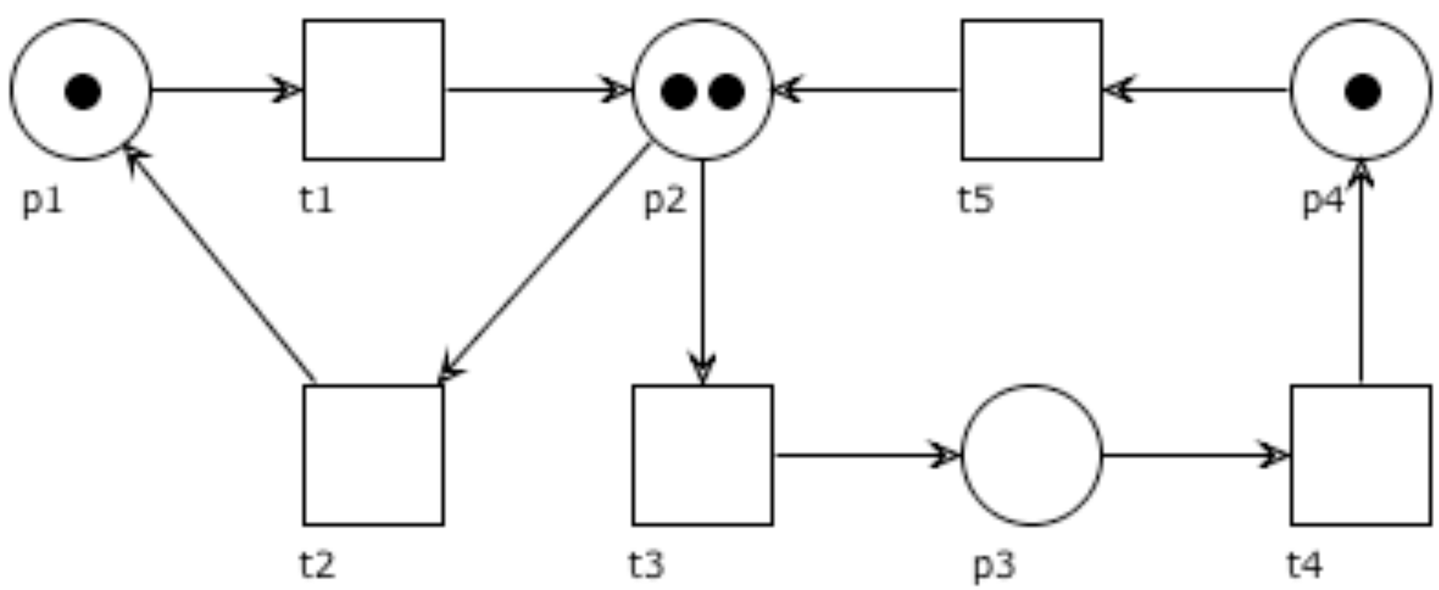


[0 0 2 2]
[1 1 1 1]
[2 2 1 1]
[2 2 2 2]

Liveness theorem for S-systems

Theorem: An S-system (N,M_0) is live **iff** N is strongly connected and M_0 marks at least one place

$\Rightarrow$) (quite obvious)
(N, M_0) is live by hypothesis and bounded (because S-system).
By the strong connectedness theorem, N is strongly connected.

Since (N, M_0) is live, then $M_0 \xrightarrow{t}$ for some t.

Assume $\bullet t = \{p\}$. Thus, $M_0(p) \geq 1$.

Liveness theorem for S-systems

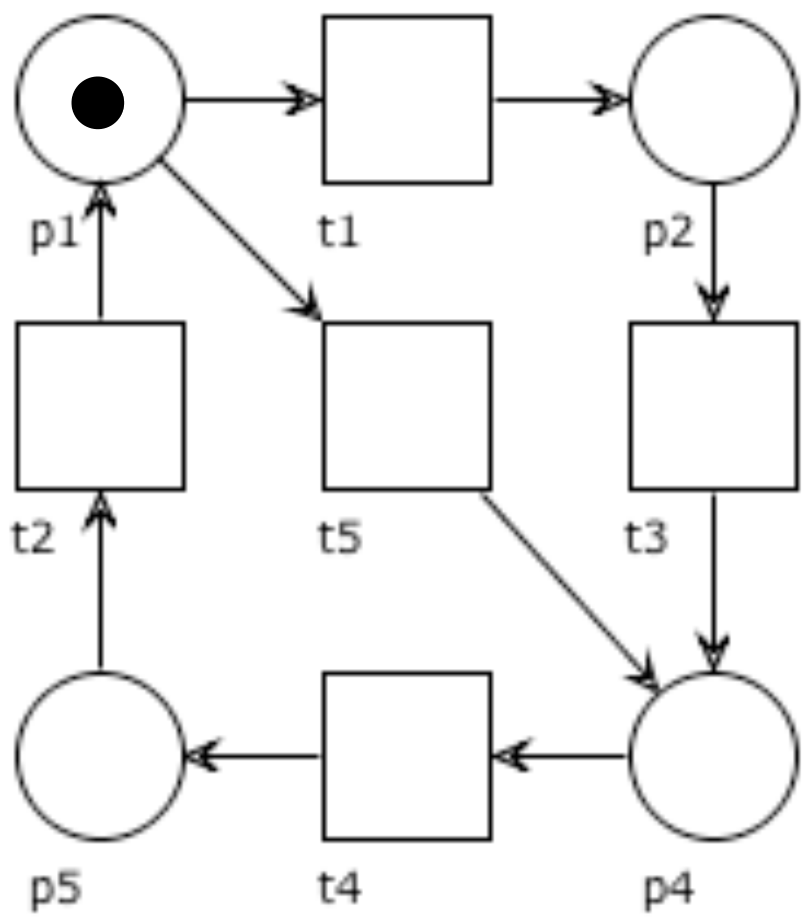
Theorem: An S-system (N,M_0) is live **iff** N is strongly connected and M_0 marks at least one place

$\Leftarrow$) (more interesting)
Take any $M \in [\, M_0 \,\rangle$ and $t \in T$.
We want to find $M' \in [\, M \,\rangle$ such that $M' \xrightarrow{t}$.

Take $p_1 \in P$ such that $M(p_1) \geq 1$ (it exists, because $M(P) = M_0(P) \geq 1$).
By strong connectedness: there is a path from p_1 to $t_n = t$
$(p_1, t_1)(t_1, p_2)(p_2, t_2)...(p_n, t_n)$

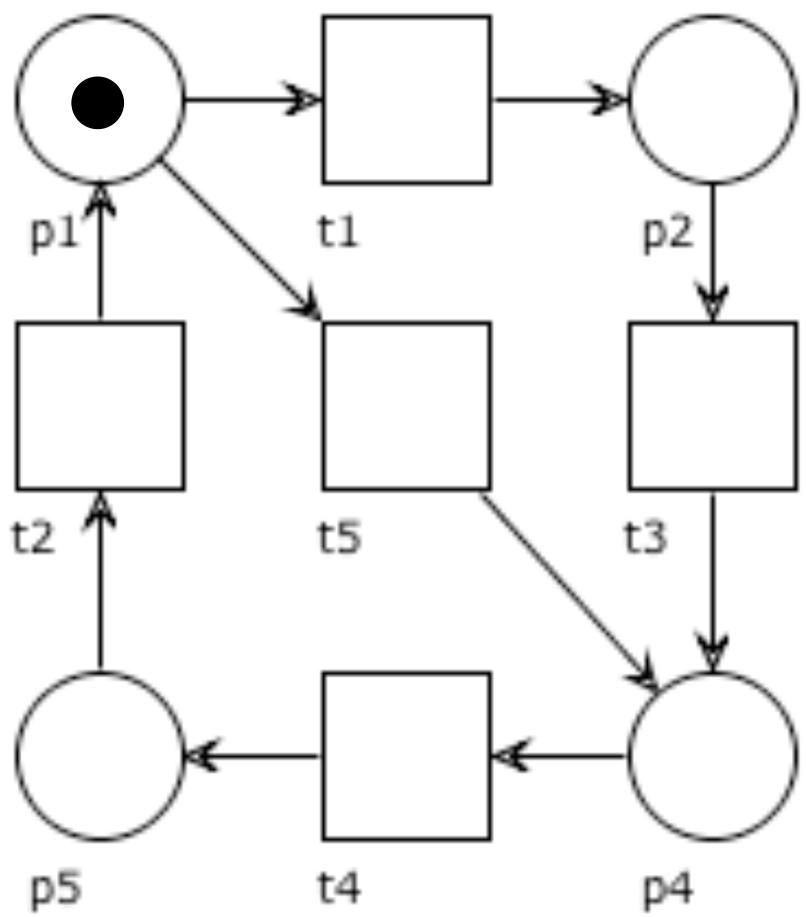
By definition of S-system: $\bullet t_i = \{p_i\}$ and $t_i \bullet = \{p_{i+1}\}$.
Thus, $M \xrightarrow{\sigma} M' \xrightarrow{t}$ for $\sigma = t_1 t_2 ... t_{n-1}$.

Question time



Is this S-system live?

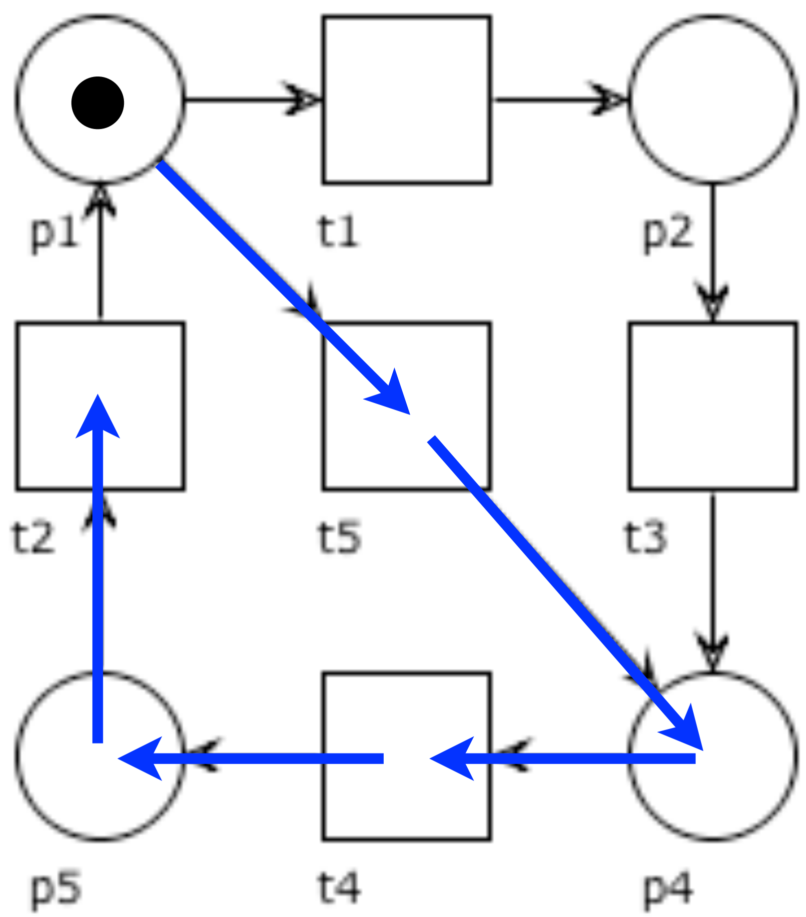
Question time



Is this S-system live?

Yes:
S-system
strongly connected
$M_0(P) > 0$

Question time

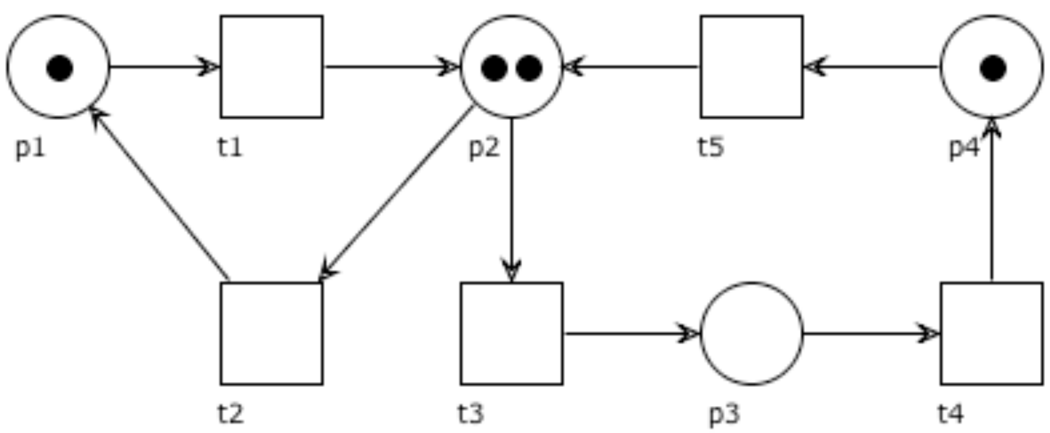


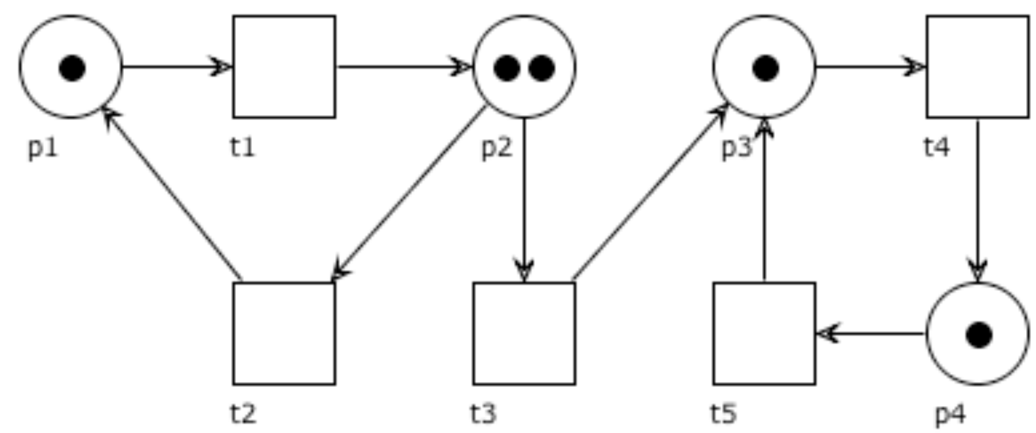
Is this S-system live?

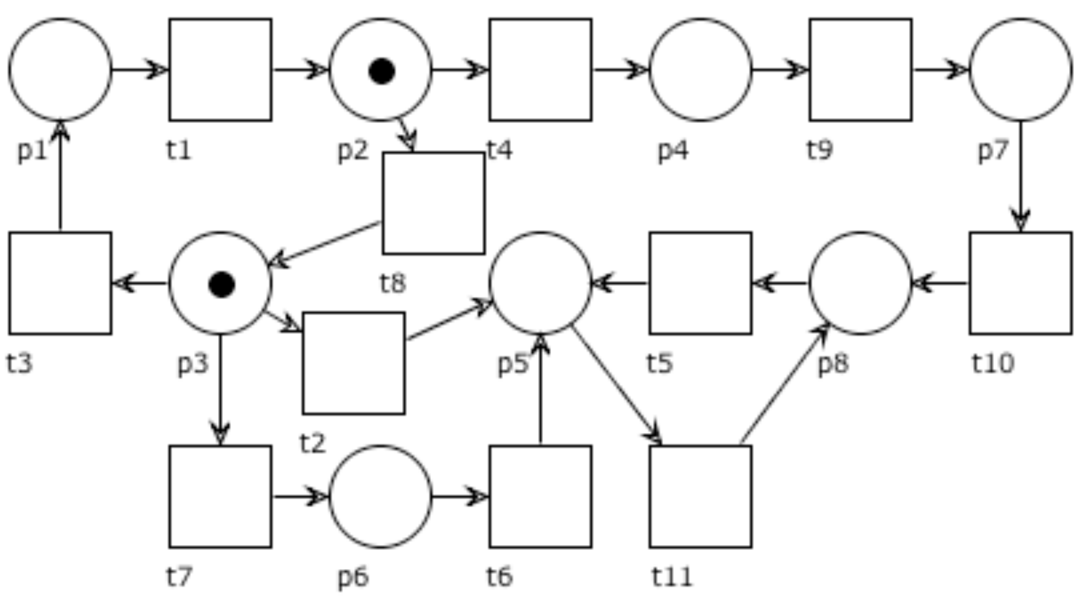
How to enable t2?
just find a path
from p1 to t2

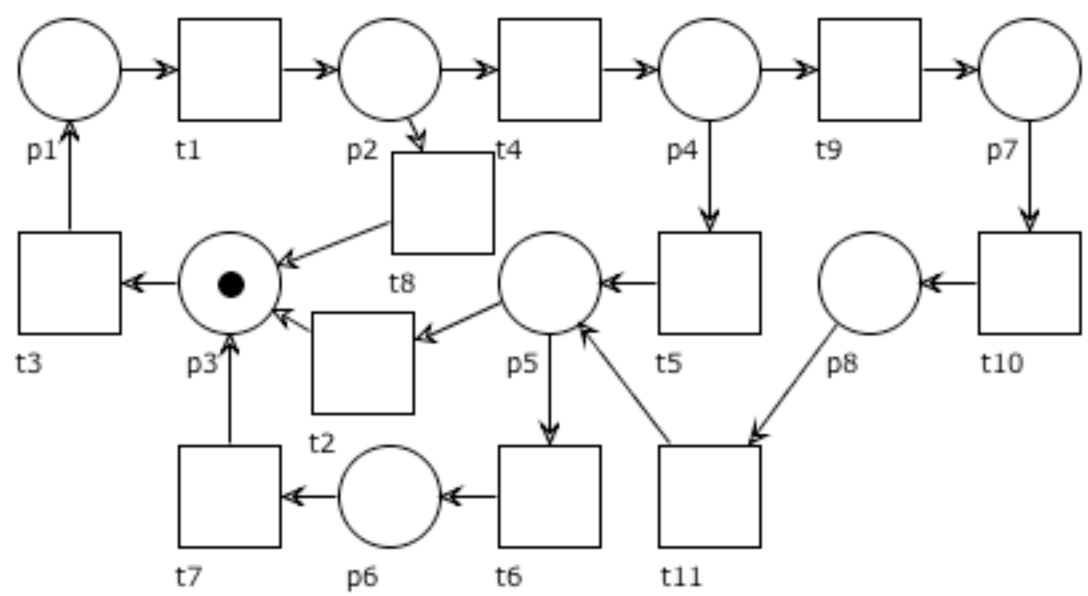
Exercises

Which of the following S-systems are live? (why?)









