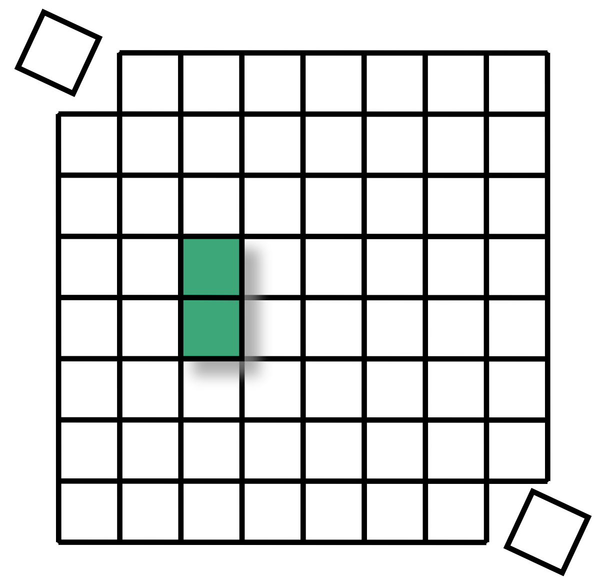
Business Processes Modelling

MPB (6 cfu, 295AA)

Roberto Bruni
http://www.di.unipi.it/~bruni

14 - Invariants

Object

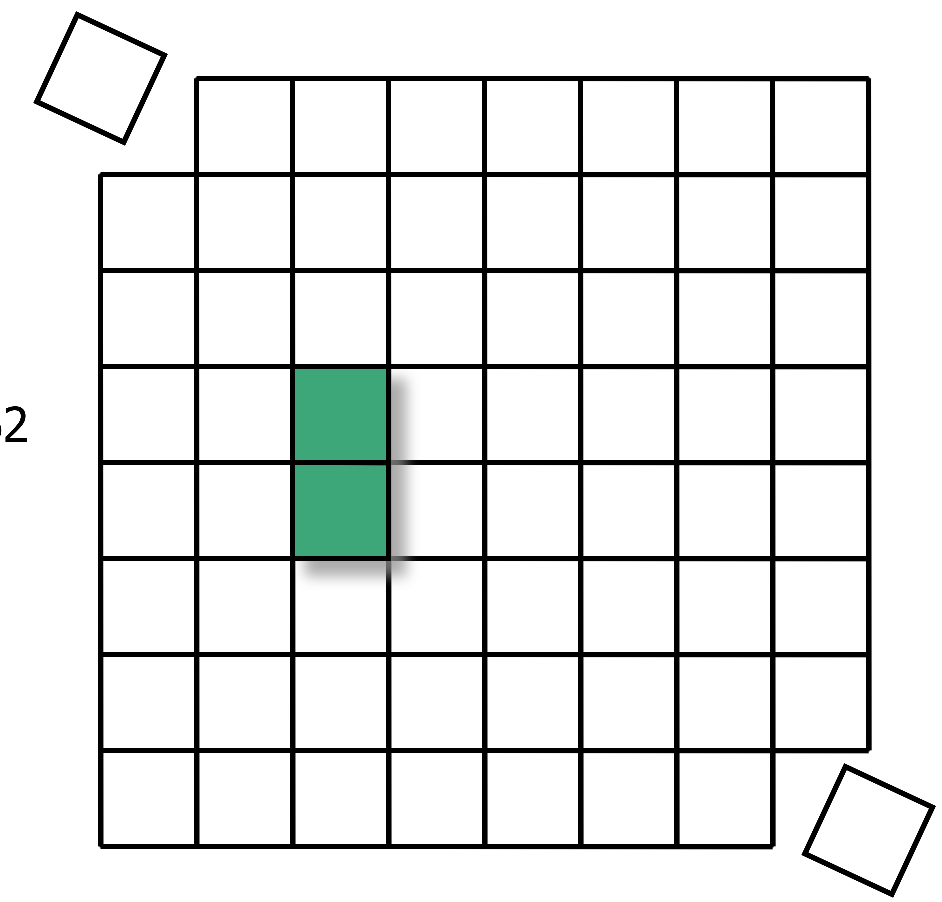


We introduce two relevant kinds of invariants for Petri nets

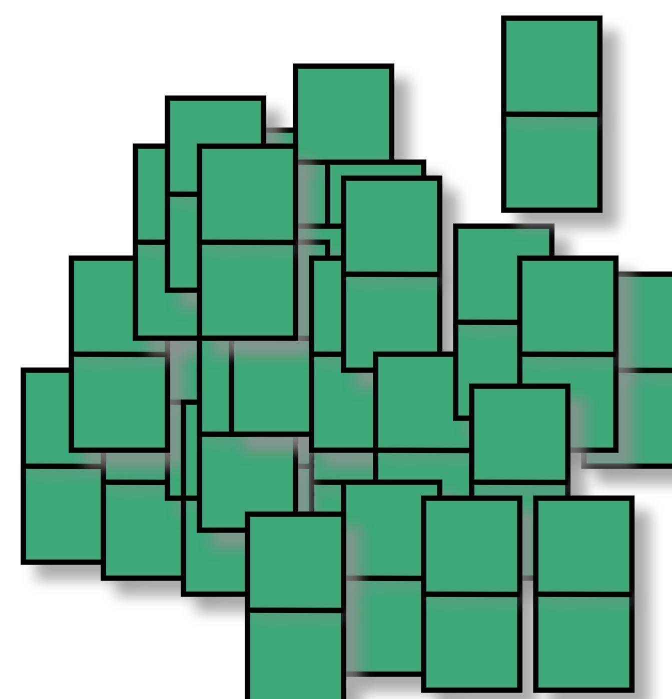
Free Choice Nets (book, optional reading)
https://www7.in.tum.de/~esparza/bookfc.html

Puzzle time: tiling a chessboard with dominoes

64-2 = 62 cells



31 dominoes



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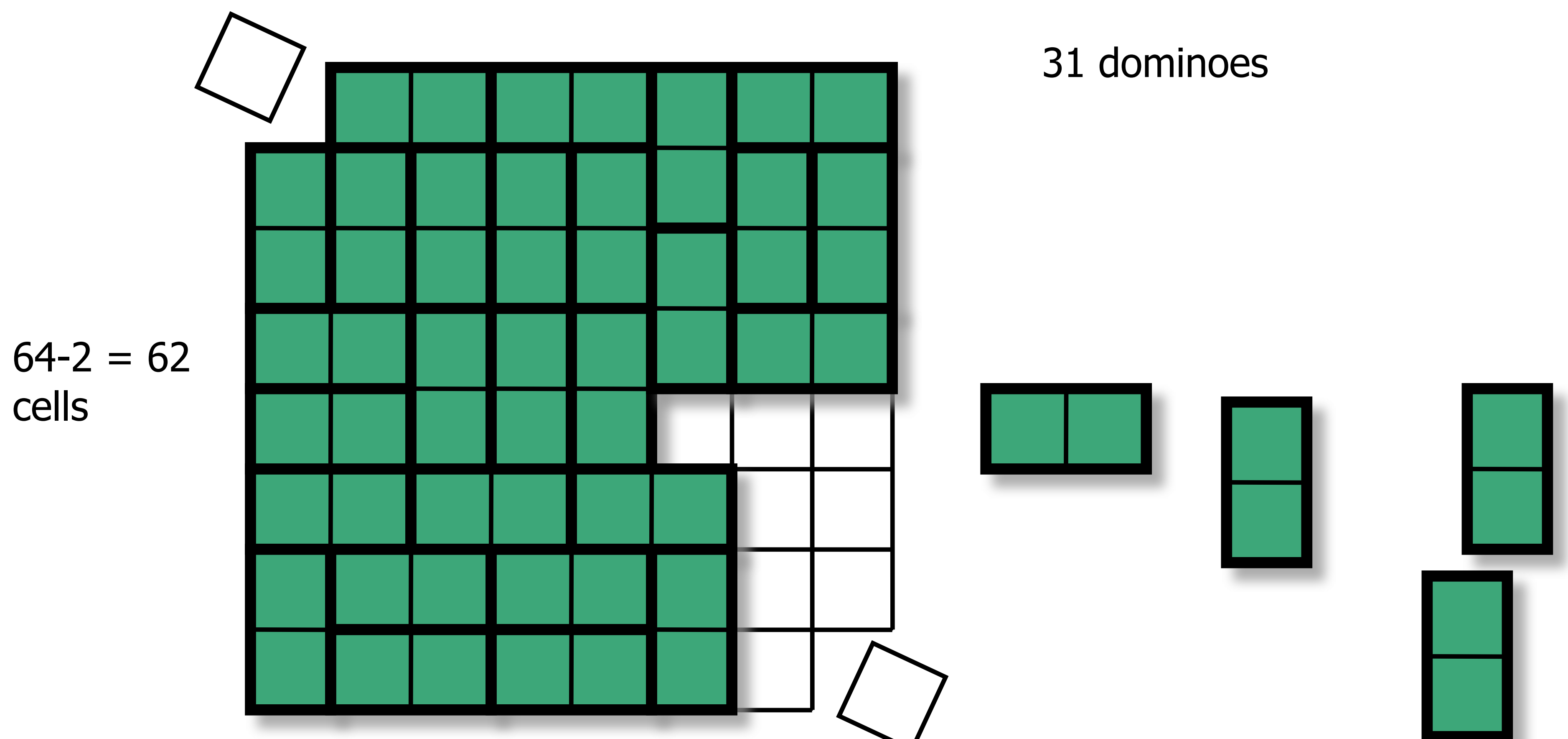
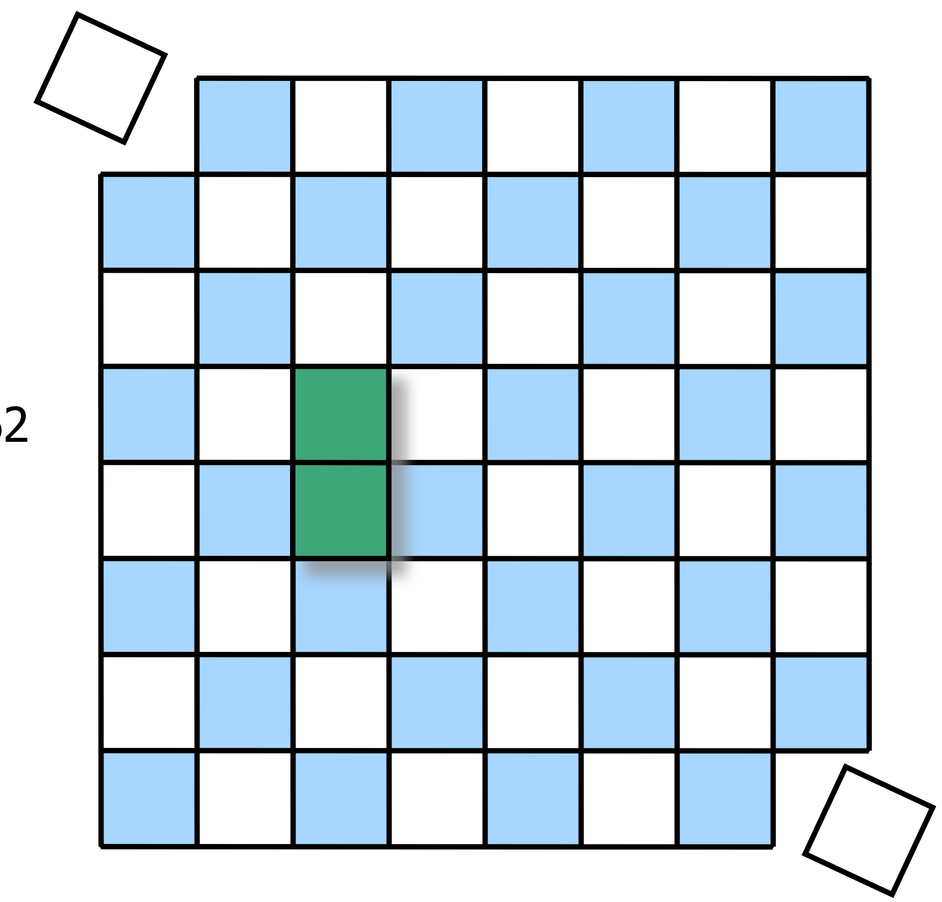
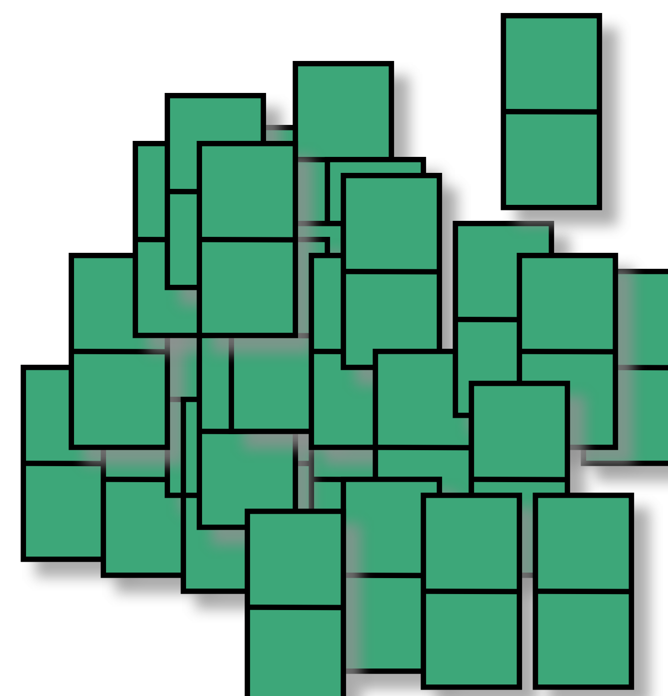