Exercises

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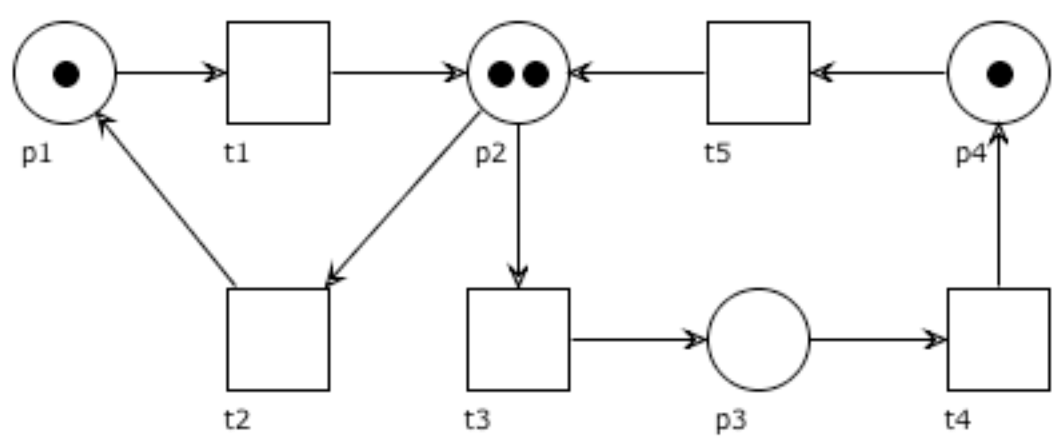


str. conn. + $M_0(P)>0$

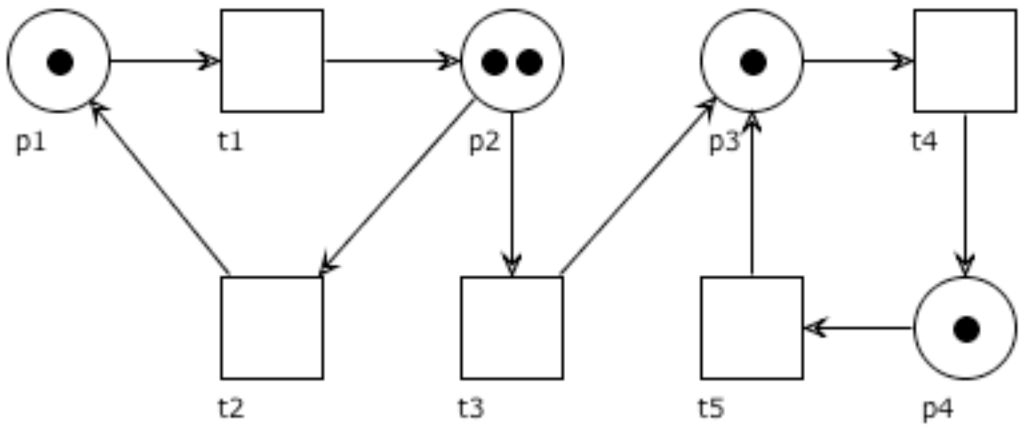
LIVE

Exercises

Which of the following S-systems are live? (why?)

not str. conn.

NON LIVE



Exercises

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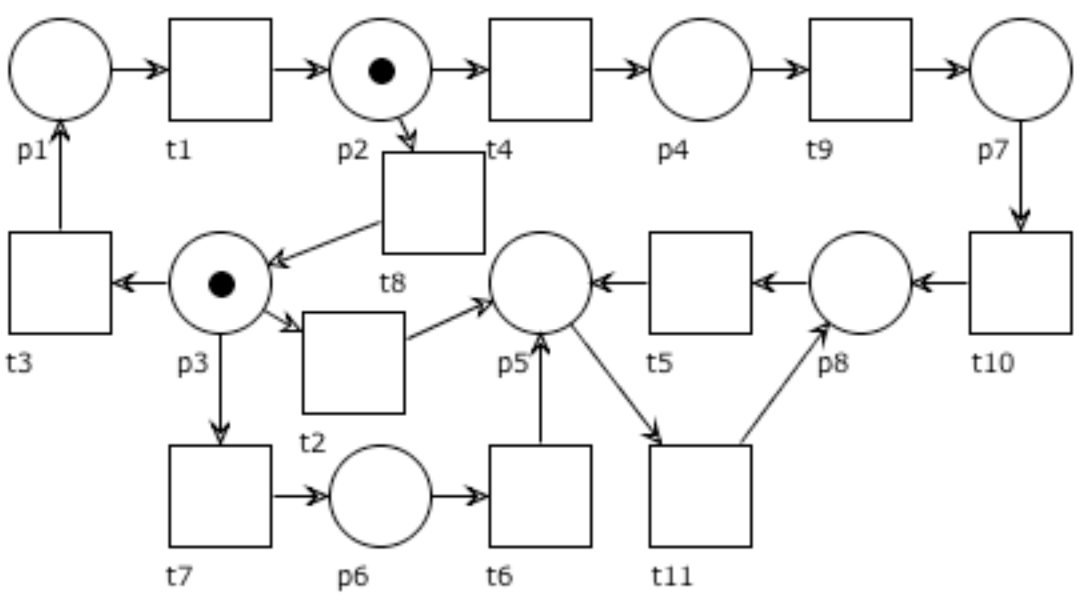
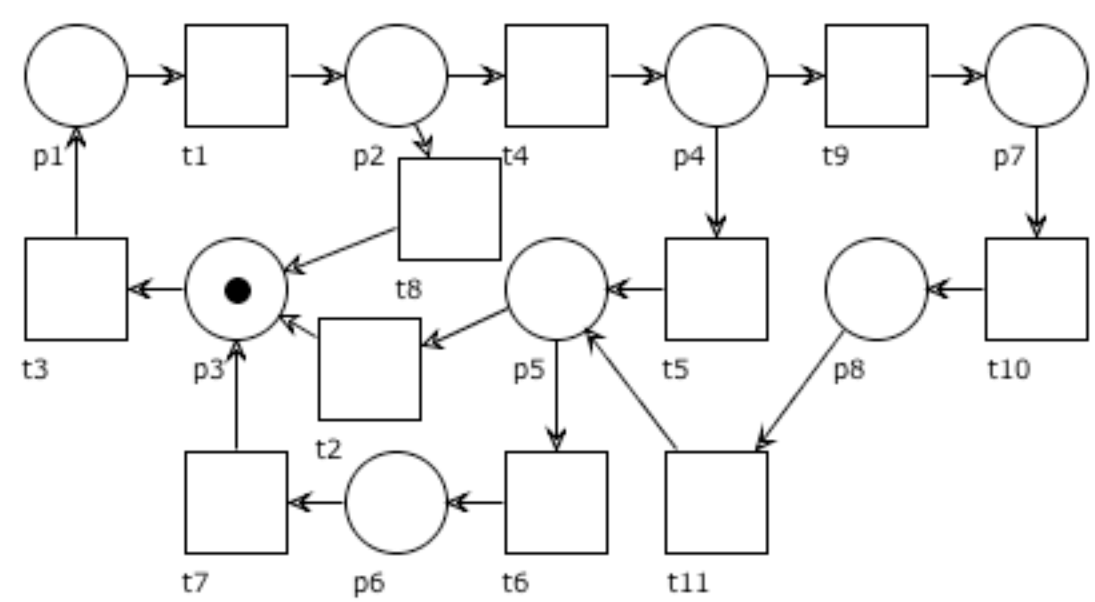


Exercises

Which of the following S-systems are live? (why?)

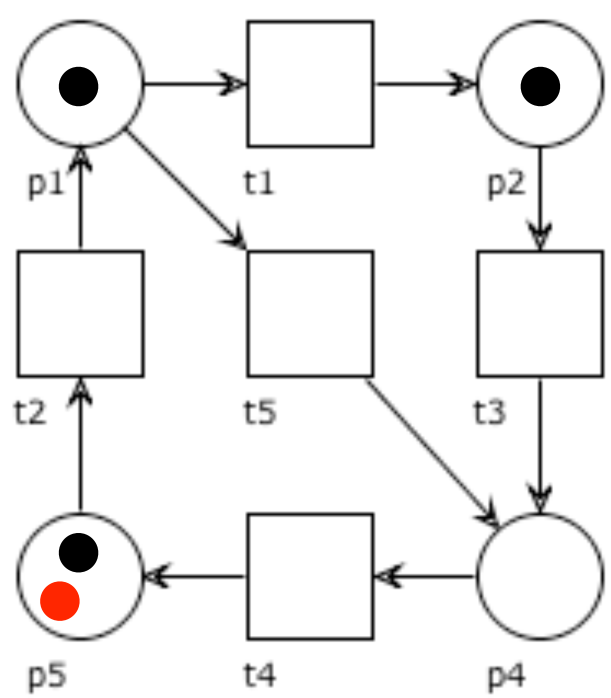
str. conn. + $M_0(P)>0$

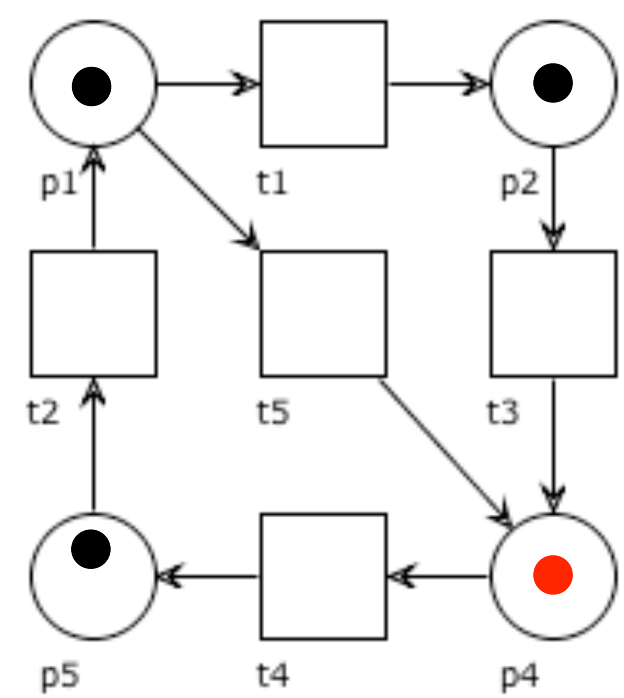
LIVE



Reachability lemma for S-nets

Lemma: Let (P,T,F) be a strongly connected S-net.
If M(P) = M'(P), then M' is reachable from M





Intuitively:
M and M' has the same number of tokens
but maybe they are in different places

Reachability lemma for S-nets

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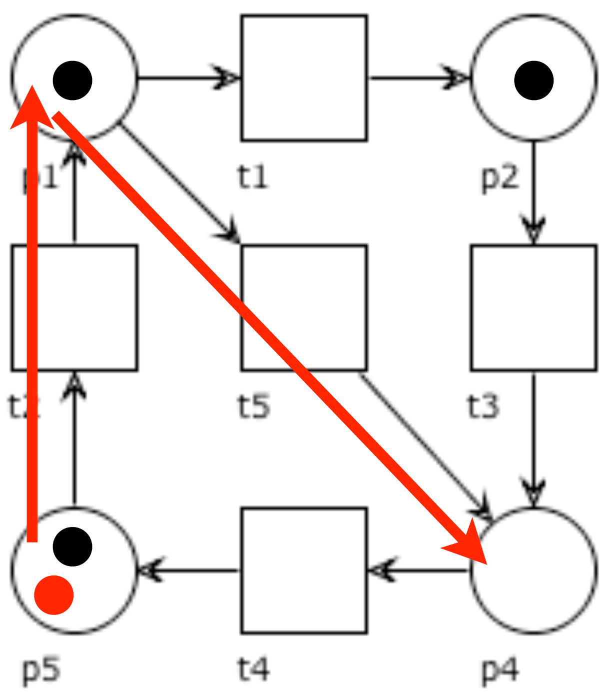
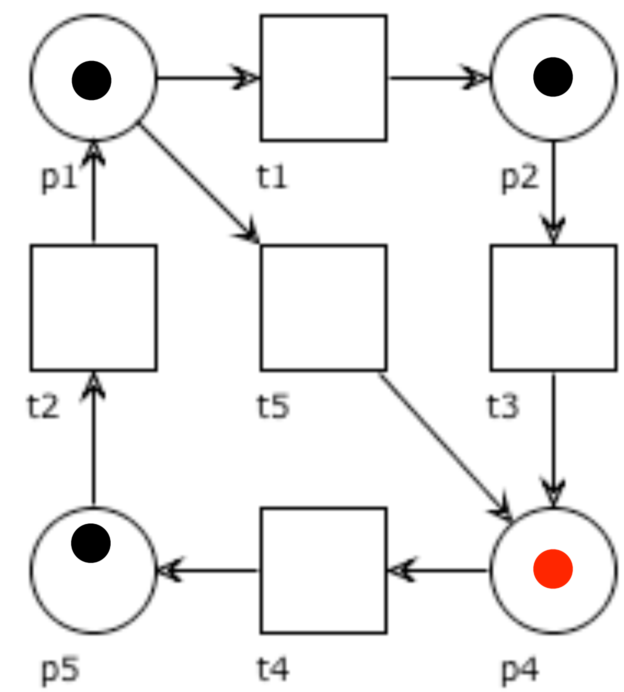


if there is only one token in the wrong place



Intuitively:
we need to move the token from p to q
by strong connectedness we have a path from p to q
by firing the transitions along the path we succeed

Reachability lemma for S-nets

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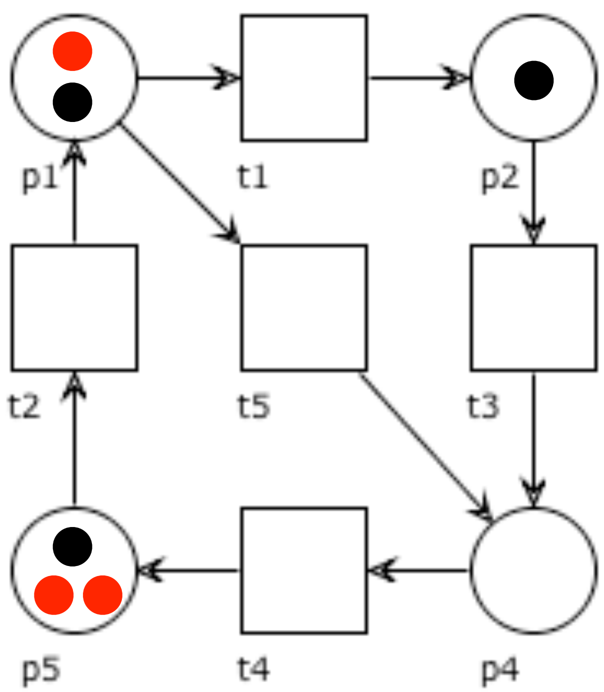
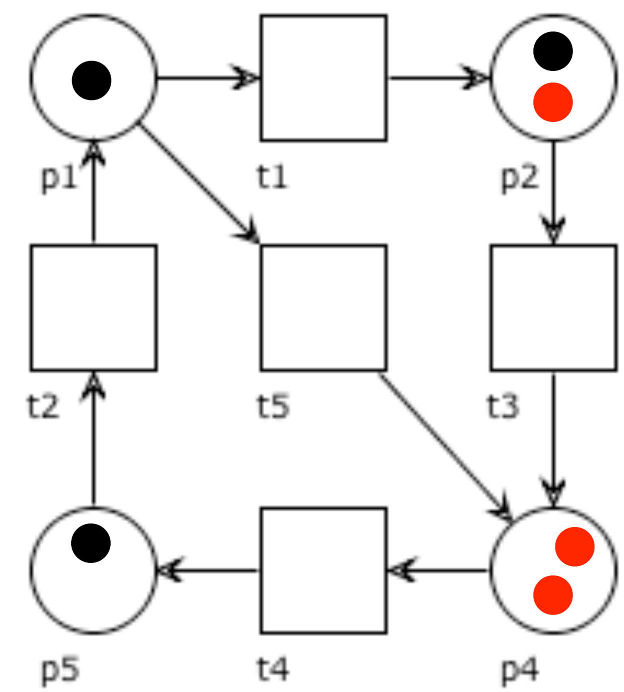


if there are many (n) tokens in the wrong places



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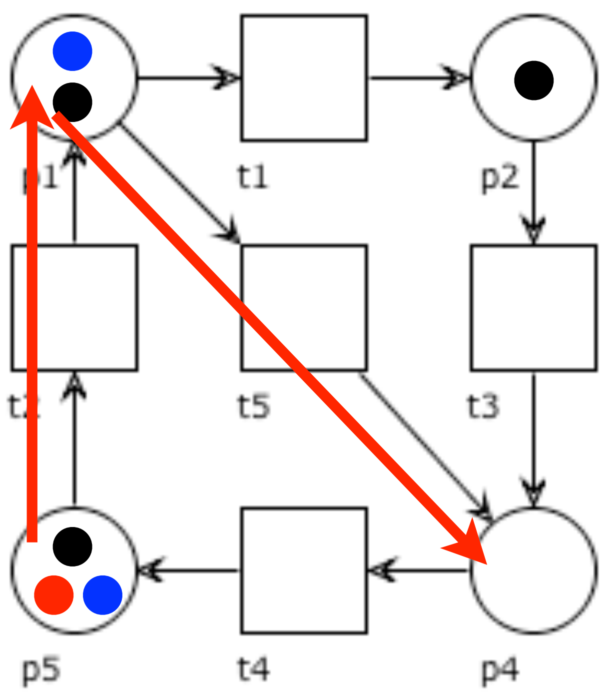

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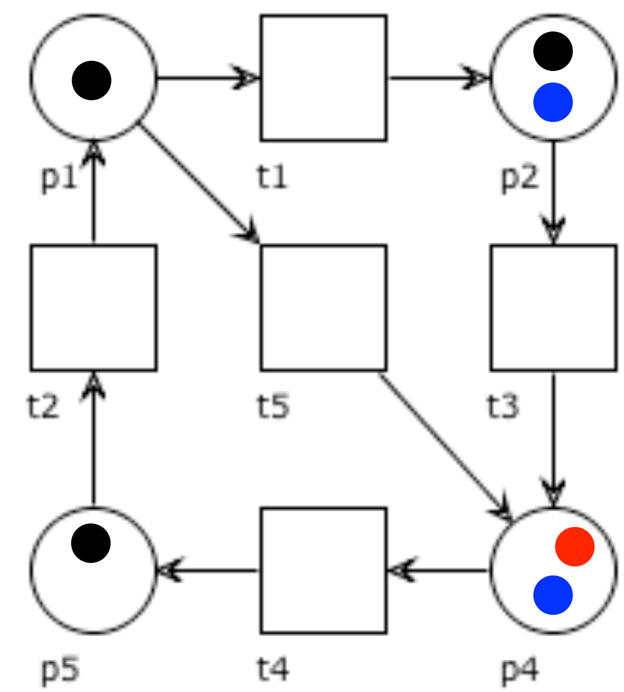


Intuitively:

we move one of them…

and inductively the remaining n-1

Reachability lemma for S-nets

Lemma: Let (P,T,F) be a strongly connected S-net. If M(P) = M'(P), then M' is reachable from M

Only for the interested reader!