Puzzle time: tiling a chessboard with dominoes

64-2 = 62
cells



31 dominoes

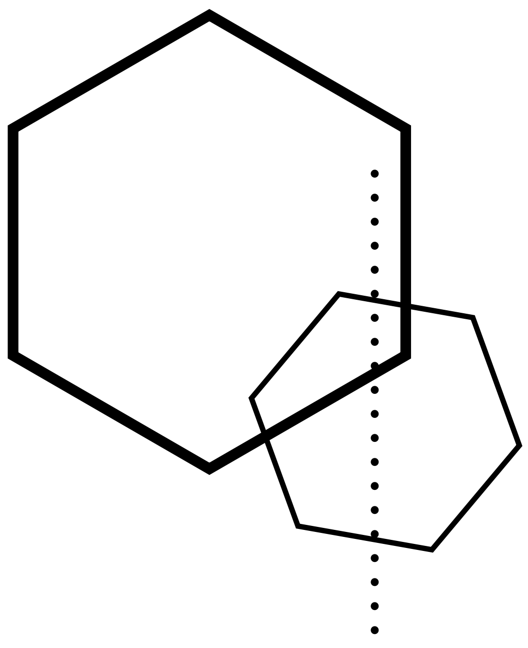


Invariant

An invariant of a dynamic system is an assertion that holds at every reachable state

Example

You have a polygon



You can rotate it

You can move it

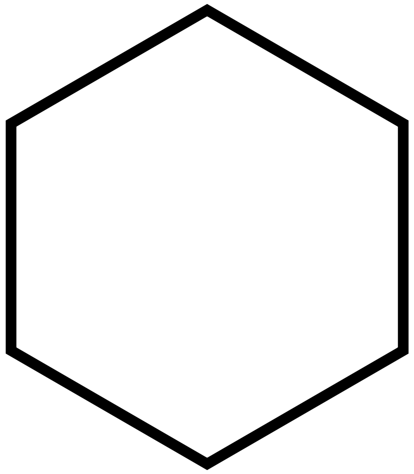
You can scale it

You can mirror it

Which invariants?

Example

You have a polygon



You can rotate it

You can move it

You can scale it

You can mirror it

Which invariants? perimeter

Example

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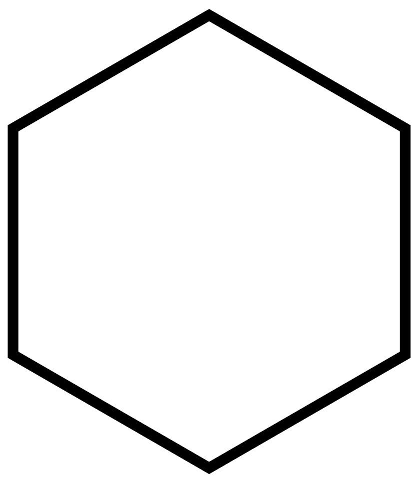


You can rotate it

You can move it

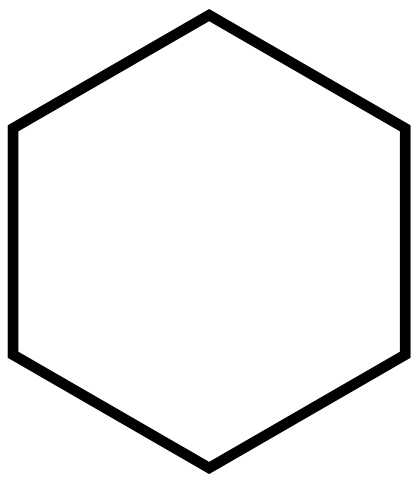
You can scale it

You can mirror it

Which invariants? area

Example

You have a polygon



You can rotate it

You can move it

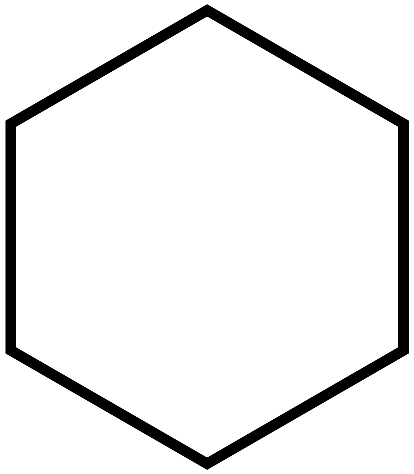
You can scale it

You can mirror it

Which invariants? number of vertices

Example

You have a polygon



You can rotate it

You can move it

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Which invariants? number of sides

Example

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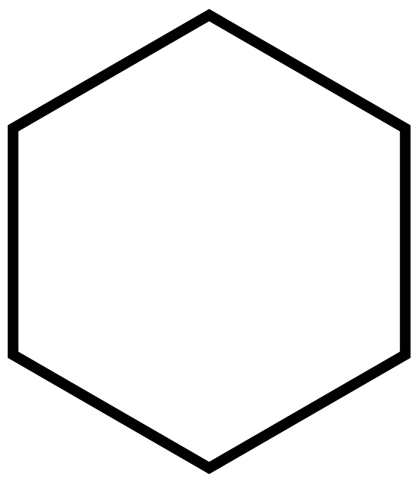

You can rotate it

You can move it

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You can mirror it

Which invariants? vertex degrees

Example

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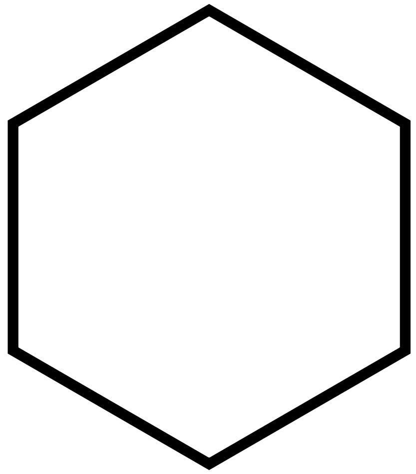

You can rotate it

You can move it

You can scale it

You can mirror it

Which invariants? convexity

Example

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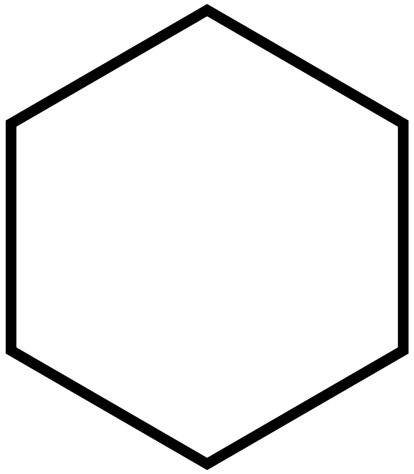

You can rotate it

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You can scale it

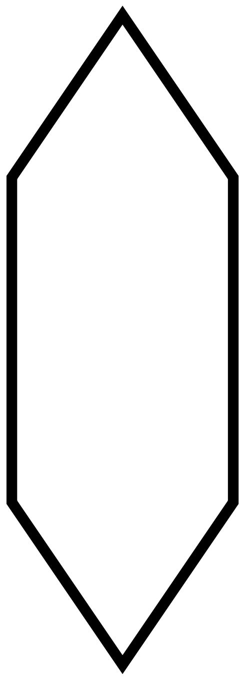
You can mirror it

Which invariants? color

Question time

You have a polygon

Which invariants?



You can rotate it

You can move it

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You can stretch it

color

convexity?

vertex degrees?

number of sides?

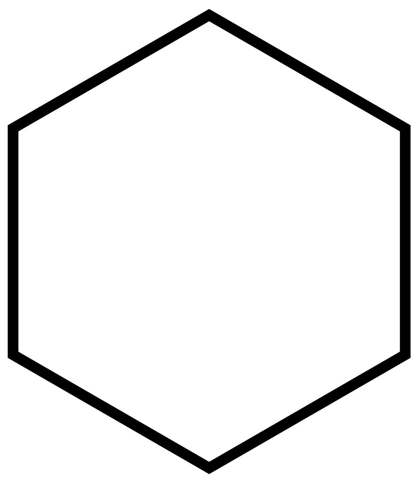
number of vertices?

area

perimeter

Question time

You have a polygon



You can rotate it

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Which invariants?

color

convexity

vertex degrees?

number of sides?

number of vertices?

area

perimeter

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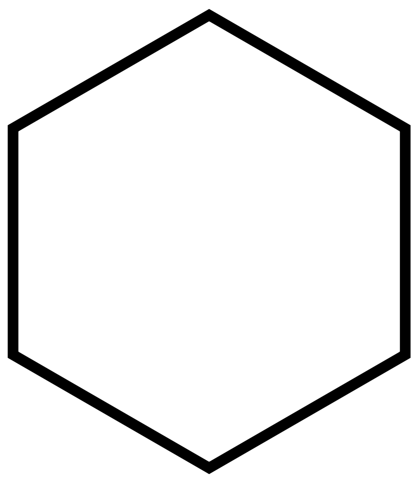

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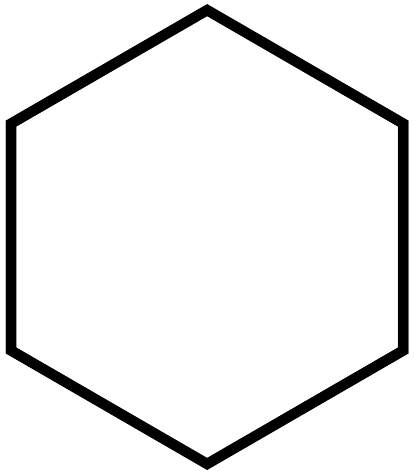

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number of vertices?

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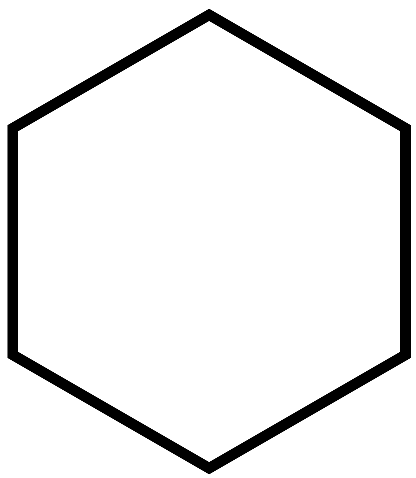

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Which invariants?

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vertex degrees

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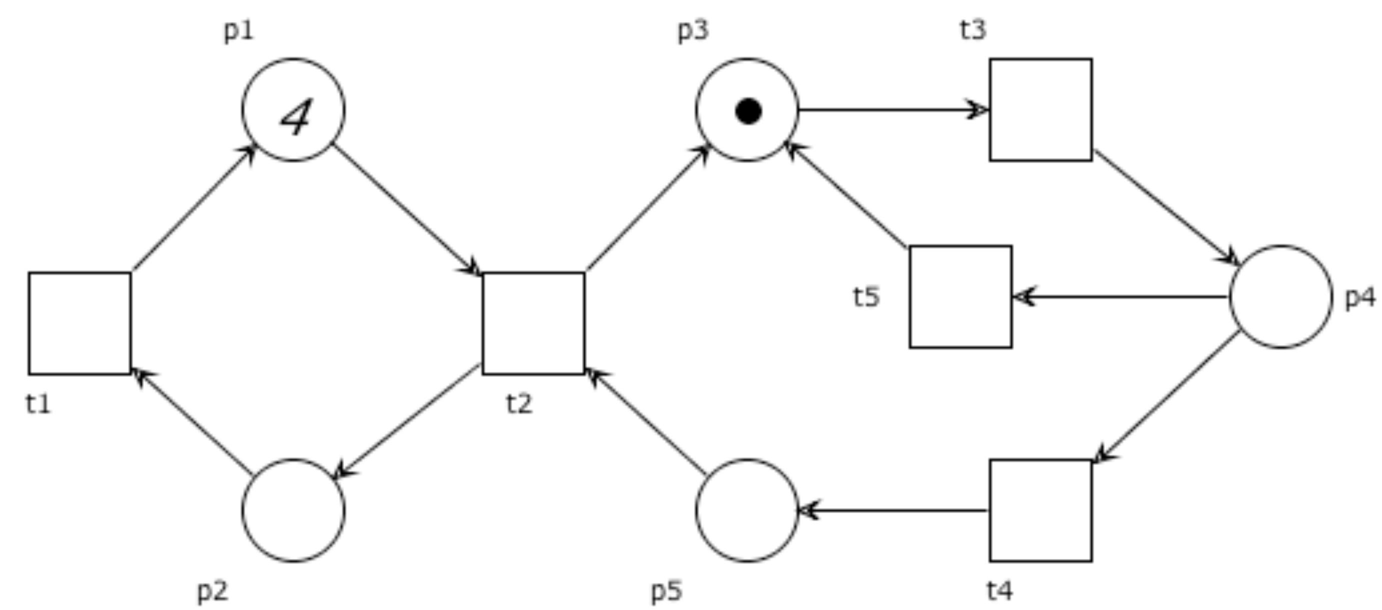
area

perimeter

Example

You have a Petri net

$$(P, T, F, M_0)$$



You can fire any currently enabled transition

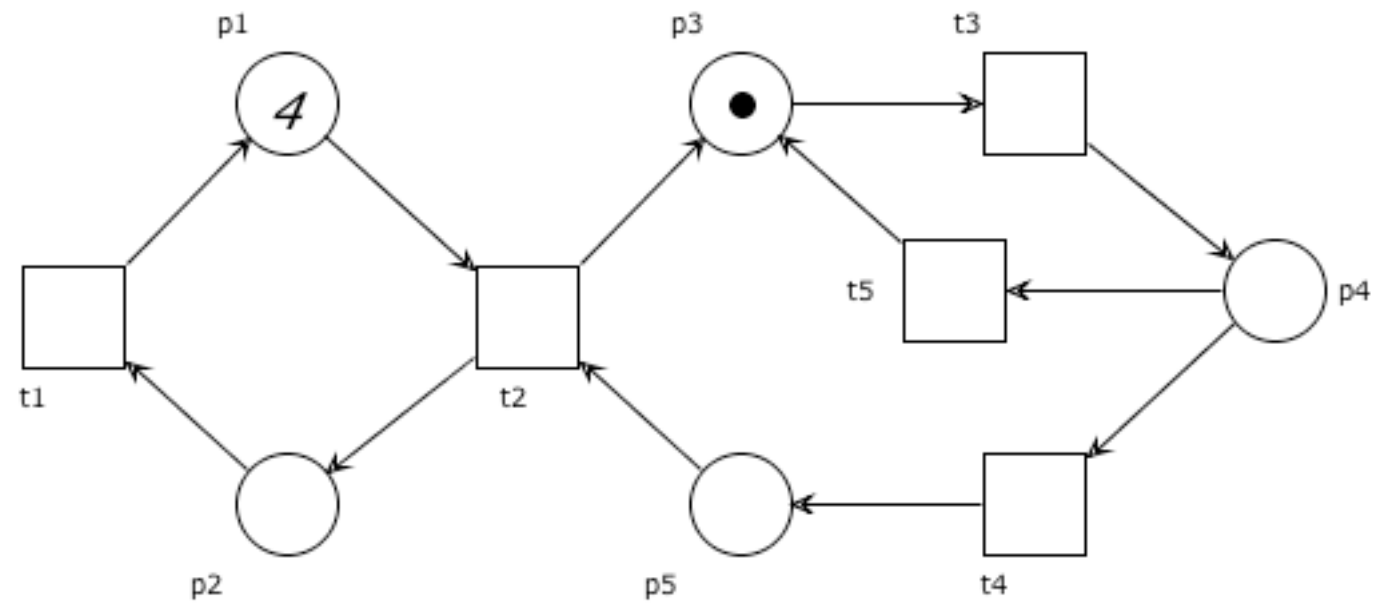
Which invariants?

Example

You have a Petri net

$$(P, T, F, M_0)$$

You can fire any currently enabled transition



Which invariants? color

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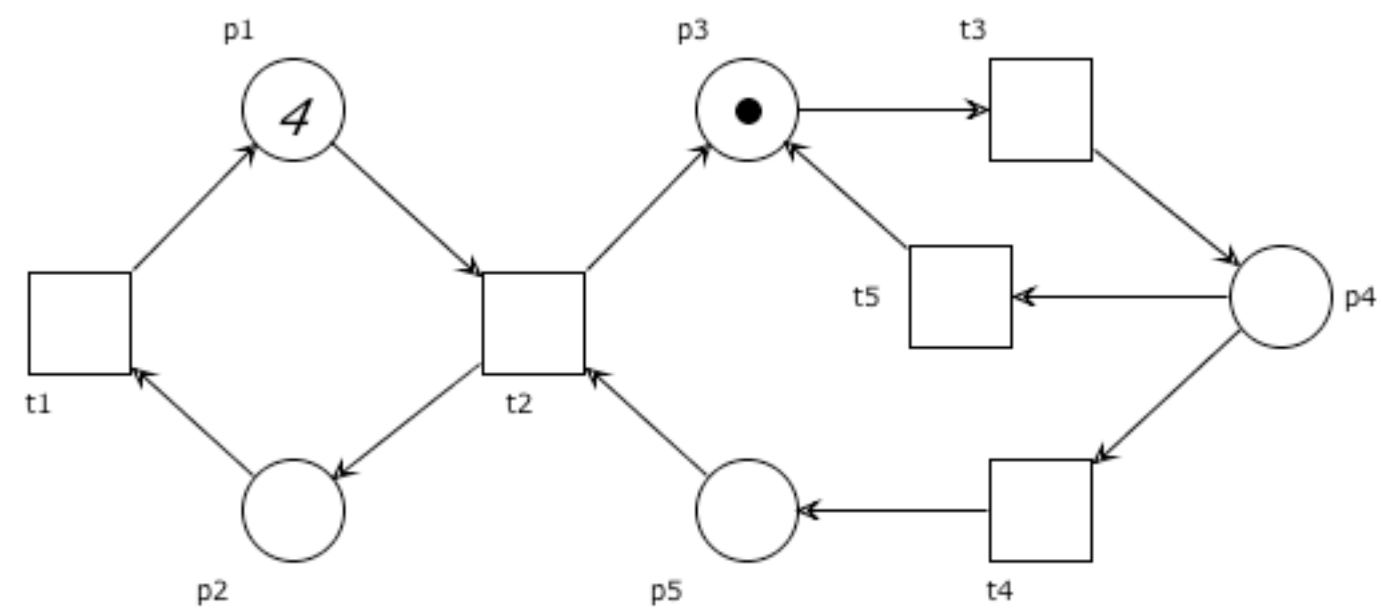