We proceed by induction on $M(P)$

base $(M(P) = M'(P) = 0)$**:** trivial $(M' = M)$

induction $(M(P) = M'(P) > 0)$**:**

Let $p, p' \in P$ be such that $M(p) > 0$ and $M'(p') > 0$.

Let $K = M - p$ and $K' = M' - p'$.

Clearly $K'(P) = K(P) < M(P) = M'(P)$.

By inductive hypothesis: $\exists \sigma$, $K \xrightarrow{\sigma} K'$

By strong connectedness: there is a path from $p_0 = p$ to $p_n = p'$

$(p_0, t_1)(t_1, p_1)(p_1, t_2)...(t_n, p_n)$

By definition of S-system: $\bullet t_i = \{p_{i-1}\}$ and $t_i \bullet = \{p_i\}$.

Thus, $p = p_0 \xrightarrow{\sigma'} p_n = p'$ for $\sigma' = t_1 t_2 ... t_n$.

By the monotonicity lemma: $M = K + p \xrightarrow{\sigma} K' + p \xrightarrow{\sigma'} K' + p' = M'$

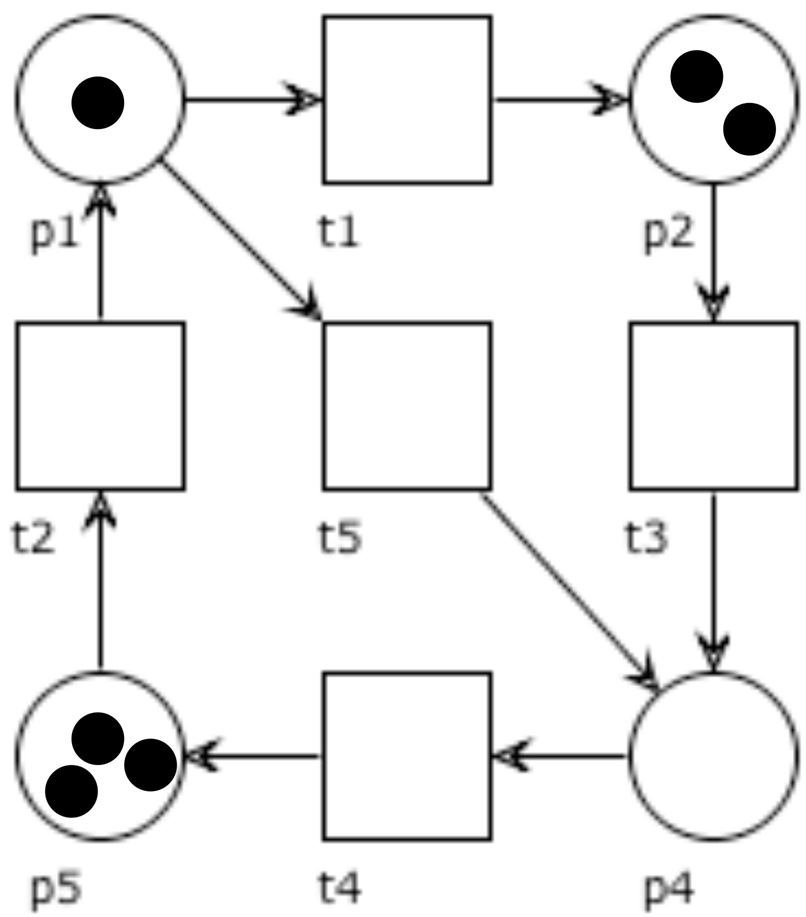
Reachability Theorem for S-systems

Theorem: Let (P,T,F,M_0) be a live S-system.
A marking M is reachable **iff** $M(P)=M_0(P)$

=>) Follows from the fundamental property of S-systems

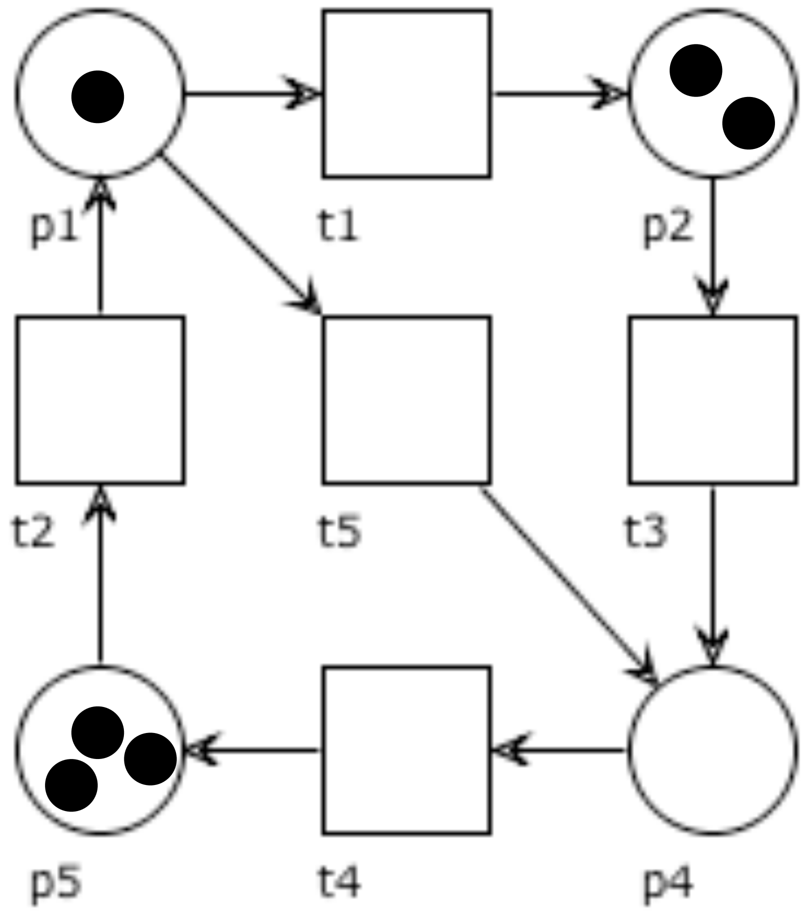
<=) By the liveness theorem for S-systems, the S-net is strongly connected.
We conclude by applying the previous reachability lemma for S-systems.

Question time



Is the marking
$M = p_2 + 4p_4 + p_5$
reachable?

Question time

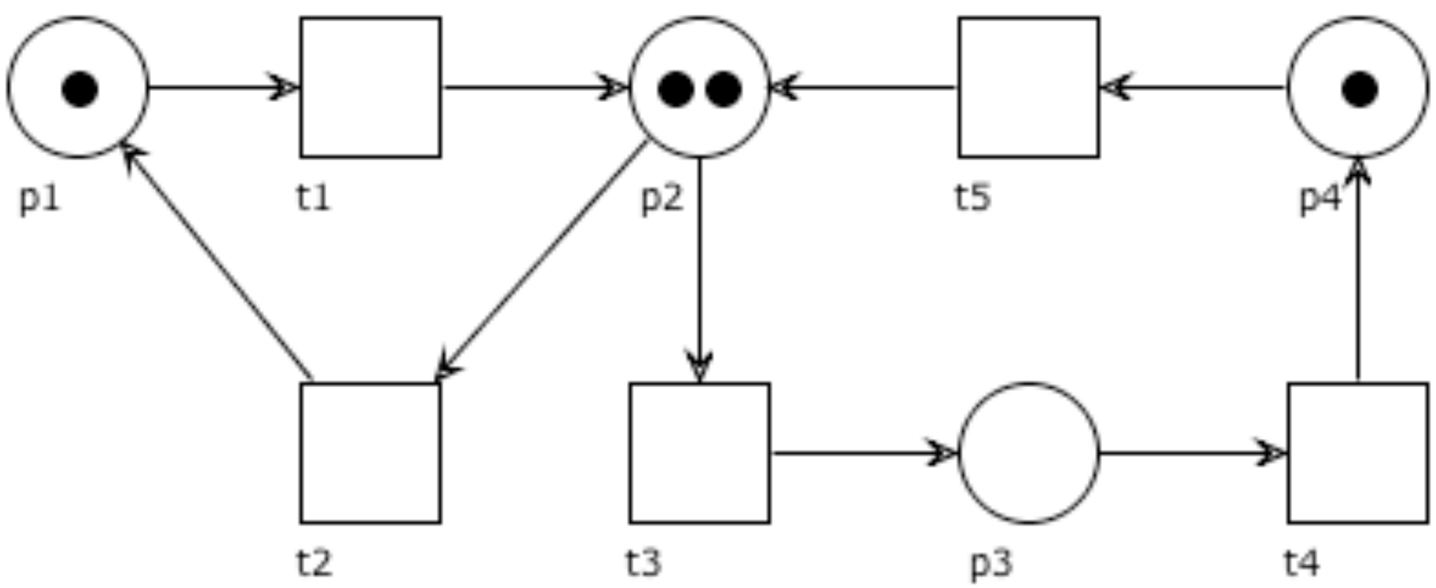


Is the marking
$M = p_2 + 4p_4 + p_5$
reachable?

Yes:
S-system
strongly connected
$M_0(P) = 6 = M(P)$

Question time

Which of the following markings are reachable? (why?)

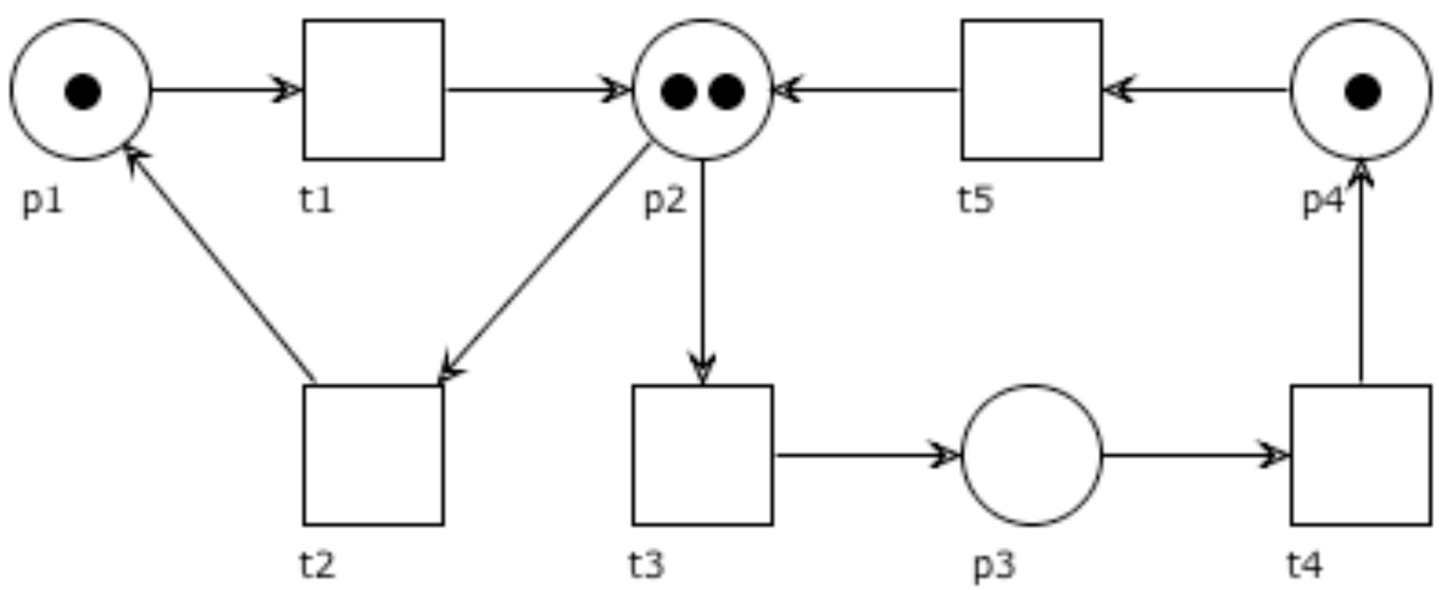


[1 1 1 1]
[2 0 2 0]
[1 2 1 2]
[4 0 0 0]

Question time

Which of the following markings are reachable? (why?)

strong connected



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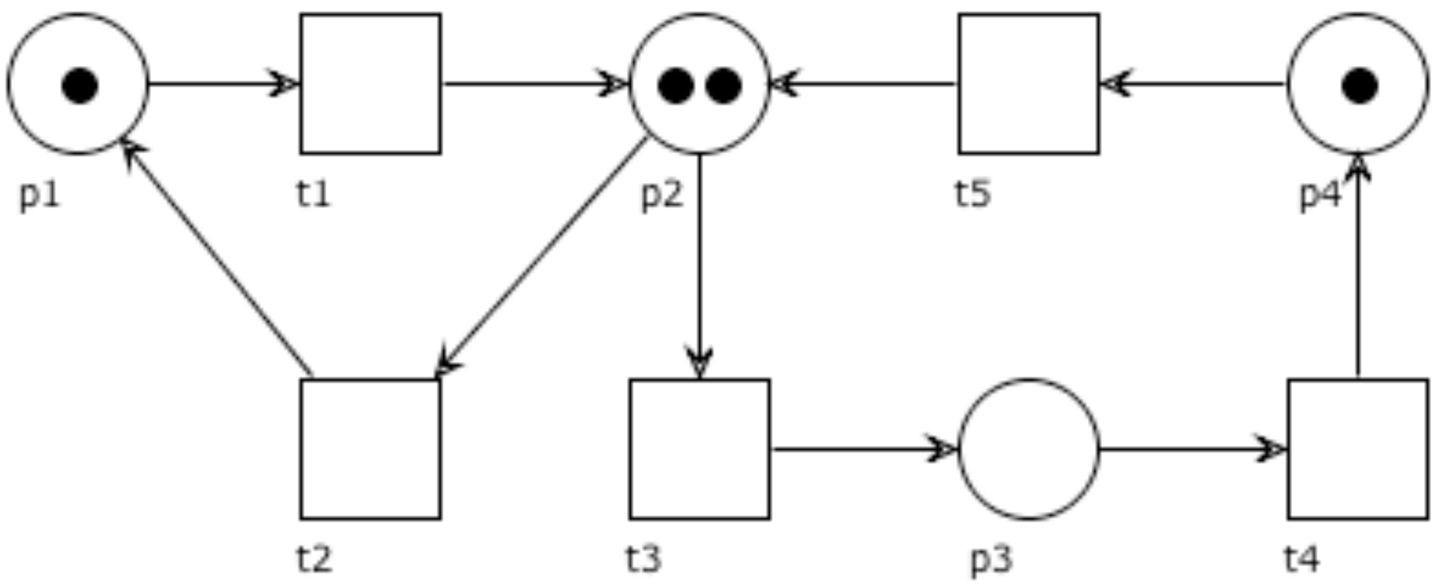

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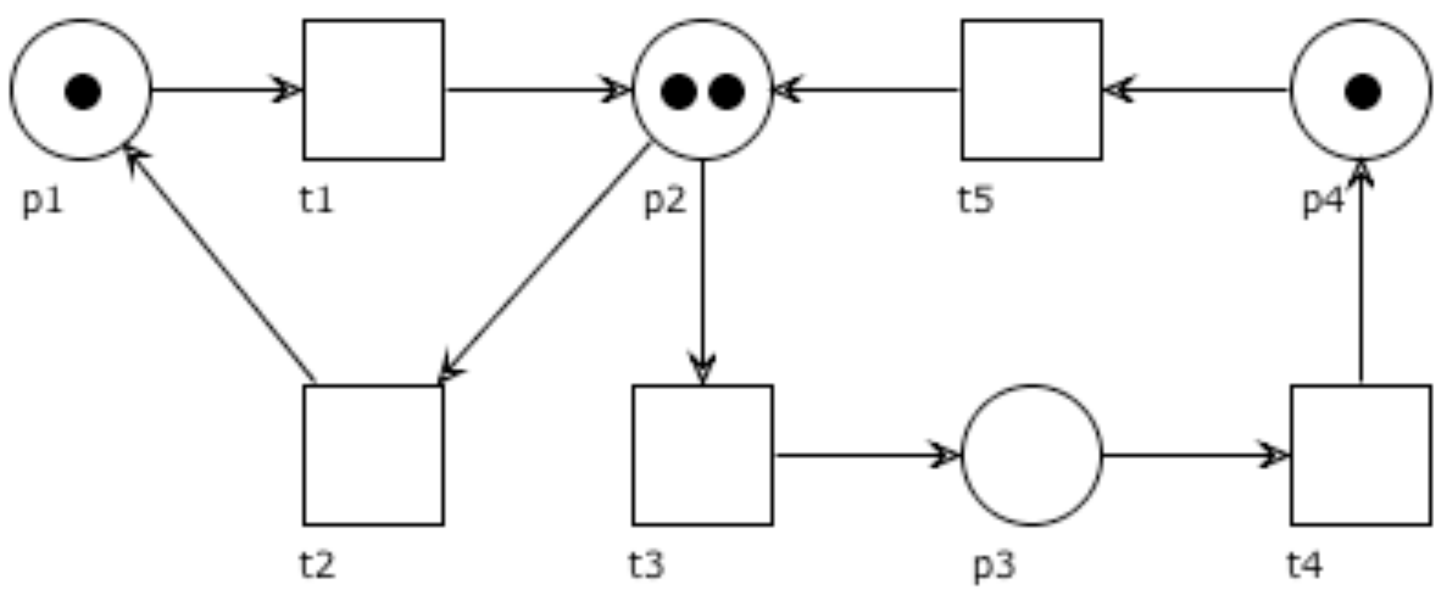

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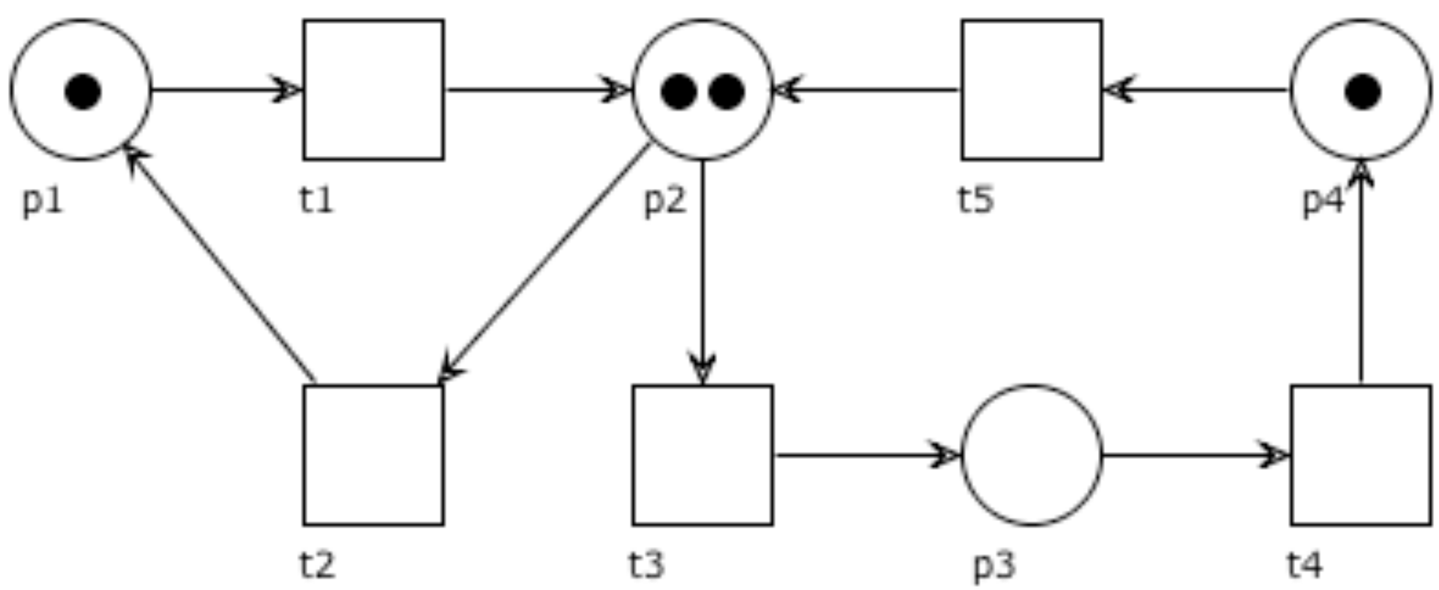


[1 1 1 1]
[2 0 2 0]
[1 2 1 2]
[4 0 0 0]

S-systems: recap

S-system $\Rightarrow$ bounded

S-system: strong conn. + $M_0(P)>0 \Leftrightarrow$ live

S-system + M reachable $\Rightarrow M(P) = M_0(P)$

S-system + str. conn.: $M(P)=M_0(P) \Leftrightarrow$ M reachable

S-system + live: $M(P)=M_0(P) \Leftrightarrow$ M reachable

S-system: S-invariant $\mathbf{I} \Leftrightarrow \mathbf{I} = [\ k\ k\ \ldots\ k\ ]$

S-net N*

Proposition: A workflow net N is an S-net
iff
N* is an S-net

N and N* differ only for the reset transition,
that has exactly one incoming arc
and exactly one outgoing arc

Workflow S-nets

Theorem: If a workflow net N is an S-system then it is sound

N is S-system <=> N* is S-system

N and N* S-systems => N and N* bounded

N workflow net => N* strong connected
$M_0(P)=1$ (initially one token in place i)
N* strong connected + $M_0(P) > 0$ <=> N* live

N* bounded and live <=> N sound

Workflow S-nets

Theorem: If a workflow net N is an S-system then it is **safe** and sound

N is S-system <=> N* is S-system

$M_0(P)=1$ (initially one token in place i)

N and N* S-systems + $M_0(P)=1$ => N and N* safe

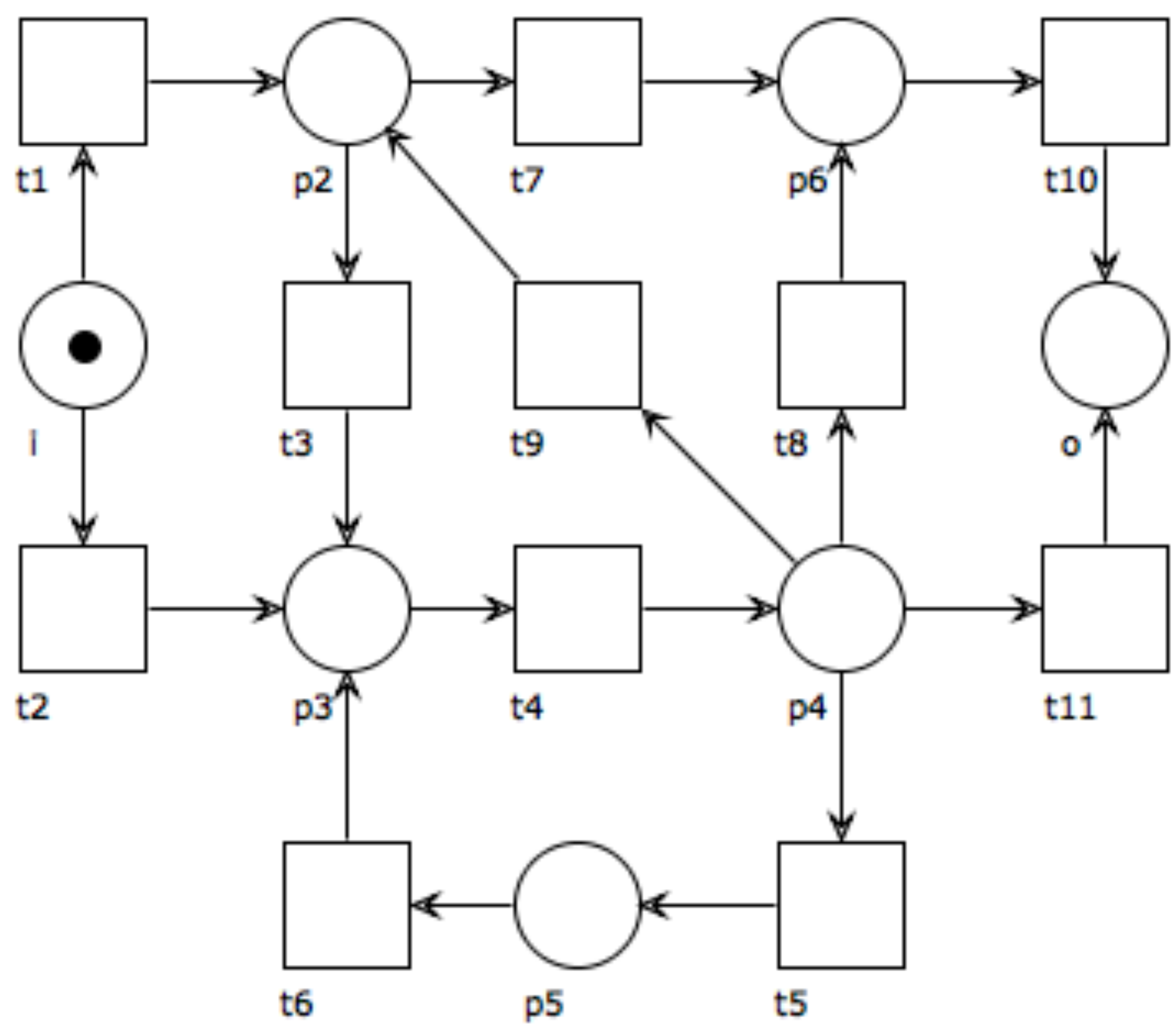
N workflow net => N* strong connected

N* strong connected + $M_0(P) = 1$ <=> N* live

N* bounded and live <=> N sound

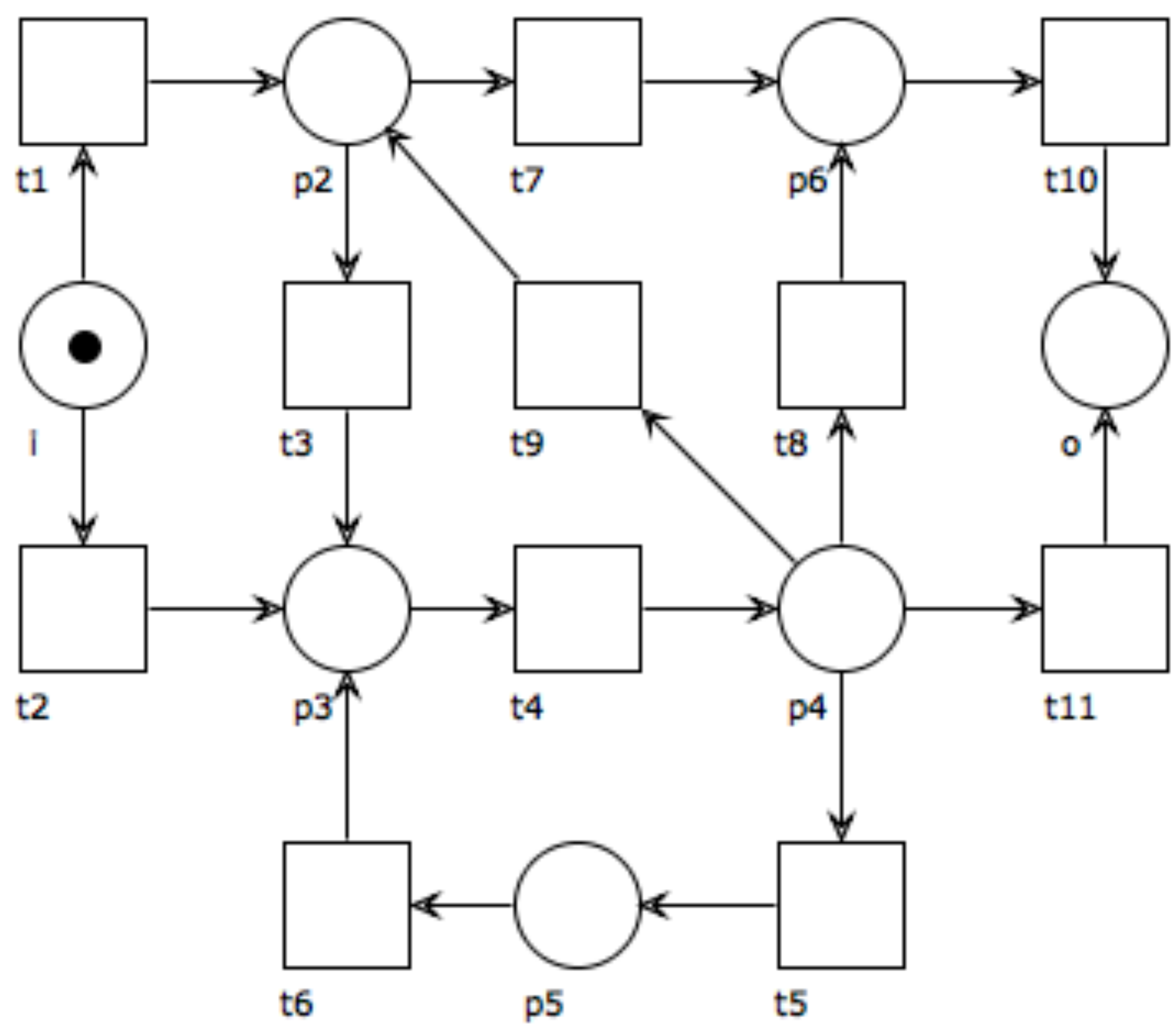
Exercise

Is the net below a workflow net?
Is it an S-net? Is it sound?



Exercise

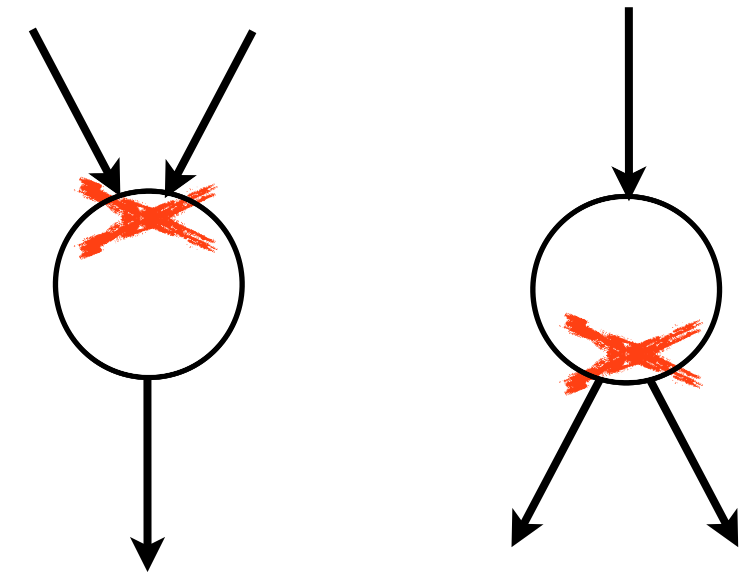
Is the net below a workflow net?
Is it an S-net? Is it sound?