Which invariants? P, T, F

Example

You have a Petri net

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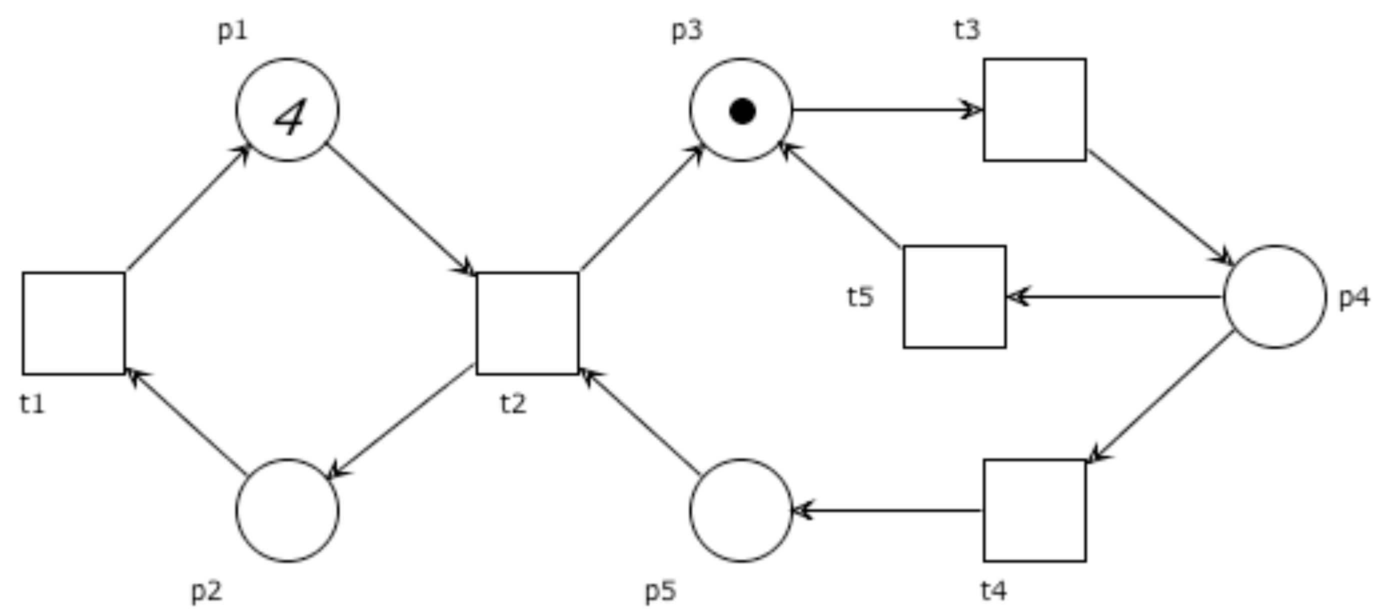


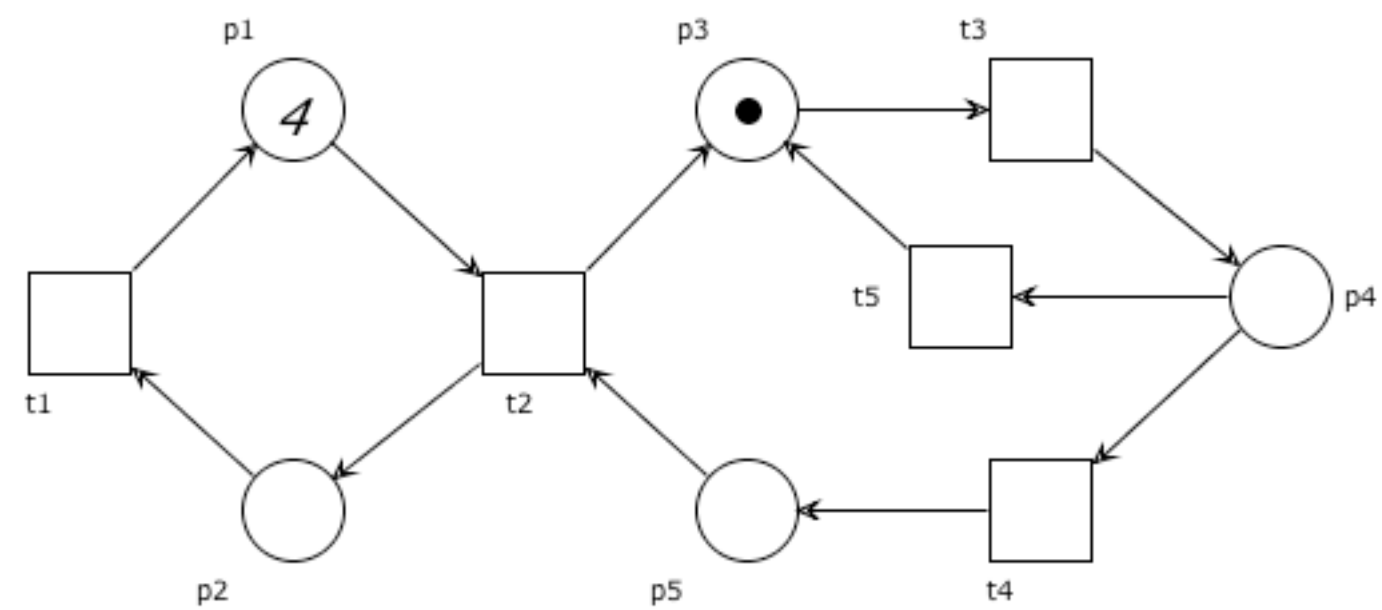
You can fire any currently enabled transition

Which invariants? number of tokens in p3

Example

You have a Petri net

$$(P, T, F, M_0)$$



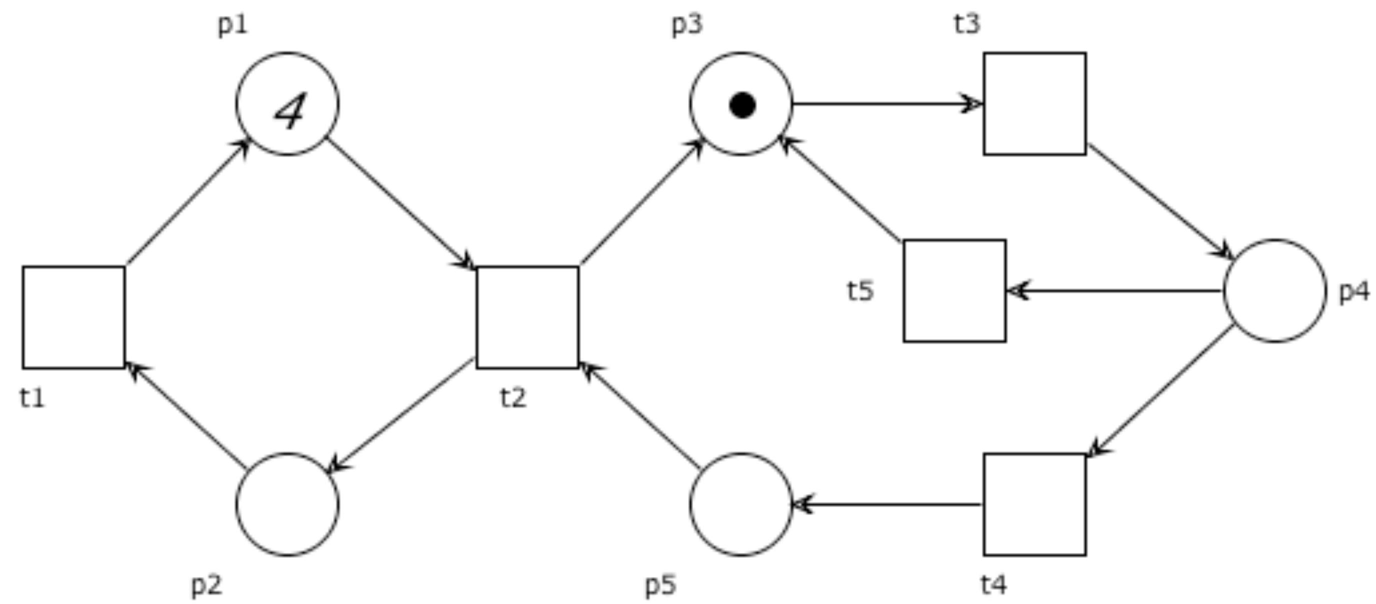
You can fire any currently enabled transition

Which invariants? number of tokens in a dead place

Example

You have a Petri net

$$(P, T, F, M_0)$$



You can fire any currently enabled transition

Which invariants?

Any property that holds for any reachable marking

Recall: Liveness, formally

$$(P, T, F, M_0)$$

$$\forall t \in T, \quad \forall M \in [\, M_0 \,\rangle, \quad \exists M' \in [\, M \,\rangle, \quad M' \xrightarrow{t}$$

Liveness as invariant

Lemma
If (P, T, F, M_0) is live and $M \in [\, M_0 \,\rangle$, then (P, T, F, M) is live.

Let $t \in T$ and $M' \in [\, M \,\rangle$.

Since $M \in [\, M_0 \,\rangle$, then $M' \in [\, M_0 \,\rangle$.

Since (P, T, F, M_0) is live, $\exists M'' \in [\, M' \,\rangle$ with $M'' \xrightarrow{t}$.

Therefore (P, T, F, M) is live.

Recall: Deadlock freedom, formally

$$(P, T, F, M_0)$$

$$\forall M \in [\, M_0 \,\rangle, \quad \exists t \in T, \quad M \xrightarrow{t}$$

Deadlock freedom as invariant

Lemma: If (P, T, F, M_0) is deadlock-free and $M \in [\, M_0 \,\rangle$, then (P, T, F, M) is deadlock-free.

Let $M' \in [\, M \,\rangle$.

Since $M \in [\, M_0 \,\rangle$, then $M' \in [\, M_0 \,\rangle$.

Since (P, T, F, M_0) is deadlock-free, $\exists t \in T$ with $M' \xrightarrow{t}$.

Therefore (P, T, F, M) is deadlock-free.

Boundedness, formally

$$(P, T, F, M_0)$$

$$\exists k \in \mathbb{N}, \quad \forall M \in [\, M_0 \,\rangle, \quad \forall p \in P, \quad M(p) \leq k$$

Boundedness as invariant

Lemma
If (P, T, F, M_0) is bounded and $M \in [\, M_0 \,\rangle$, then (P, T, F, M) is bounded.

Since (P, T, F, M_0) is bounded, it must be k-bounded for some $k \in \mathbb{N}$

Let $M' \in [\, M \,\rangle$.

Since $M \in [\, M_0 \,\rangle$, then $M' \in [\, M_0 \,\rangle$.

Since (P, T, F, M_0) is k-bounded, $M'(p) \leq k$ for all $p \in P$.

Therefore (P, T, F, M) is (k-)bounded.

Cyclicity, formally

Definition:
A net system (P, T, F, M_0) is **cyclic** if $\forall M \in [\, M_0 \,\rangle .\, M_0 \in [\, M \,\rangle$.

Cyclicity as invariant

Lemma
If (P, T, F, M_0) is cyclic and $M \in [\, M_0 \,\rangle$, then (P, T, F, M) is cyclic.

Let $M' \in [\, M \,\rangle$, then $M' \in [\, M_0 \,\rangle$.

Since (P, T, F, M_0) is cyclic, there exists σ' such that $M' \xrightarrow{\sigma'} M_0$.

Since $M \in [\, M_0 \,\rangle$, there exists σ such that $M_0 \xrightarrow{\sigma} M$.

Then $M' \xrightarrow{\sigma' \sigma} M$, hence (P, T, F, M) is cyclic.

Structural invariants

In the case of Petri nets, it is possible to compute certain vectors of **rational** numbers(*) (directly from the structure of the net) (independently from the initial marking) which induce nice invariants, called

S-invariants

T-invariants

(*) it is not necessary to consider real-valued solutions, because incidence matrices only have integer entries

Why invariants?

Can be calculated efficiently
(polynomial time for a basis)

Independent of initial marking

Structural property with behavioural consequences

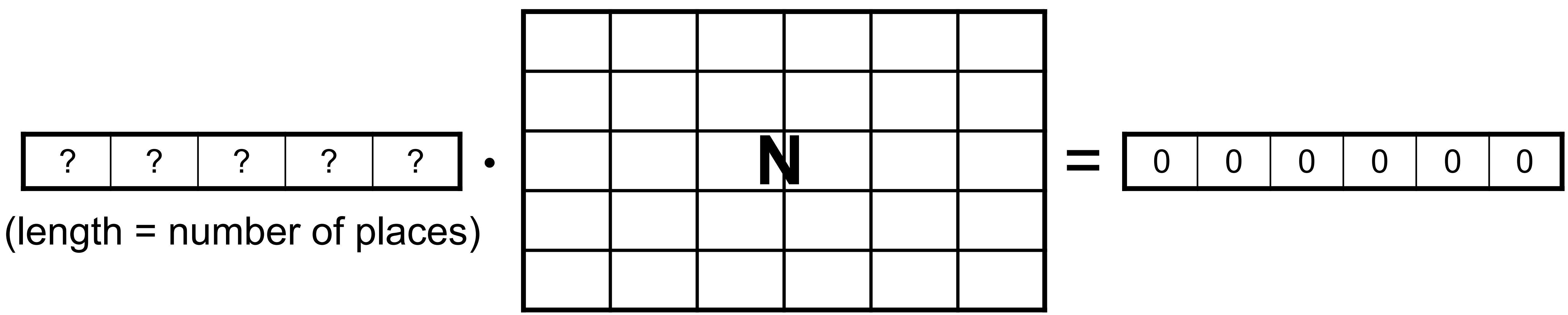
However, the main reason is didactical!
You only truly understand a model if you think
about it in terms of invariants!

S-invariants

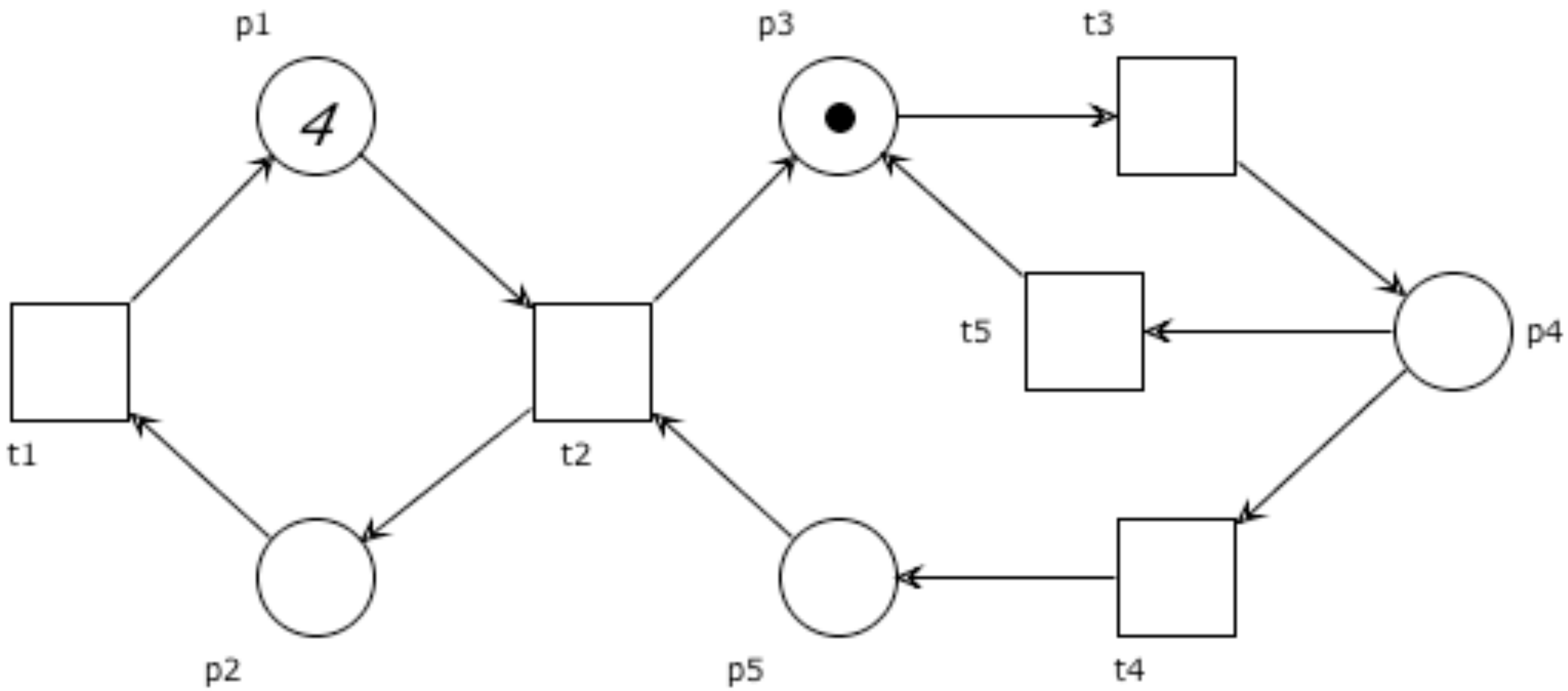
S-invariant
(aka place-invariant)

Definition: An **S-invariant** of a net N=(P,T,F) is a rational-valued solution **x** of the equation

$$\mathbf{x} \cdot \mathbf{N} = \mathbf{0}$$

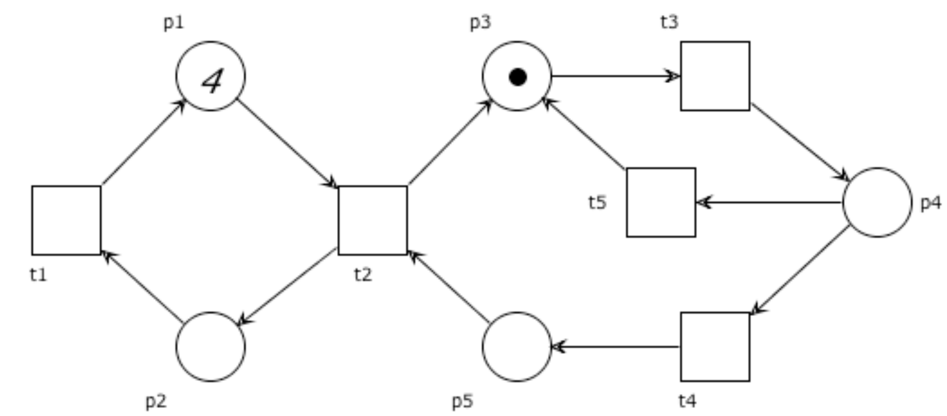


Example



Find some/all S-invariants for the net above

Example



Find some/all S-invariants for the net above

$$\overset{\mathbf{x}}{\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}} \cdot \overset{N}{\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix}} = \mathbf{0}$$

$$\begin{cases} x_1 - x_2 = 0 \\ -x_1 + x_2 + x_3 - x_5 = 0 \\ -x_3 + x_4 = 0 \\ -x_4 + x_5 = 0 \\ x_3 - x_4 = 0 \end{cases}$$

$$x_1 = x_2$$

$$x_3 = x_4$$

$$x_4 = x_5$$

$$\begin{bmatrix} n & n & m & m & m \end{bmatrix}$$

Homogeneous systems of linear equations

$$\begin{cases} a_{1,1}x_1 & + & a_{1,2}x_2 & + & a_{1,n}x_n & = & 0 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & a_{2,n}x_n & = & 0 \\ \cdots \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + & a_{m,n}x_n & = & 0 \end{cases}$$

where $x_1, x_2, \ldots, x_n$ are the "unknowns"

trivial solution: $x_1 = x_2 = \ldots = x_n = 0$

if $\mathbf{x}$ and $\mathbf{x}'$ are solutions, then $\mathbf{x} + \mathbf{x}'$ is a solution

if $\mathbf{x}$ is a solution, then $k\mathbf{x}$ is a solution

Linear combination

Proposition:
Any linear combination of S-invariants is an S-invariant

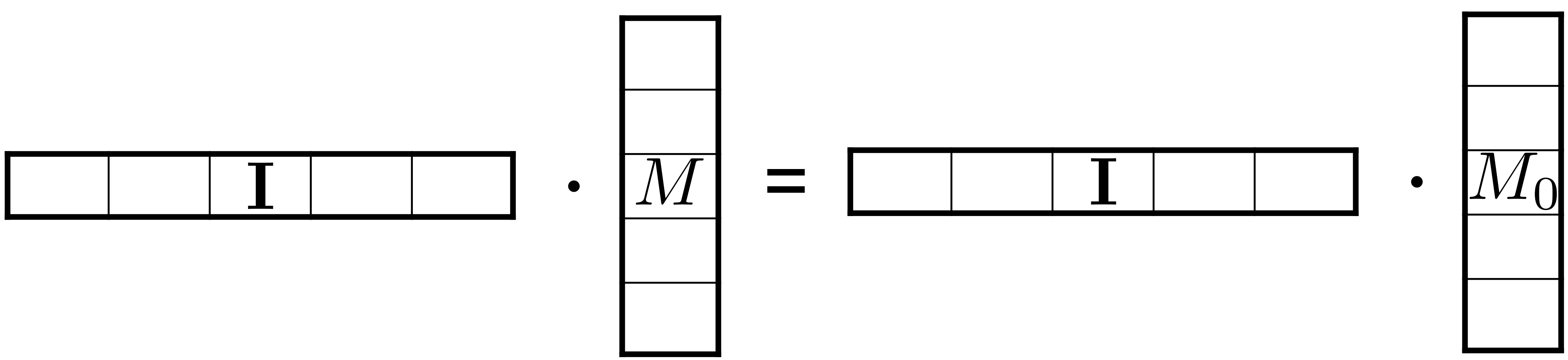
Take any two S-Invariants $\mathbf{I}_1$ and $\mathbf{I}_2$ and any two values k_1, k_2.
We want to prove that $k_1\,\mathbf{I}_1 + k_2\,\mathbf{I}_2$ is an S-invariant.

$$\begin{aligned}(k_1\,\mathbf{I}_1 + k_2\,\mathbf{I}_2)\cdot\mathbf{N} &= k_1\,\mathbf{I}_1\cdot\mathbf{N} + k_2\,\mathbf{I}_2\cdot\mathbf{N}\\ &= k_1\,\mathbf{0} + k_2\,\mathbf{0}\\ &= \mathbf{0}\end{aligned}$$

Fundamental property of S-invariants

Proposition: Let $\mathbf{I}$ be an invariant of N.

For any $M \in [\, M_0 \,\rangle$ we have $\mathbf{I} \cdot M = \mathbf{I} \cdot M_0$



Fundamental property of S-invariants

Proposition: Let $\mathbf{I}$ be an invariant of N.