Yes:
WF net
S-net
=> sound

T-systems

T-system



Definition: We recall that a net N is a **T-net** if each place has exactly one input transition and exactly one output transition

$$\forall p \in P, \qquad |\bullet p| = 1 = |p \bullet|$$

A system (N,M_0) is a **T-system** if N is a T-net

T-net N*

Is the following conjecture true?
A workflow net N is a T-net
iff N* is a T-net

T-net N*

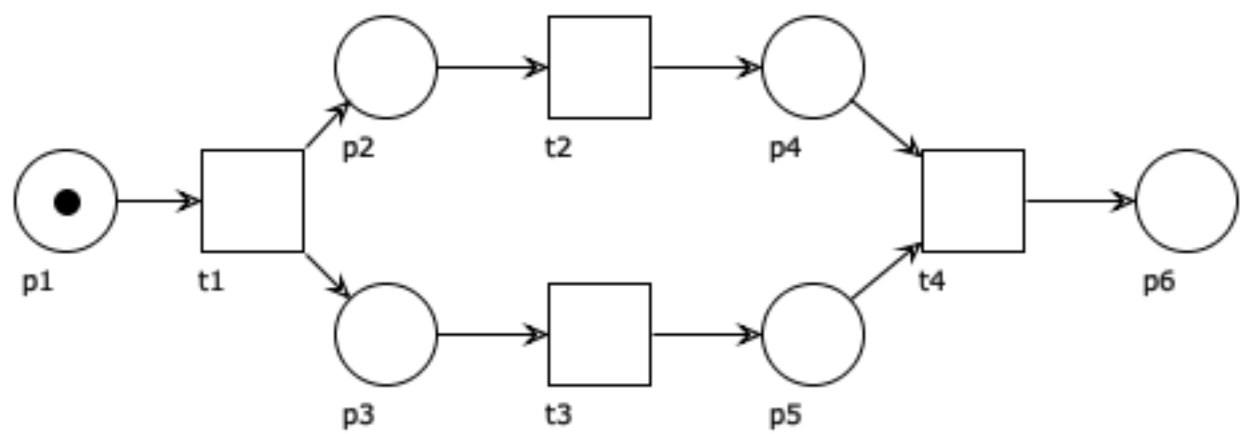
Is the following conjecture true?
A workflow net N is a T-net
iff N* is a T-net

No, a workflow net cannot be a T-net because
the place i has no incoming arc
and the place o has no outgoing arc

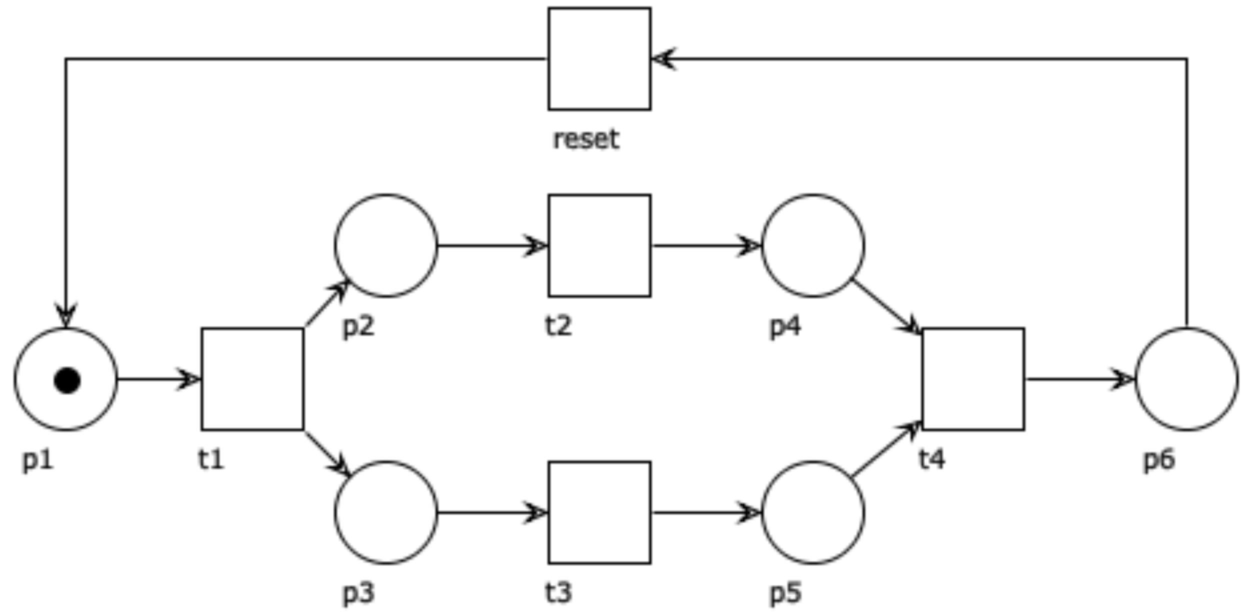
(but **N* can be a T-net**)

T-net N*: example

N is a workflow net
but not a T-net



N* is a T-net
(not a workflow net)



Take-home message

Any workflow net N such that N* is a T-net
is **safe and sound**
iff
every circuit of N* is marked

Fundamental property of T-systems

The token count of a circuit is invariant under any firing.

Notation: token count of a circuit

Let $\gamma = (x_1, y_1)(y_1, x_2)(x_2, y_2)...(x_n, y_n)$ be a circuit.

Let $P_{|\gamma} \subseteq P$ be the set of places in γ.

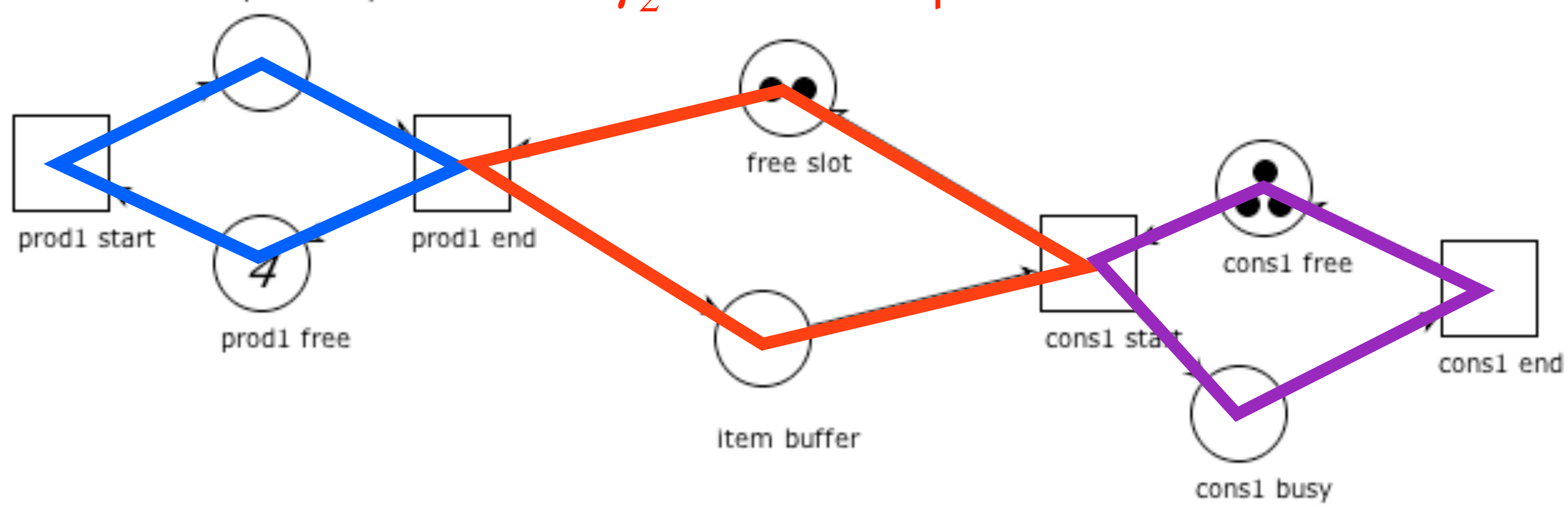
$$M(\gamma) = M(P_{|\gamma}) = \sum_{p \in P_{|\gamma}} M(p)$$

We say that γ is **marked at** M if $M(\gamma) > 0$

Example

$\gamma_1 =$ prod1busy prod1end prod1free prod1start

$\gamma_2 =$ freeslot prod1end itembuffer cons1start



$\gamma_3 =$ cons1busy cons1end cons1free cons1start

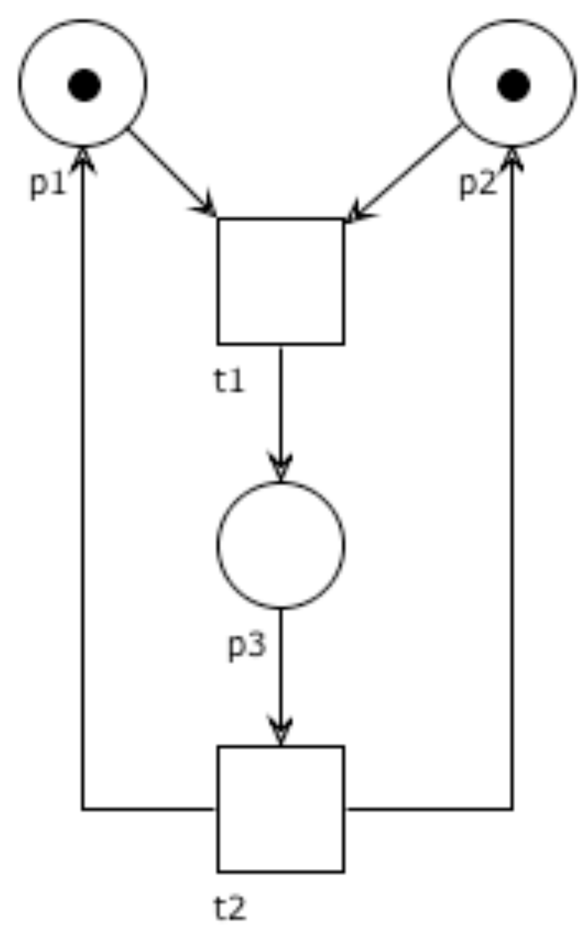
$$M(\gamma_1) = 4$$

$$M(\gamma_2) = 2$$

$$M(\gamma_3) = 3$$

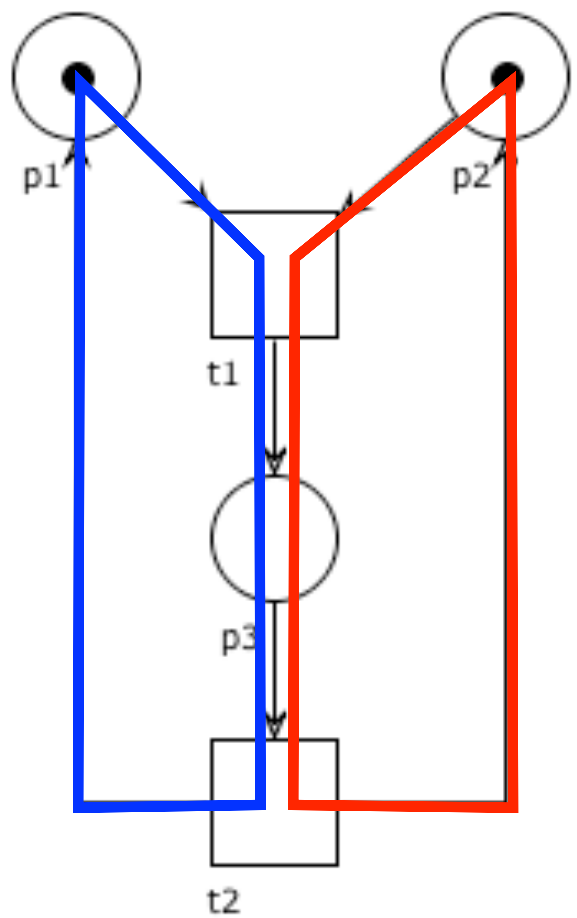
Question time

Trace two circuits over the T-system below



Question time

Trace two circuits over the T-system below



Fundamental property of T-systems

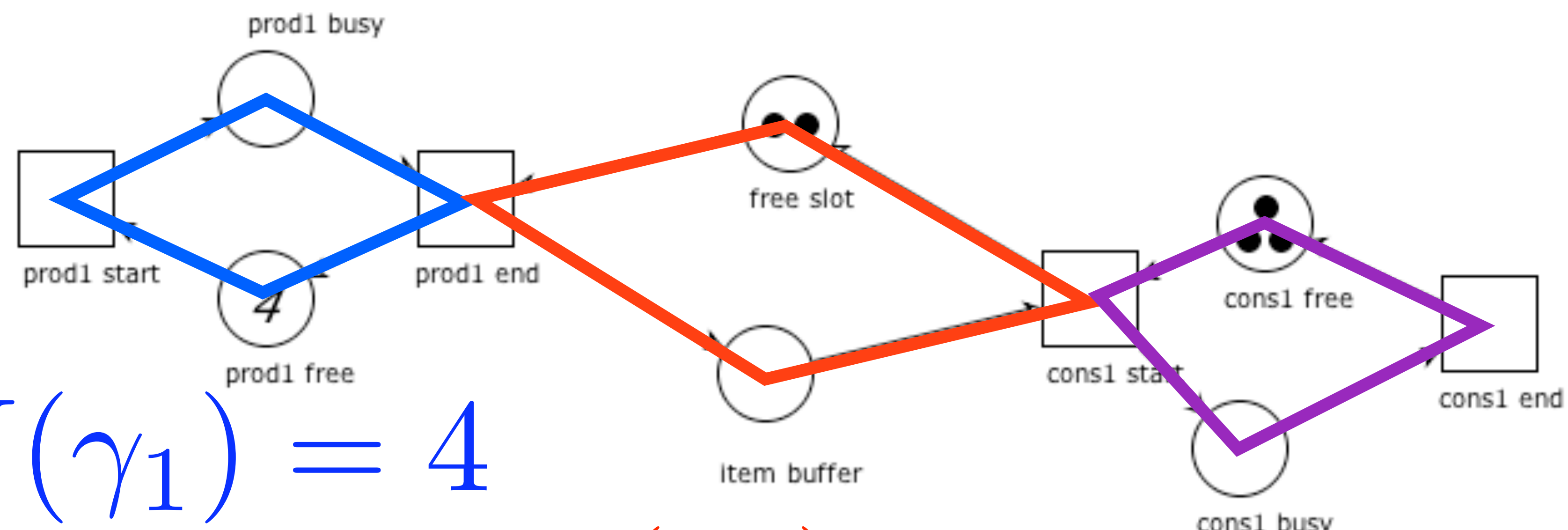
Proposition: Let γ be a circuit of a T-system (P, T, F, M_0). If M is a reachable marking, then $M(\gamma) = M_0(\gamma)$

Take any $t \in T$: either $t \notin \gamma$ or $t \in \gamma$.

If $t \notin \gamma$, then no place in $\bullet t \cup t\bullet$ is in γ
(otherwise, by definition of T-nets, t would be in γ).
Then, an occurrence of t does not change the token count of γ.

If $t \in \gamma$, then exactly one place in $\bullet t$ and one place in $t\bullet$ are in γ.
Then, an occurrence of t does not change the token count of γ.

Example



$$M(\gamma_1) = 4$$

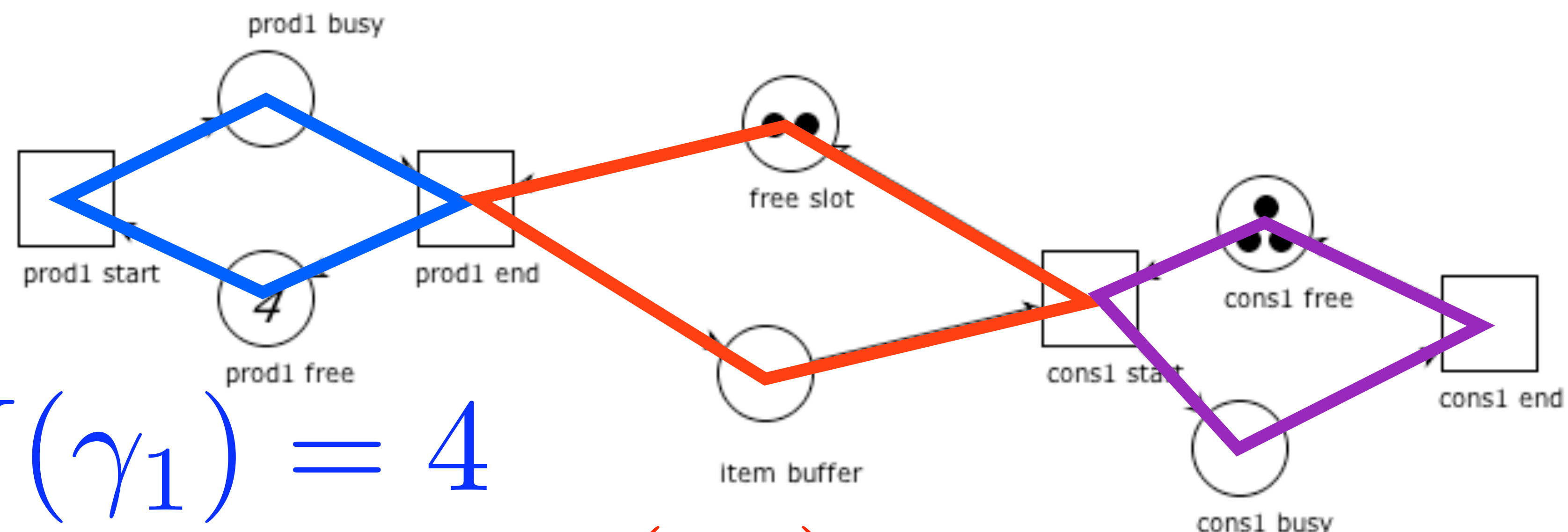
$$M(\gamma_2) = 2$$

$$M(\gamma_3) = 3$$

M_0 = [0 4 2 0 3 0]

M = [2 2 1 2 2 1] Not reachable!

Example



$M(\gamma_1) = 4$

$M(\gamma_2) = 2$

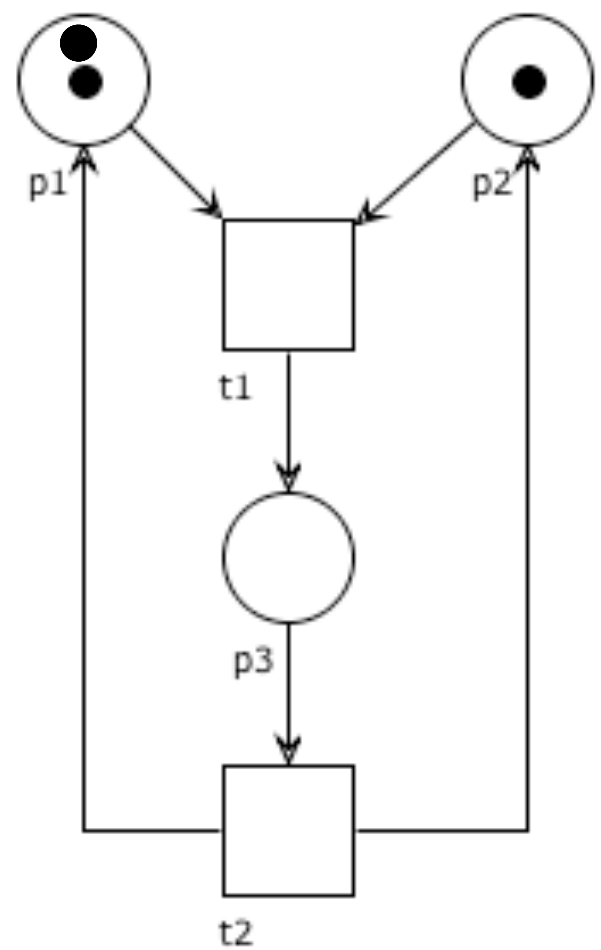
$M(\gamma_3) = 3$

M_0 = [0 4 2 0 3 0]

M’ = [2 1 1 1 2 2] Not reachable!