For any $M \in [\, M_0 \,\rangle$ we have $\mathbf{I} \cdot M = \mathbf{I} \cdot M_0$

Since $M \in [\, M_0 \,\rangle$, there is σ s.t. $M_0 \xrightarrow{\sigma} M$
By the marking equation: $M = M_0 + \mathbf{N} \cdot \vec{\sigma}$

$$\text{Therefore:} \quad \begin{aligned} \mathbf{I} \cdot M &= \mathbf{I} \cdot (M_0 + \mathbf{N} \cdot \vec{\sigma}) \\ &= \mathbf{I} \cdot M_0 + \mathbf{I} \cdot \mathbf{N} \cdot \vec{\sigma} \\ &= \mathbf{I} \cdot M_0 + \mathbf{0} \cdot \vec{\sigma} \\ &= \mathbf{I} \cdot M_0 \end{aligned}$$

A necessary condition for reachability

reachability problem: is M reachable from M_0? $M \stackrel{?}{\in} [M_0\rangle$

decidable, but computationally expensive (EXPSPACE-hard)

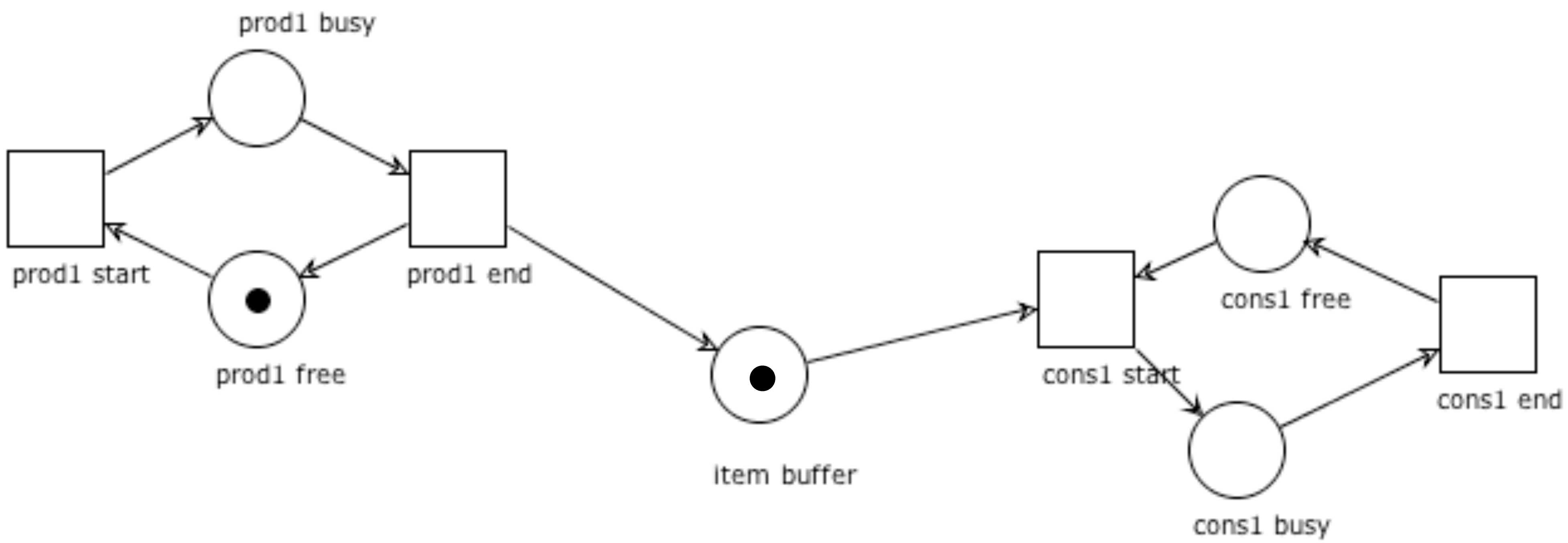
S-invariants provide a preliminary check that can be computed more efficiently

Let (P, T, F, M_0) be a system.

If there is an S-invariant $\mathbf{I}$ s.t. $\mathbf{I} \cdot M \neq \mathbf{I} \cdot M_0$ then $M \notin [\, M_0 \,\rangle$

Example

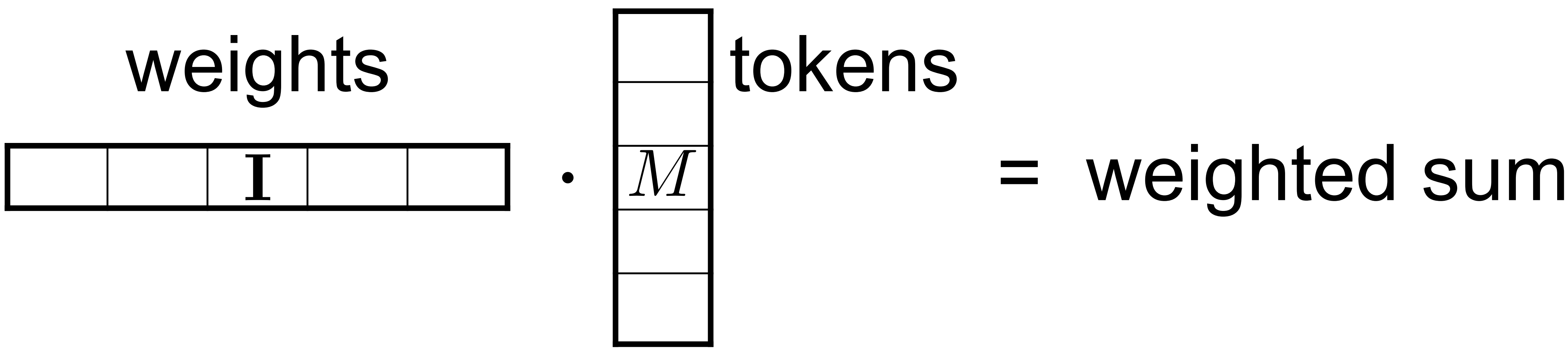
Prove that the marking
M = prod1free + cons1busy
is not reachable



$\mathbf{I} = [\ 0\ 0\ 0\ 1\ 1\ ]$

$\mathbf{I} \cdot M_0 = 0 \neq 1 = \mathbf{I} \cdot M$

Place-invariant, intuitively



Place-invariant, intuitively

A place-invariant assigns a **weight to each place** such that the weighted token sum remains constant during any computation

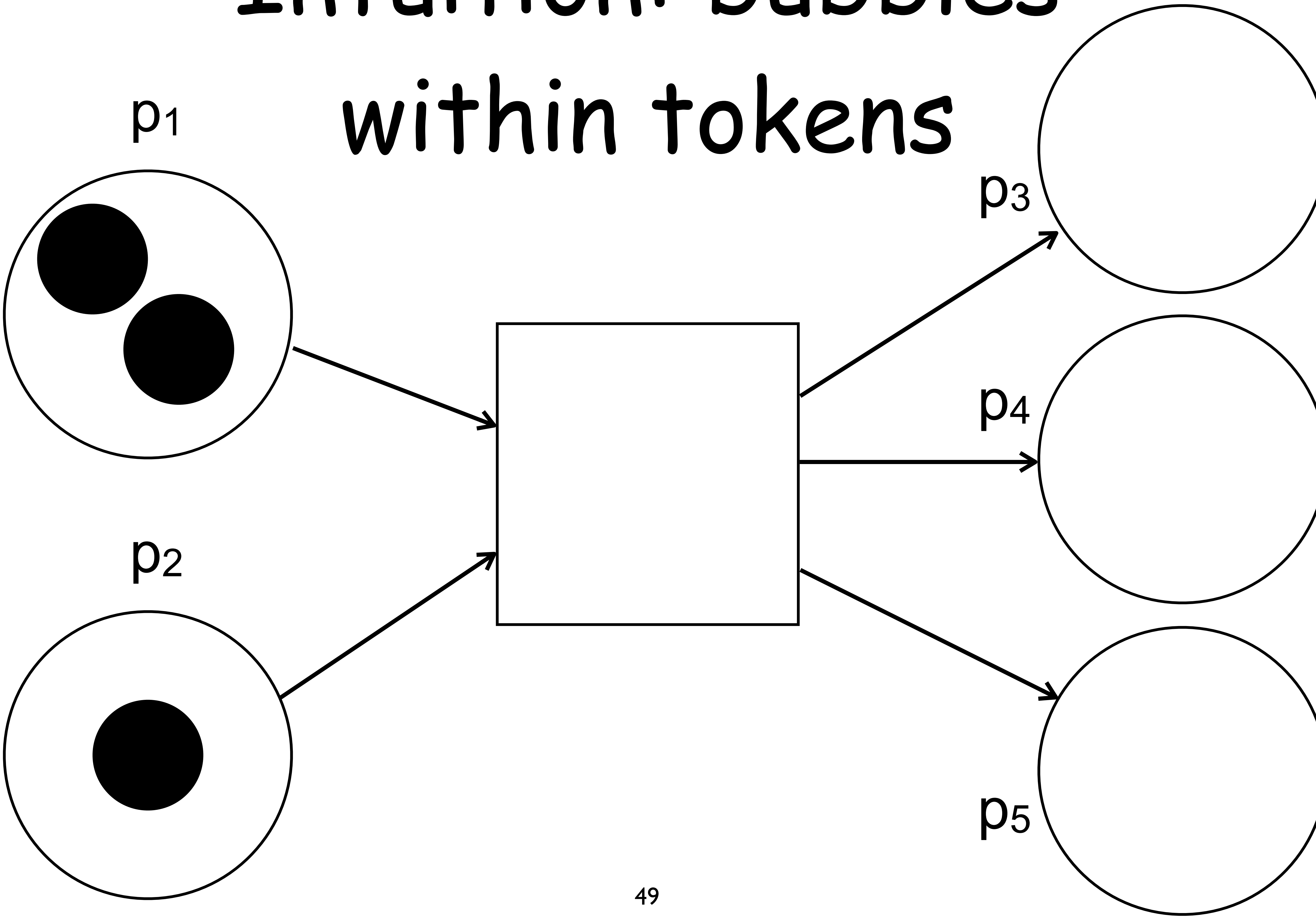
For example, you can imagine that tokens are coins,
places are the different kinds of available coins,
the S-invariant assigns a value to each coin:
the value of a marking is the sum of the values of the tokens/coins in it and it is not changed by firings

Place-invariant, intuitively

A place-invariant assigns a **weight to each place** such that the weighted token sum remains constant during any computation

For example, you can imagine that tokens are molecules,
places are different kinds of molecules,
the S-invariant assigns the number of atoms needed to form each molecule:
the overall number of atoms is not changed by firings