Question time

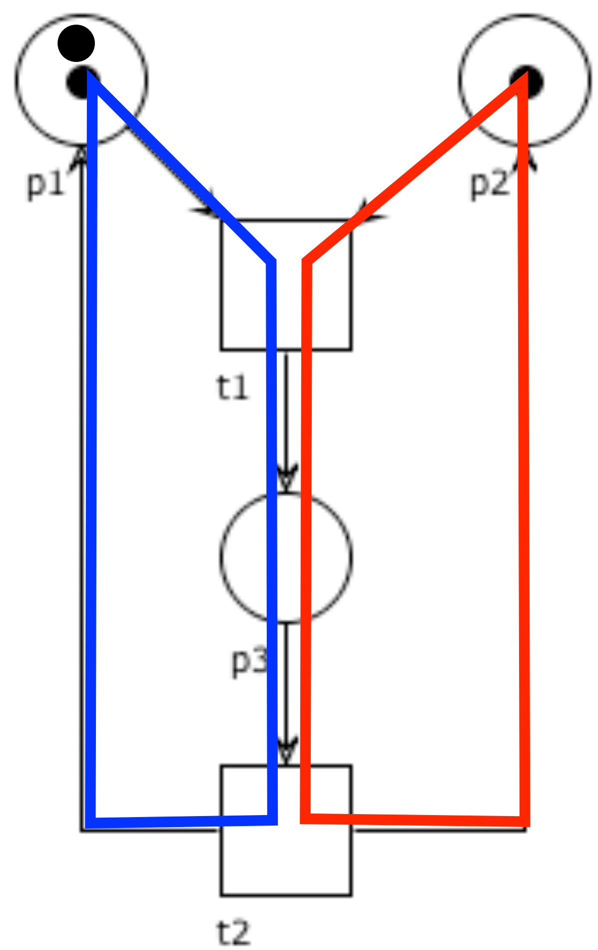
Is the marking $p_1 + 2p_2$ reachable? (why?)



Question time

Is the marking $p_1 + 2p_2$ reachable? (why?)

No
the token count in
the left (blue) circuit
must remain 2
and
the token count in
the right (red) circuit
must remain 1



T-invariants of T-nets

Proposition: Let N=(P,T,F) be a (connected) T-net. **J** is a T-invariant of N **iff** **J**=[k ... k] for some value k

(the proof is dual to the analogous proposition for S-invariants of S-nets)

T-systems: an observation

Notably, computation in T-systems is concurrent, but essentially deterministic: the firing of a transition t in M cannot disable another transition t’ enabled at M

Determination of control: the transitions responsible for enabling t are exactly one for each input place of t

Boundedness in strongly connected T-systems

Lemma: If a T-system (N,M_0) is strongly connected, then it is bounded

Let Γ be the set of the circuits of N and let $k = \max_{\gamma \in \Gamma} M_0(\gamma)$.

Since N is strongly connected, every place p belongs to some circuit γ_p.

By the fundamental property of T-systems: token count of γ_p is invariant.

Thus, for any reachable marking M, we have $M(p) \leq M(\gamma_p) = M_0(\gamma_p) \leq k$.
Hence the net is k-bounded.

Safeness in strongly connected T-systems

Corollary: If a T-system (N,M_0) is strongly connected and $M_0(P)=1$, then it is safe

Let Γ be the set of the circuits of N and let $k = \max_{\gamma\in\Gamma} M_0(\gamma)$=1

Since N is strongly connected, every place p belongs to some circuit γ_p.

By the fundamental property of T-systems: token count of γ_p is invariant.

Thus, for any reachable marking M, we have $M(p) \leq M(\gamma_p) = M_0(\gamma_p) \leq k$.
Hence the net is k-bounded. =1

Safeness in strongly connected T-systems

Corollary: If a T-system (N, M_0) is strongly connected and for any circuit γ $M_0(\gamma) \leq 1$, then it is safe

Let Γ be the set of the circuits of N and let $k = \max_{\gamma \in \Gamma} M_0(\gamma)$. ≤ 1

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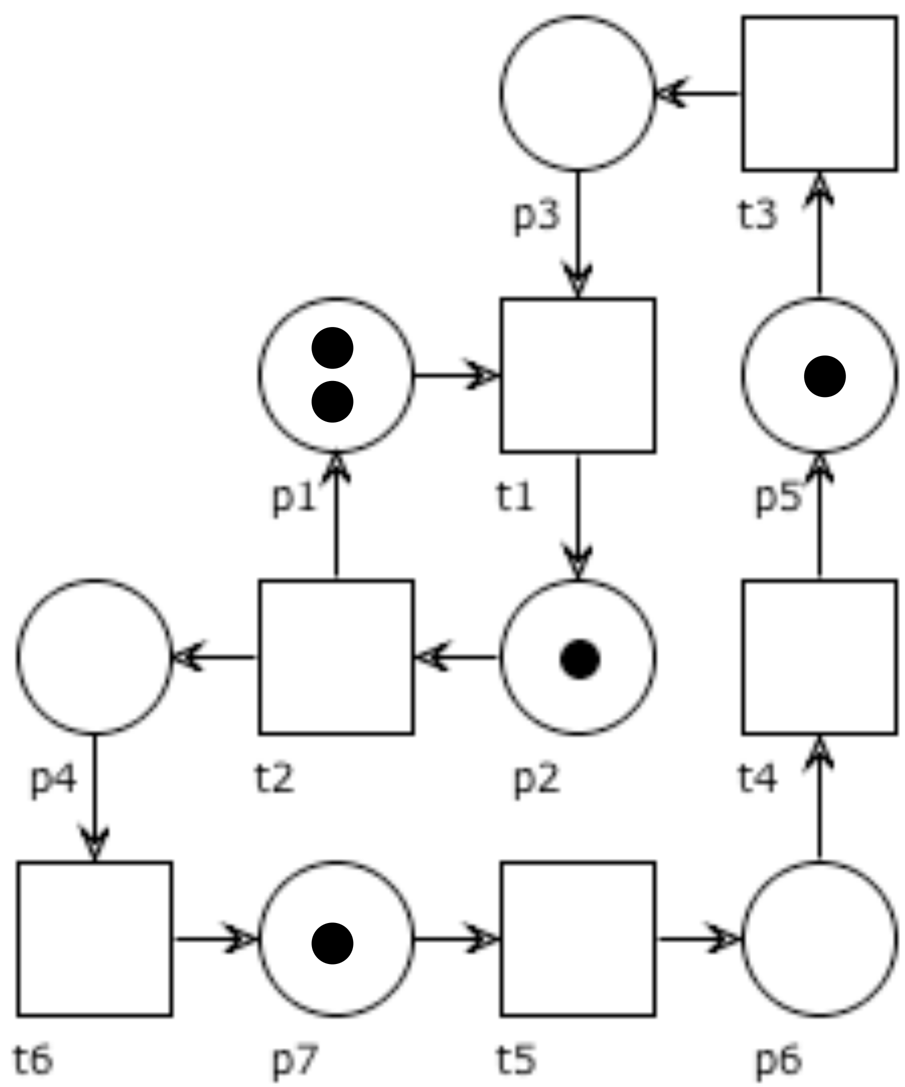
By the fundamental property of T-systems: token count of γ_p is invariant.

Thus, for any reachable marking M, we have $M(p) \leq M(\gamma_p) = M_0(\gamma_p) \leq k$.
Hence the net is k-bounded. ≤ 1

Question time

Is the T-systems below bounded? (why?)



Question time

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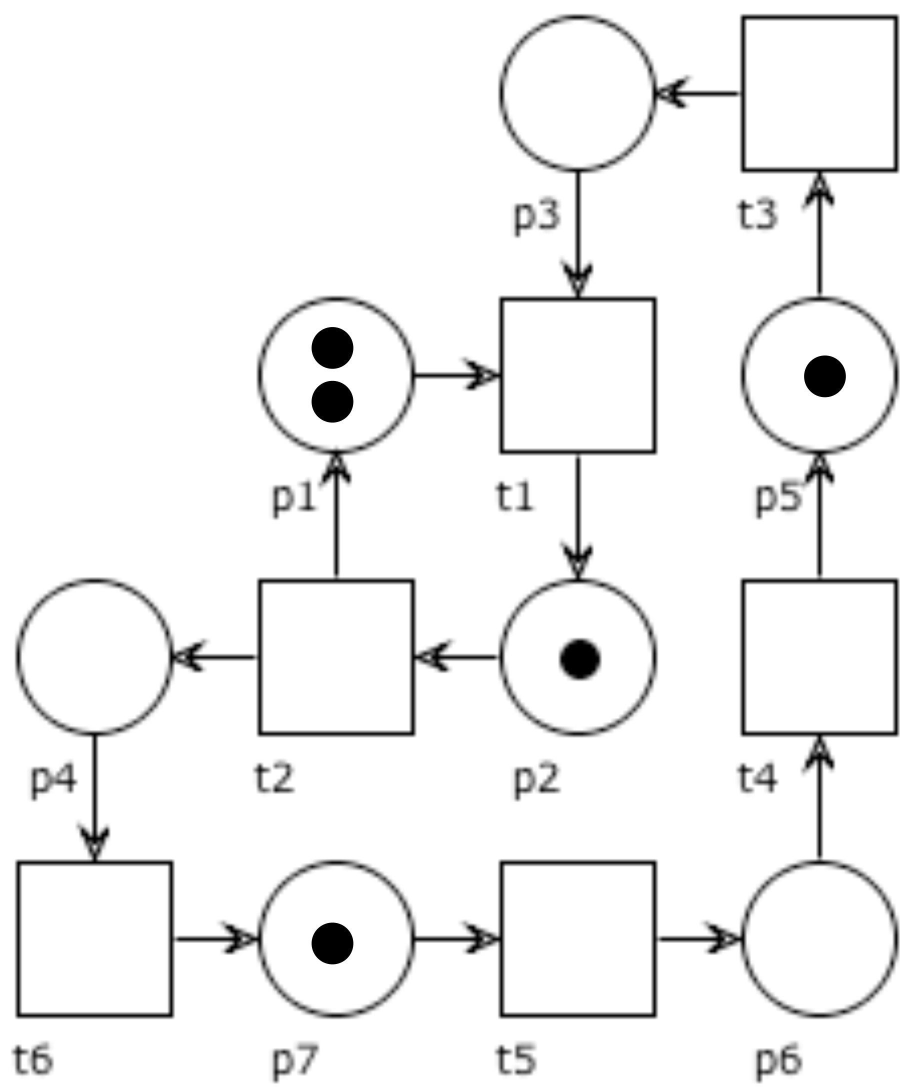
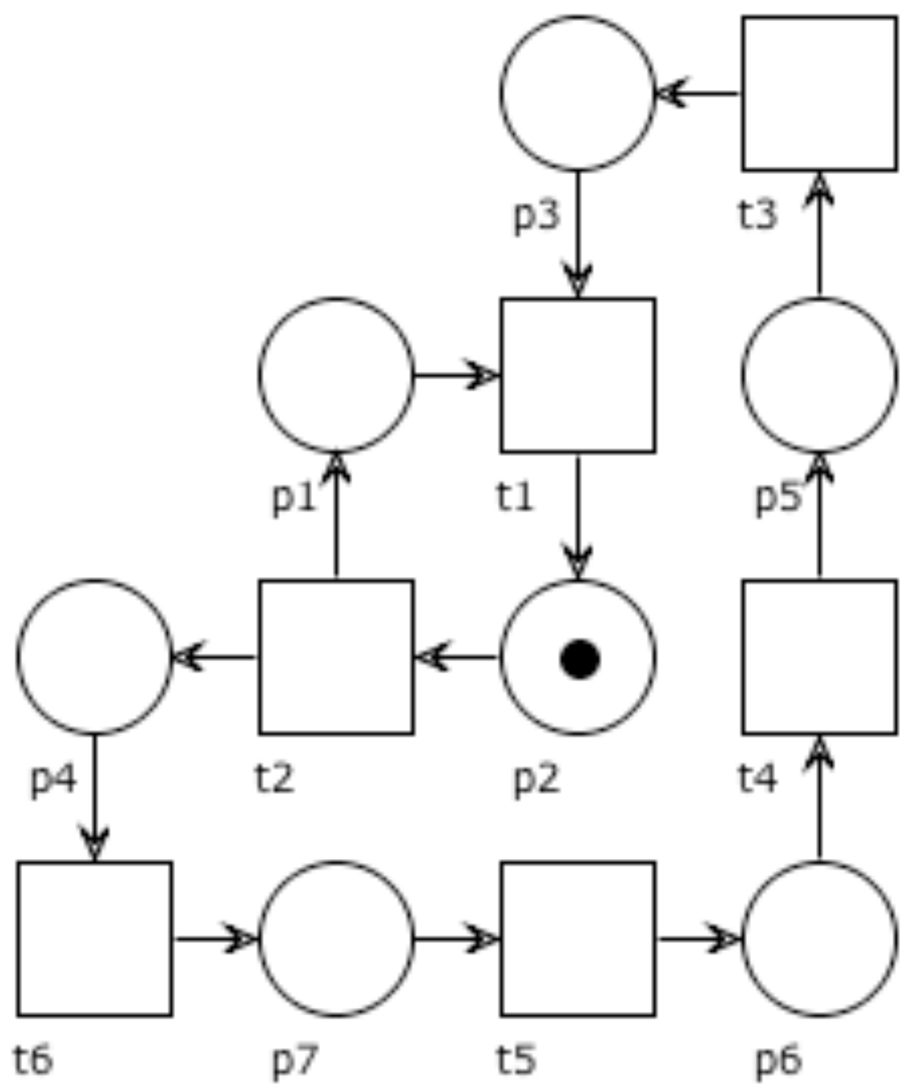


Strongly connected
=> bounded

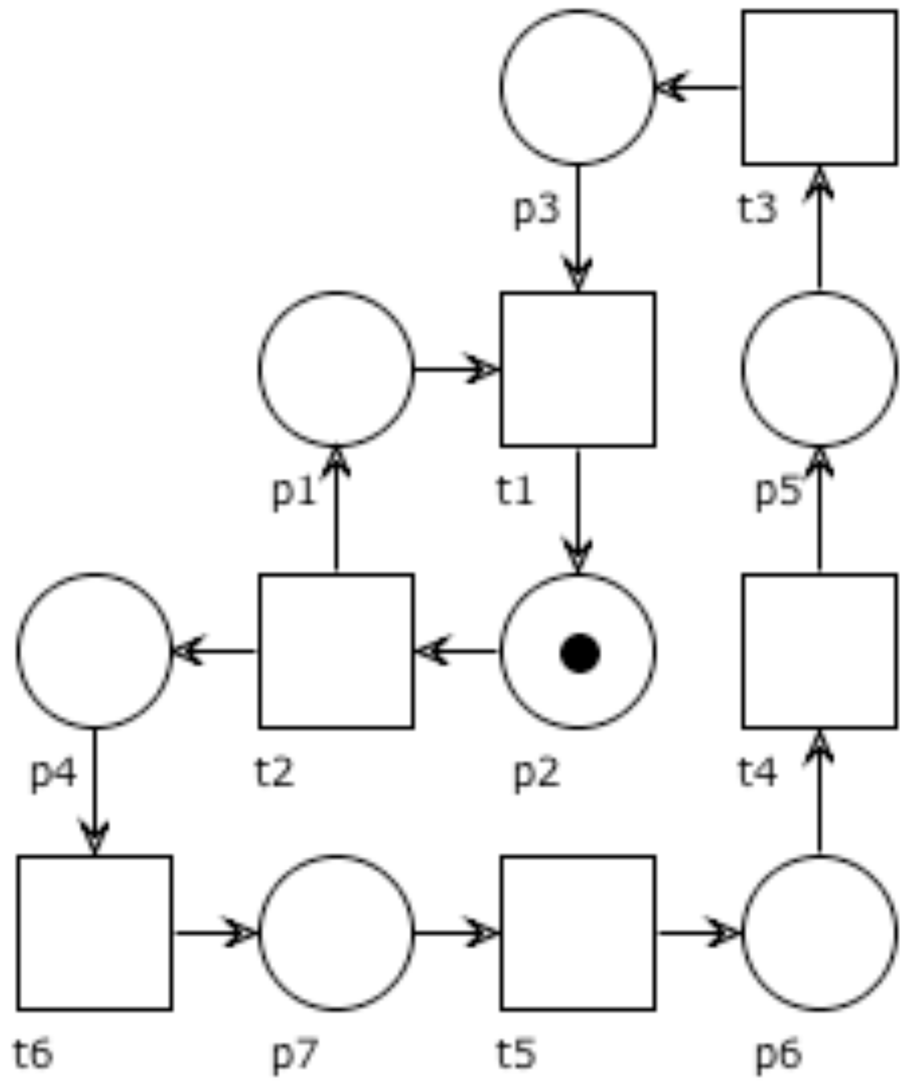
Question time

Is the T-systems below safe? (why?)



Question time

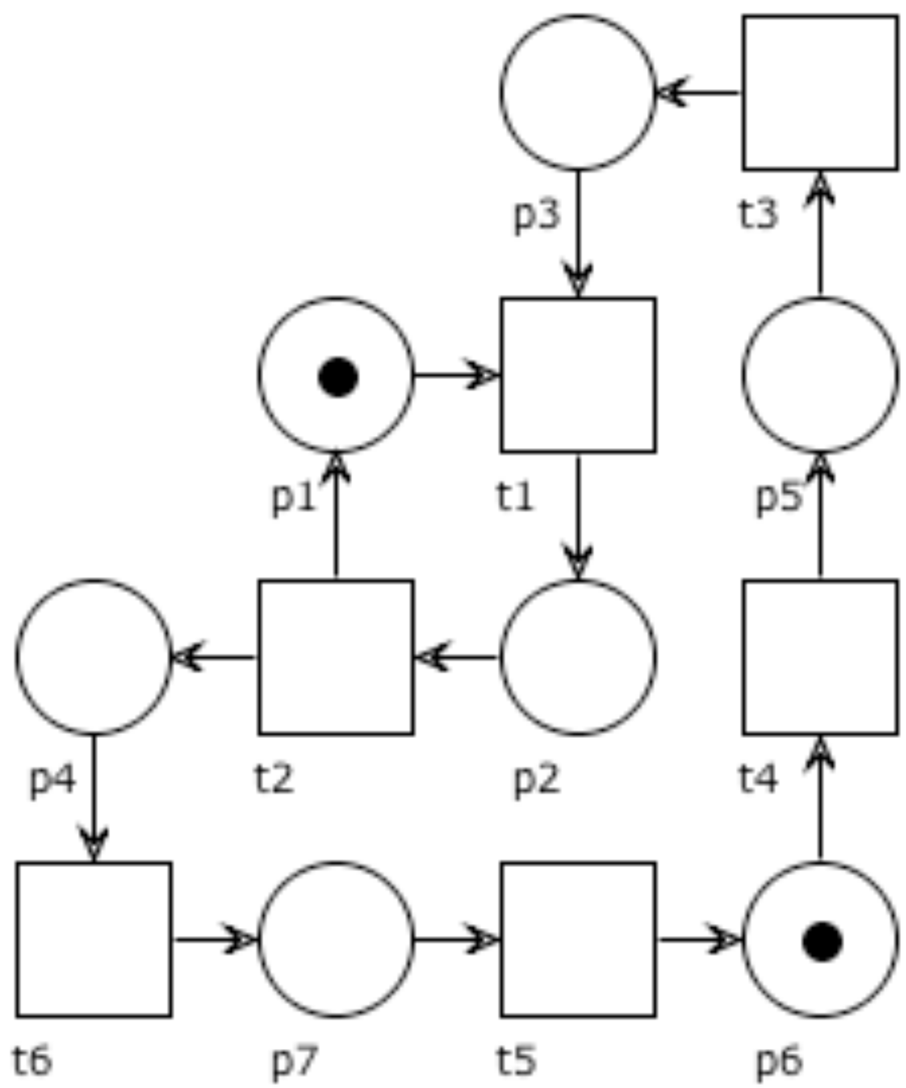
Is the T-systems below safe? (why?)



Strongly connected
+ $M_0(P)=1$
=> safe

Question time

Is the T-systems below safe? (why?)



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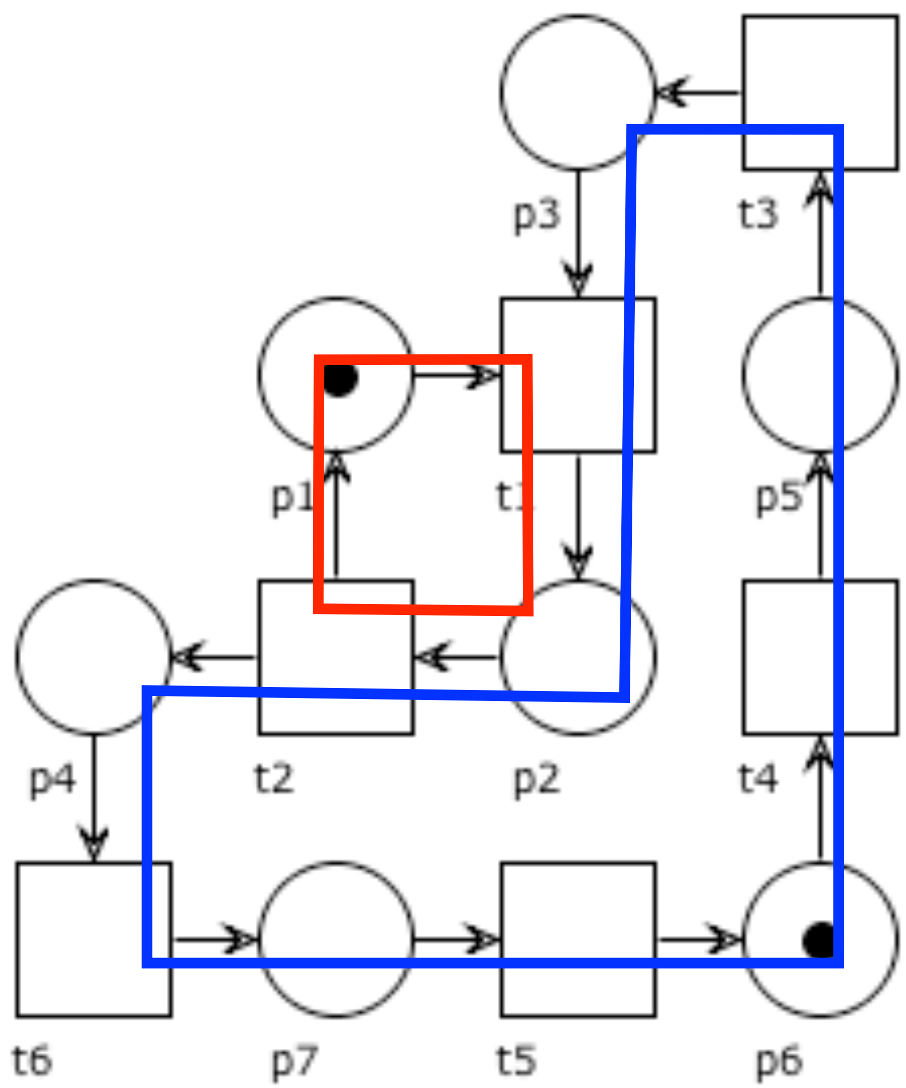


Strongly connected
+ $\forall \gamma$. $M_0(\gamma) \leq 1$
=> safe

Liveness theorem for T-systems

Theorem: A T-system (N, M_0) is live **iff** every circuit of N is marked at M_0

$\Rightarrow$) (quite obvious)
By contradiction, let γ be a circuit with $M_0(\gamma) = 0$.
By the fundamental property of T-systems: $\forall M \in [\, M_0 \,\rangle,\ M(\gamma) = 0$.

Take any $t \in T_{|\gamma}$ and $p \in P_{|\gamma} \cap \bullet t$.

For any $M \in [\, M_0 \,\rangle$, we have $M(p) = 0$.
Hence t is never enabled and the T-system is not live.

Liveness theorem for T-systems

Theorem: A T-system (N,M_0) is live **iff** every circuit of N is marked at M_0

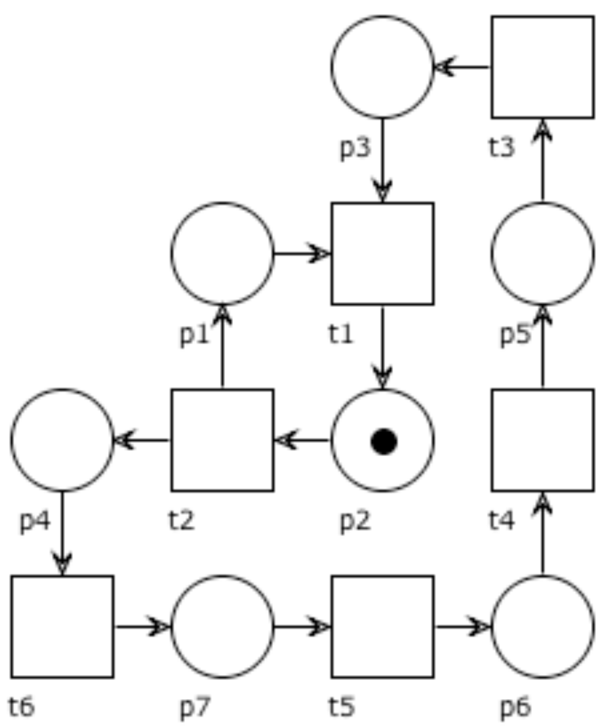
$\Leftarrow$) (more involved)
Take any $t \in T$ and $M \in [\, M_0 \,\rangle$.
We need to show that some marking M' reachable from M enables t.

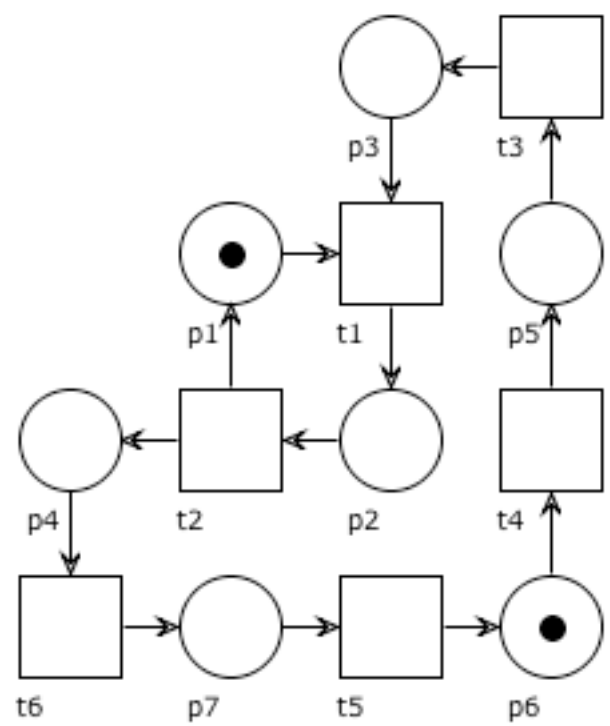
The key idea is to collect the places that control the firing of t:
$p \in P_{M,t}$ if there is a path from p to t through places unmarked at M.
We then proceed by induction on the size of $P_{M,t}$.

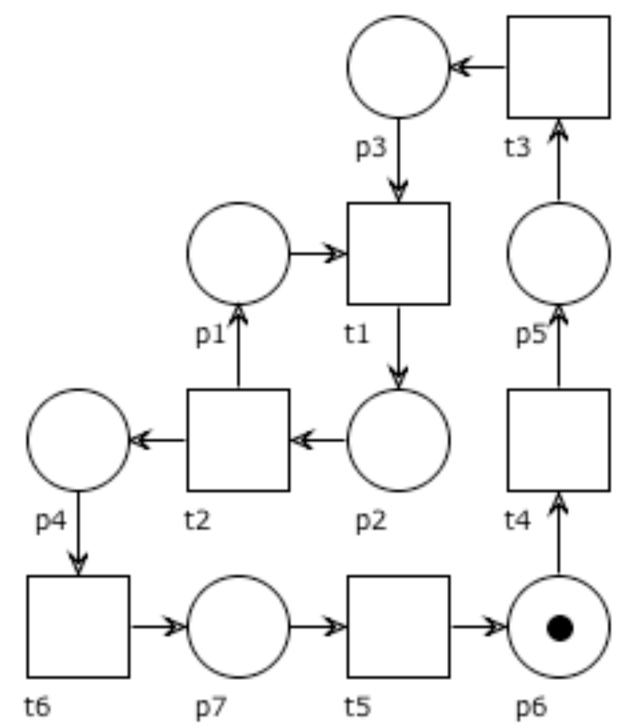
Rest of the proof left to your ingenuity

Question time

Which of the T-systems below is live? (why?)







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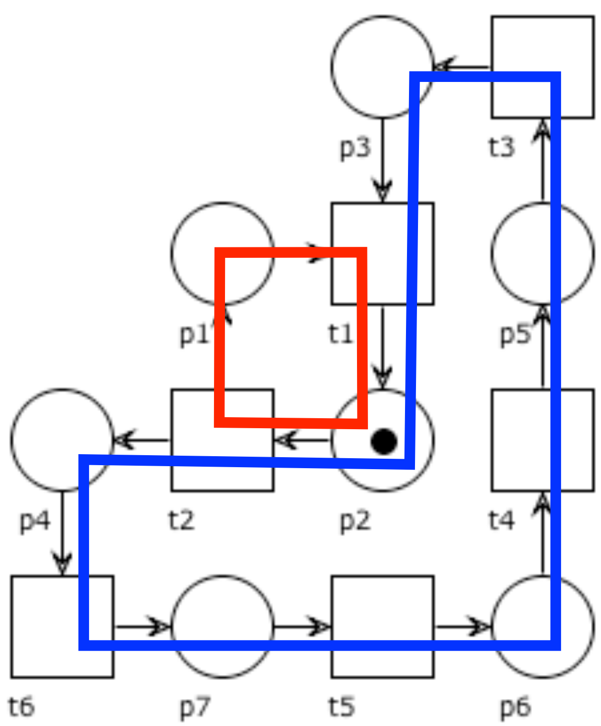
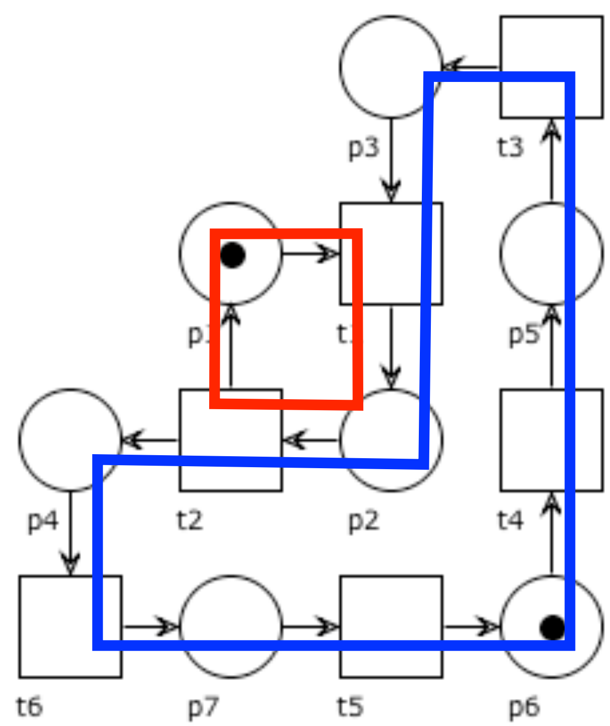


Both circuits are marked
=> live



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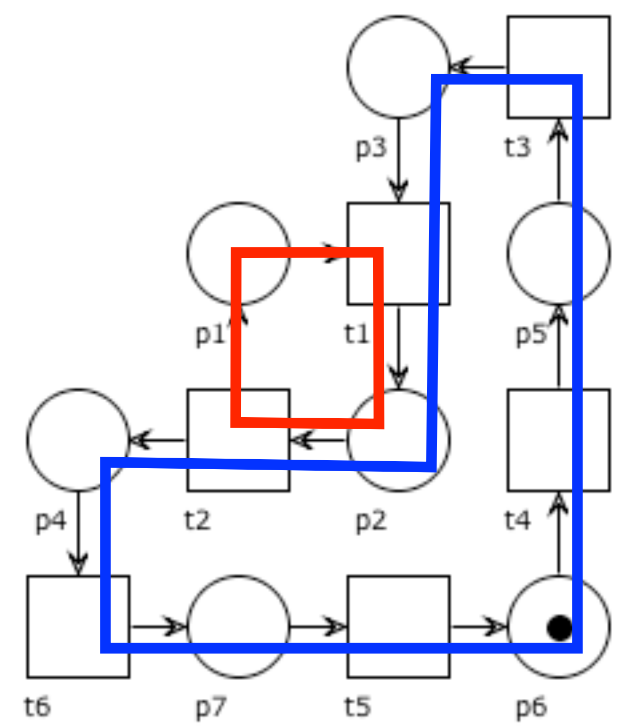


red circuit is not marked
=> not live

T-systems: recap

T-system + γ circuit + M reachable => $M(\gamma) = M_0(\gamma)$

T-system + γ circuit + $M(\gamma) \neq M_0(\gamma)$ => M not reachable

T-system + $\gamma_1 \ldots \gamma_n$ circuits: $\exists i.\ p \in \gamma_i$ <=> p bounded