Intuition: bubbles within tokens



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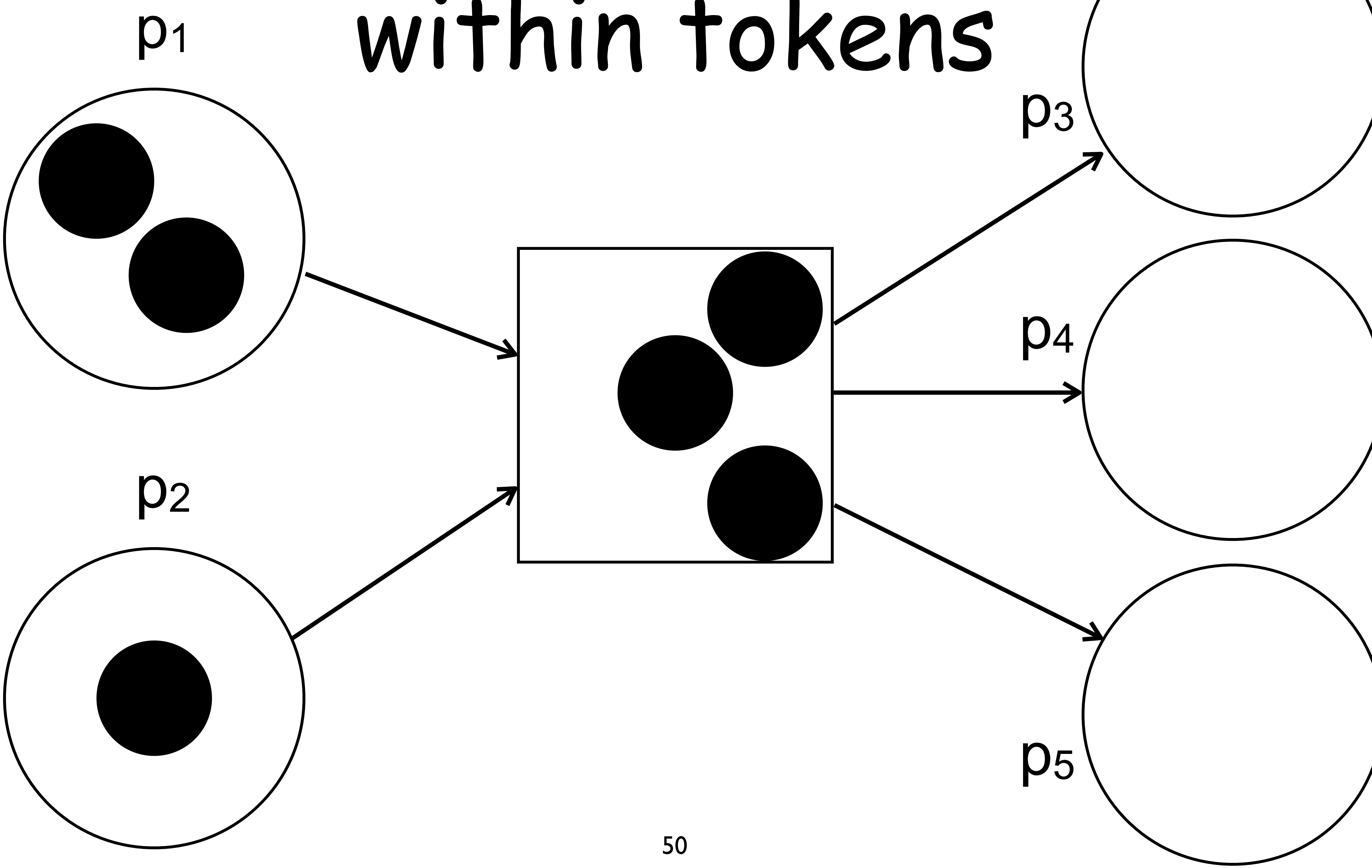

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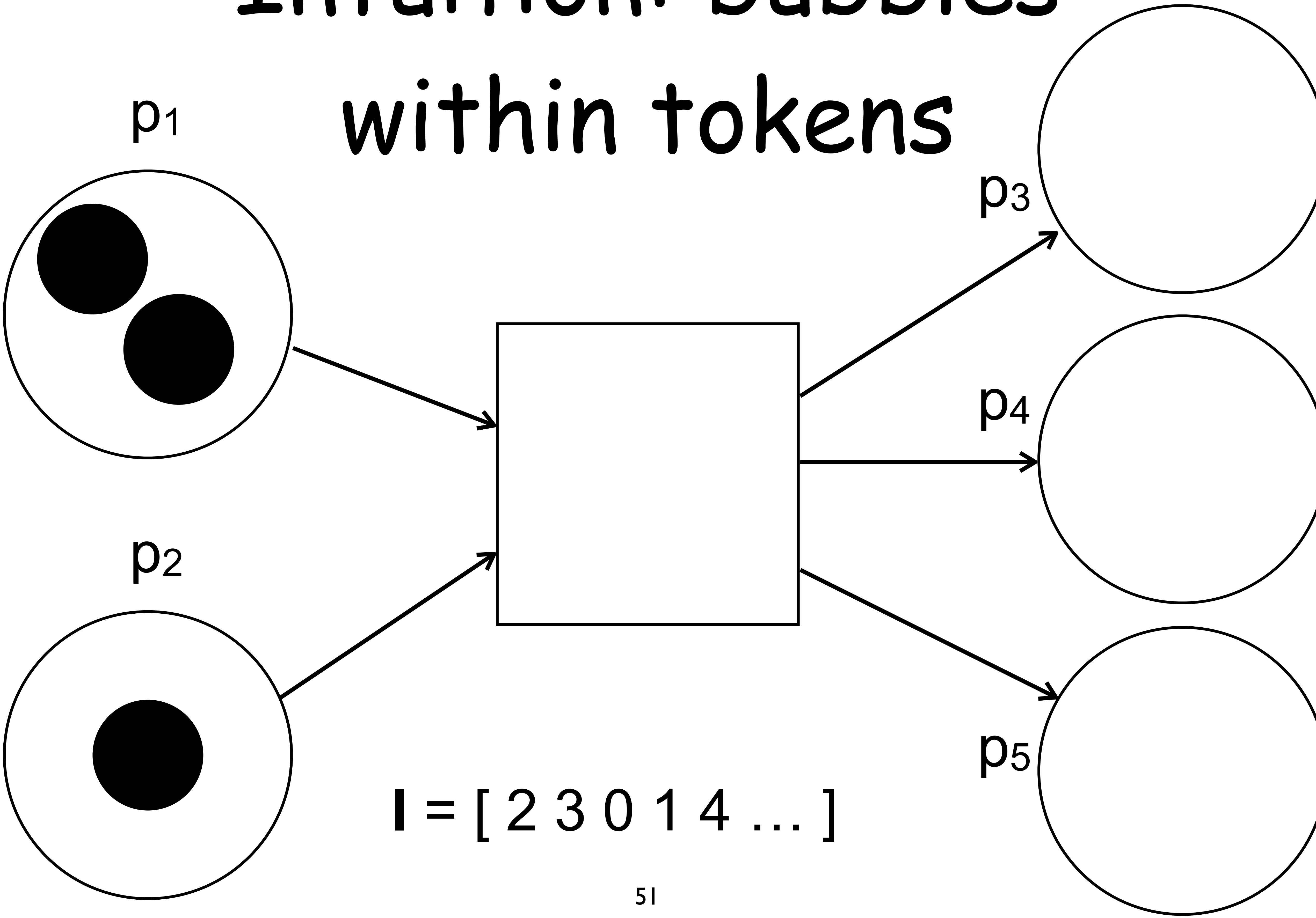
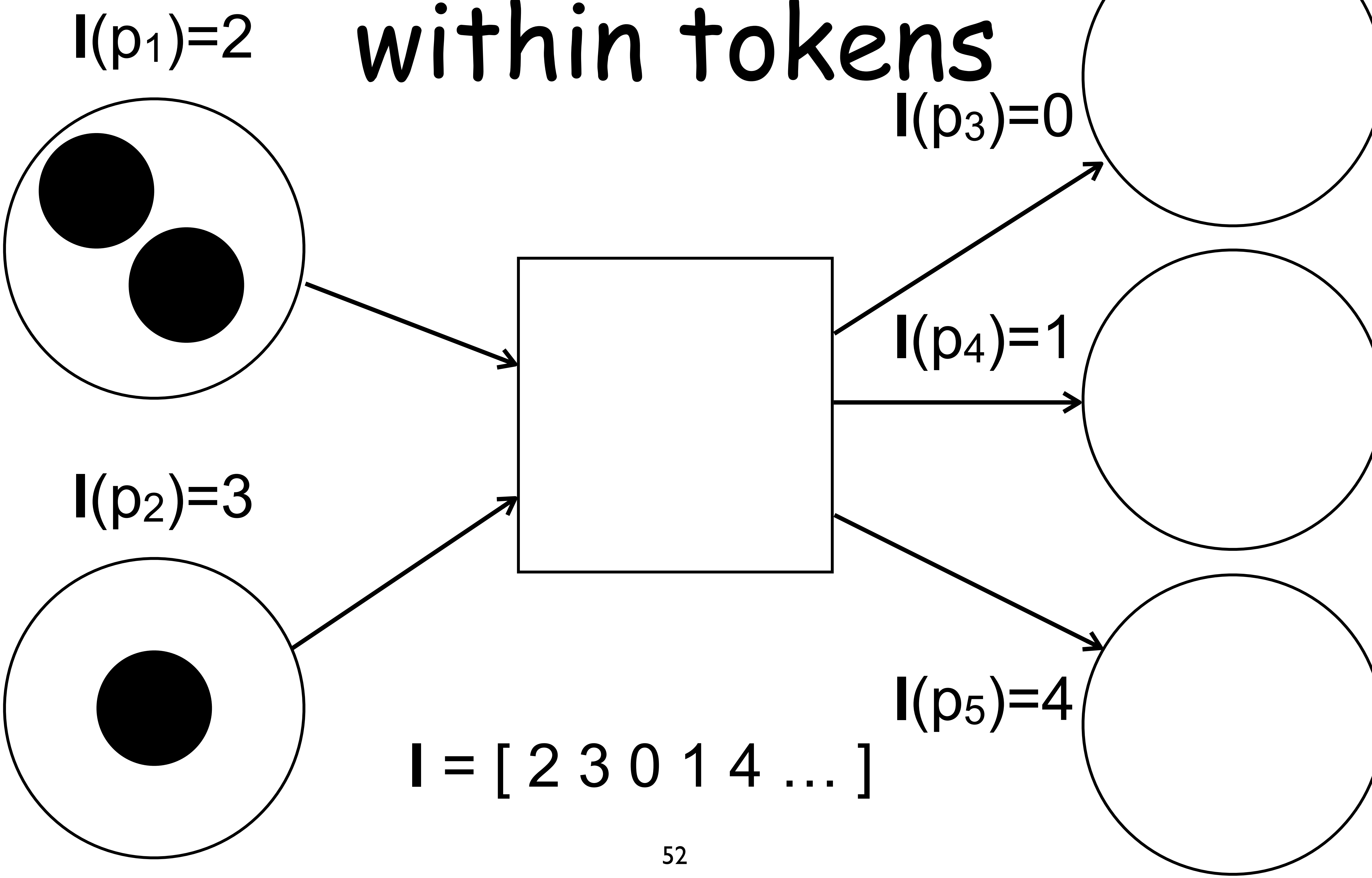


I = [2 3 0 1 4 …]