T-system: $M_0(\gamma) > 0$ for all circuits γ <=> live

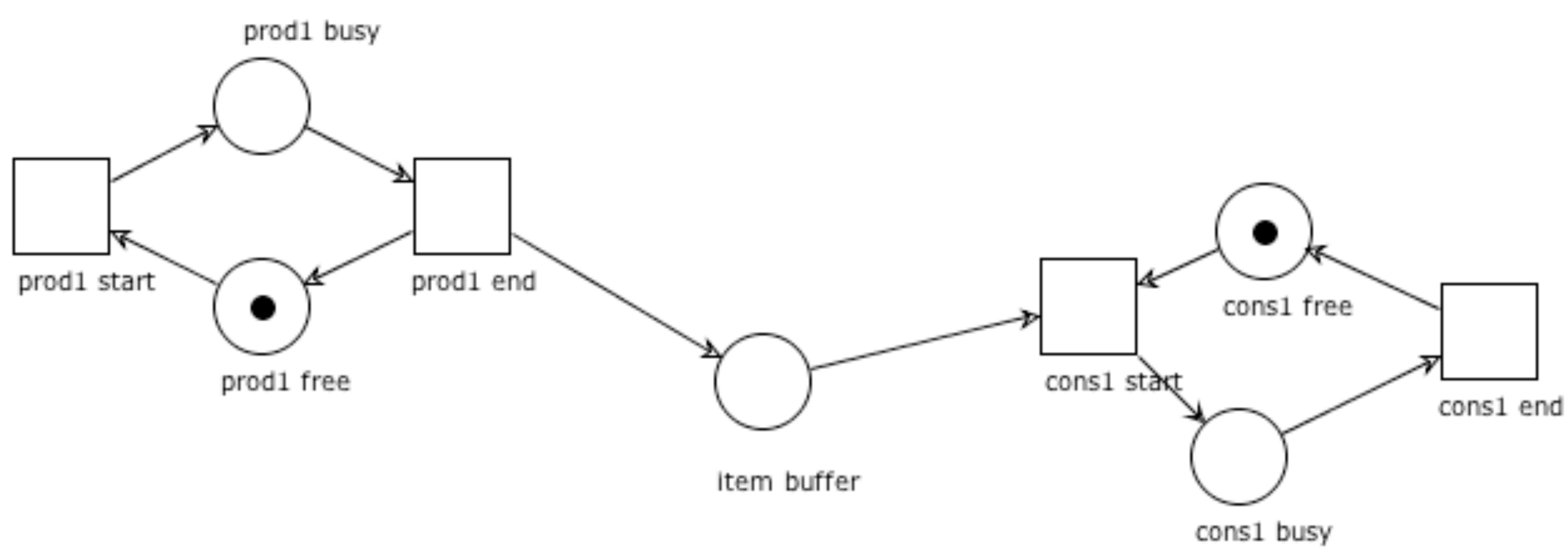
T-system: strongly connected => bounded

T-system + live: strongly connected <=> bounded

T-system: T-invariant **J** <=> **J** = [k k ... k]

Exercises

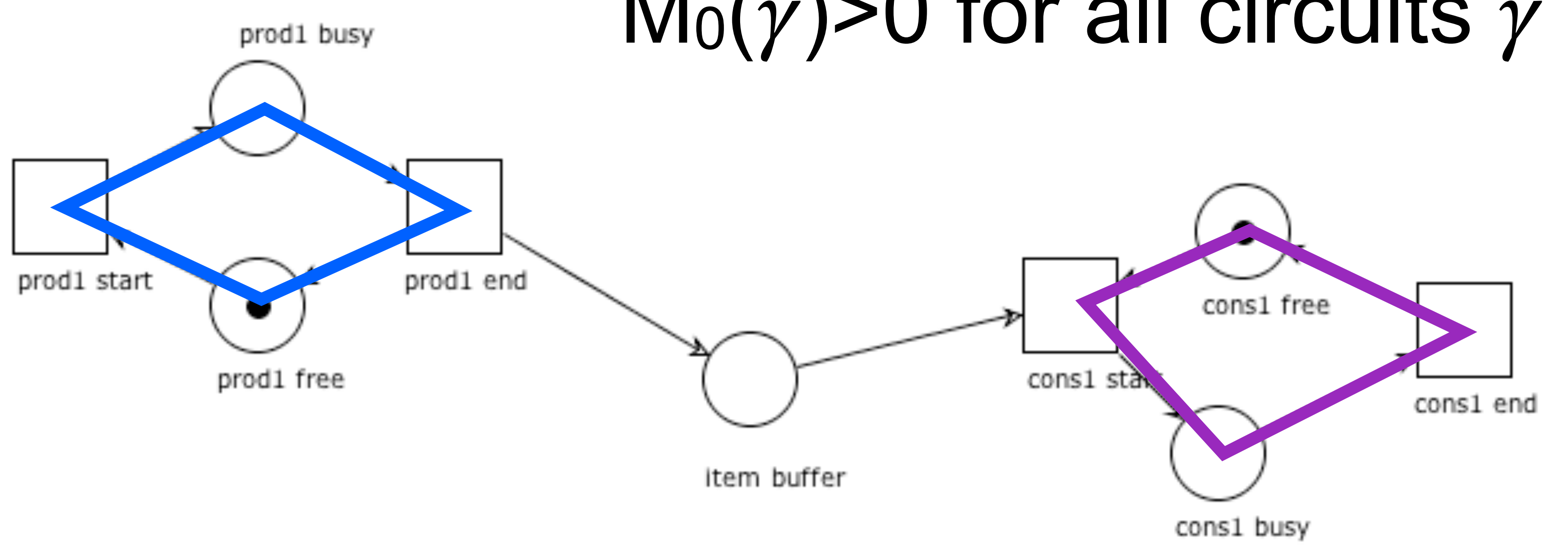
Which are the circuits of the T-system below?
Is the T-system below live? (why?)
Which places are bounded? (why?)
Assign a bound to each bounded place.



Exercises

Which are the circuits of the T-system below?
Is the T-system below live? (why?)
Which places are bounded? (why?)
Assign a bound to each bounded place.

$M_0(\gamma)>0$ for all circuits γ <=> LIVE



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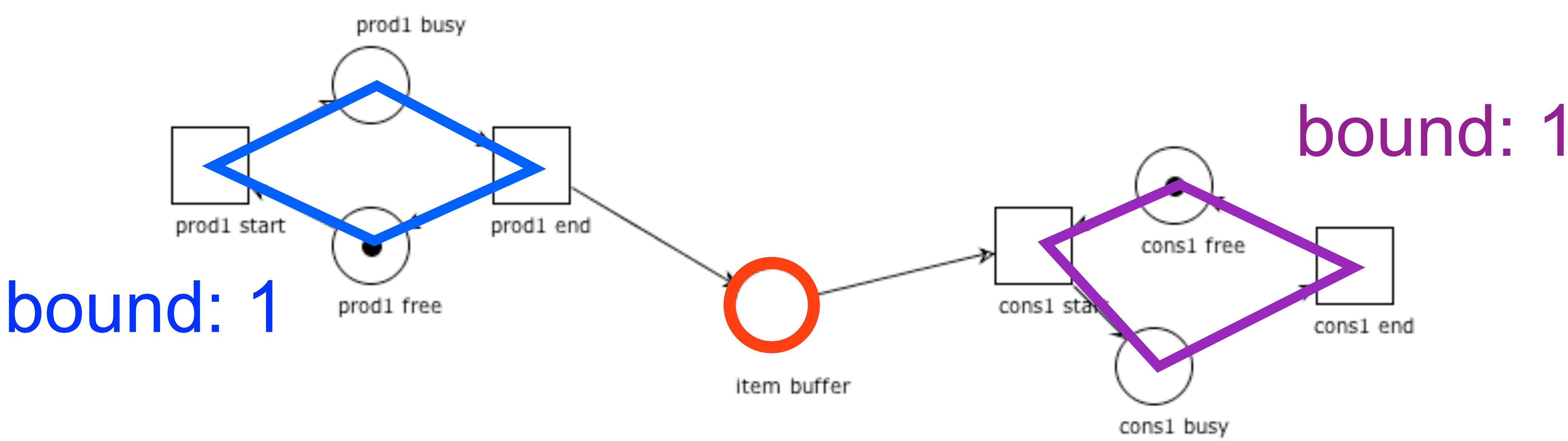


$\exists i.\ p \in \gamma_i \Leftrightarrow p$ bounded

Consequences on workflow nets

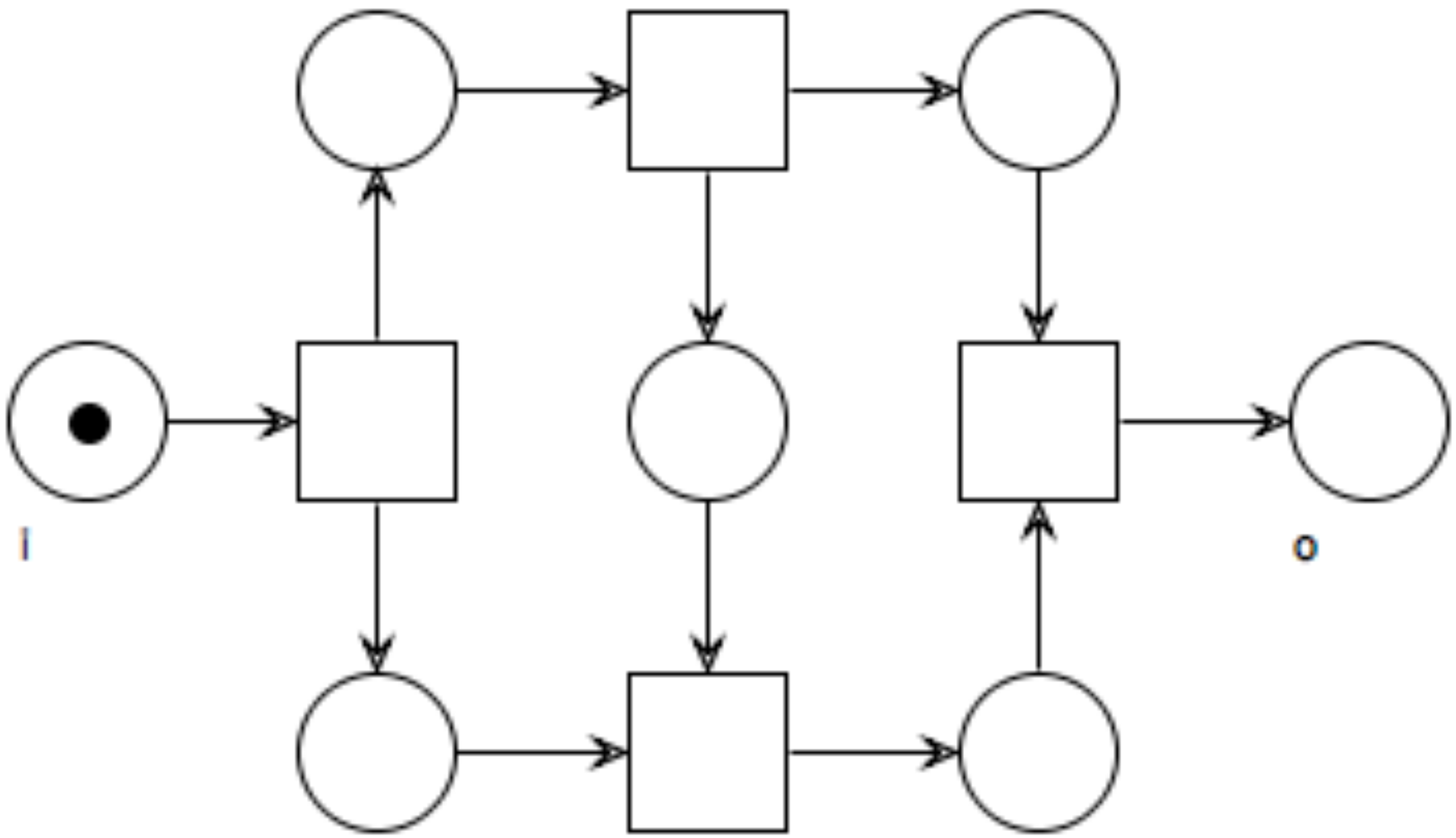
Theorem: If N is a workflow net s.t. N* is a T-system then
N is safe and sound **iff**
every circuit of N* is marked

N workflow net => N* strongly connected
N* strongly connected + T-system => N* bounded
N* strongly connected + T-system + $M_0(P)=1$ => N* safe

every circuits of N* is marked <=> N* live

Exercise

Is the net below a workflow net?
Is it sound?



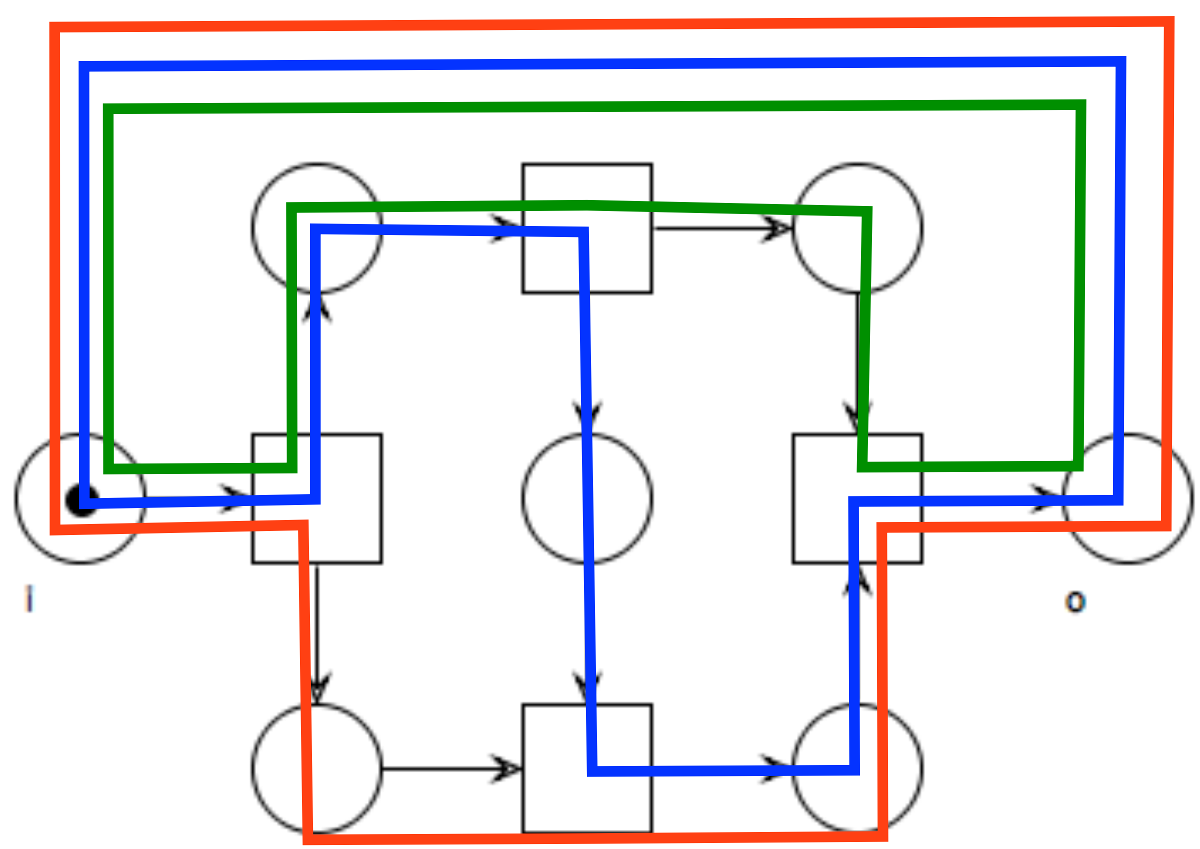
Exercise

Is the net below a workflow net? Yes

Is it sound? N* is a T-net

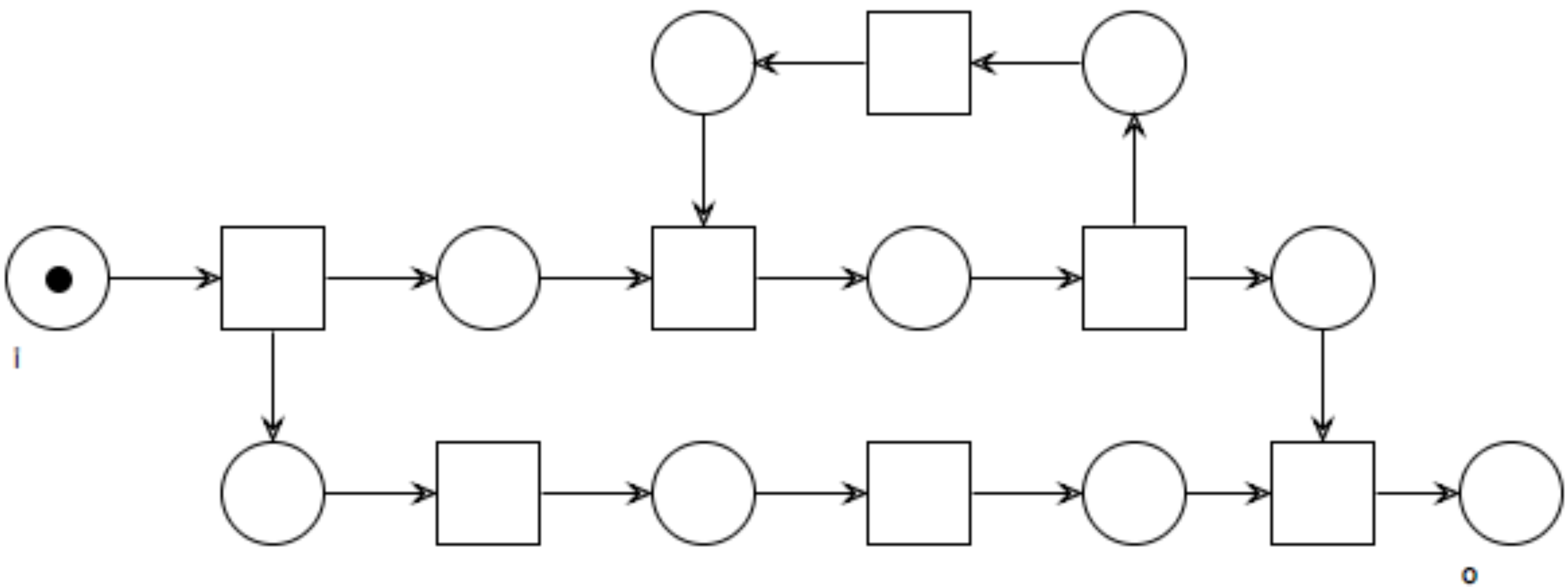
Every circuit is marked

=> Yes



Exercise

Is the net below a workflow net?
Is it sound?



Exercise

Is the net below a workflow net? Yes

Is it sound?

N* T-net

Unmarked circuit

=> No