Intuition: bubbles within tokens



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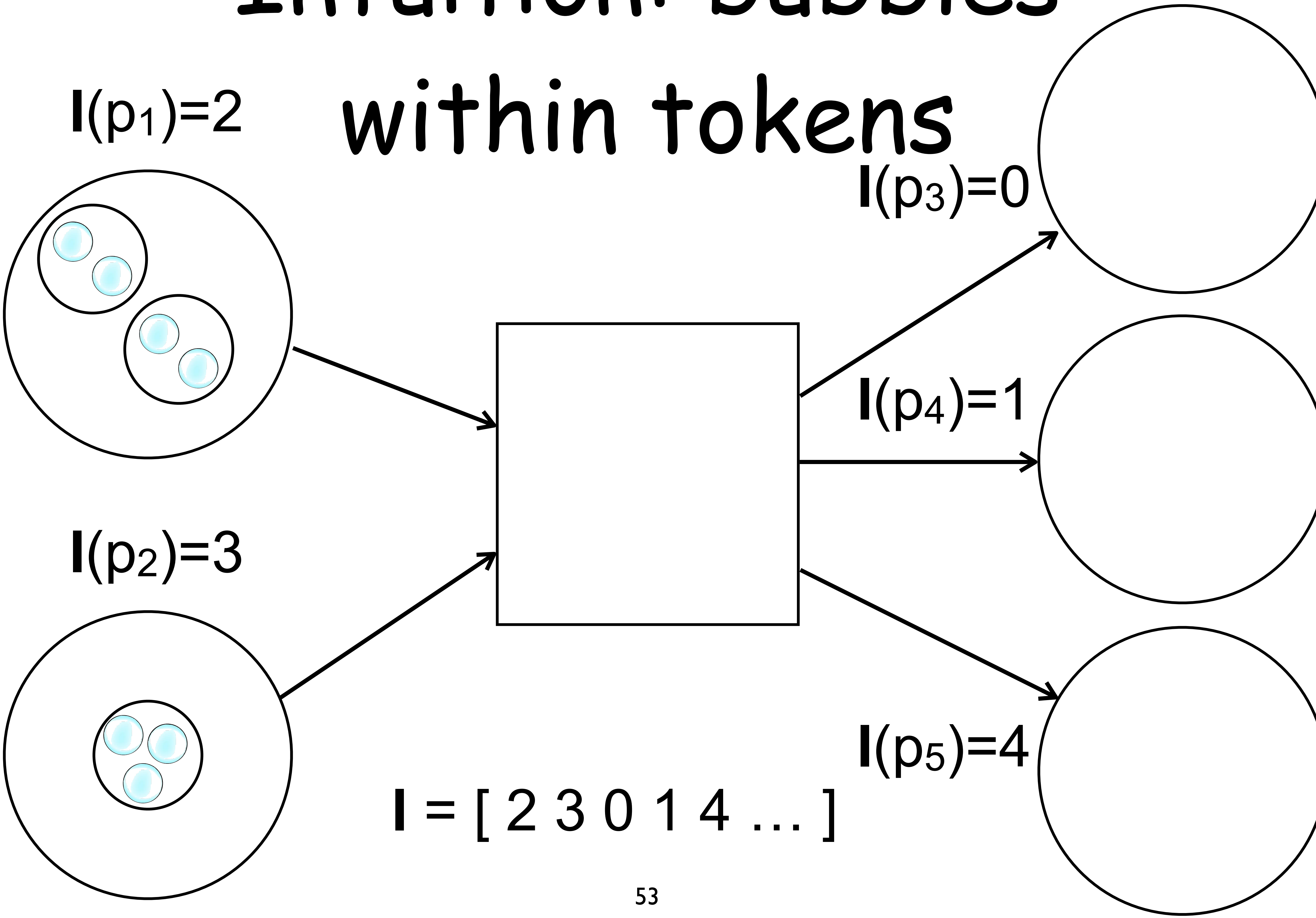

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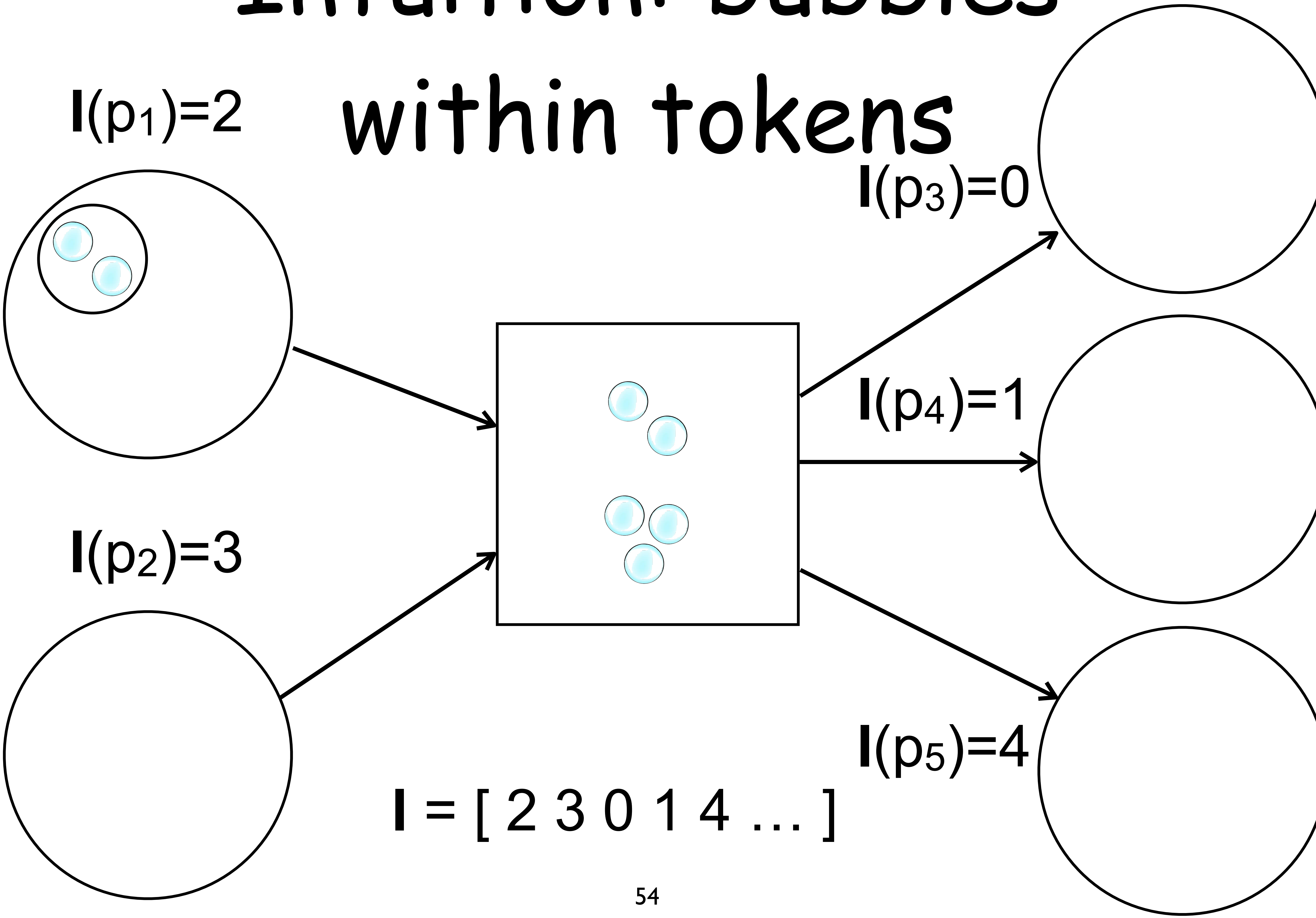

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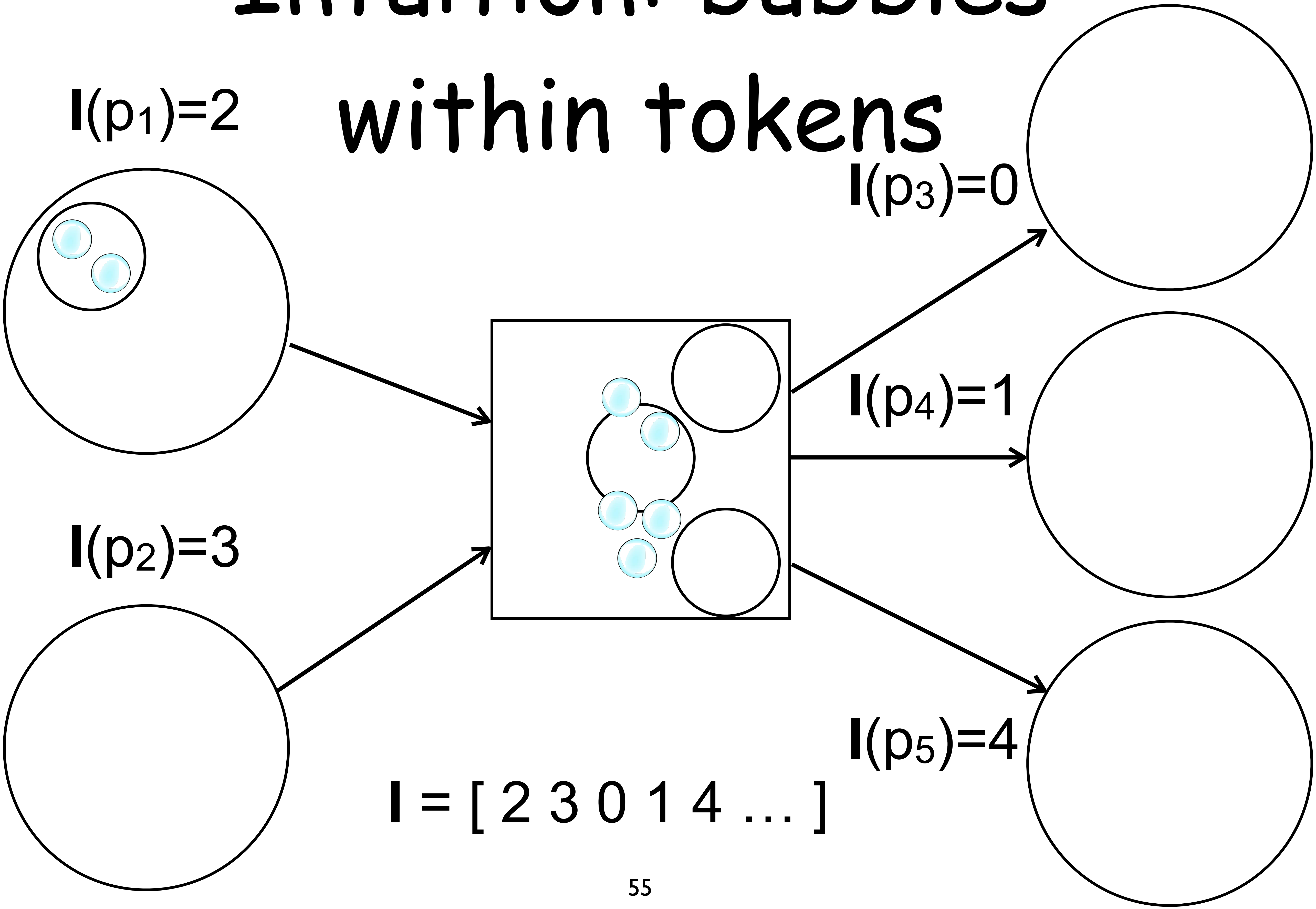

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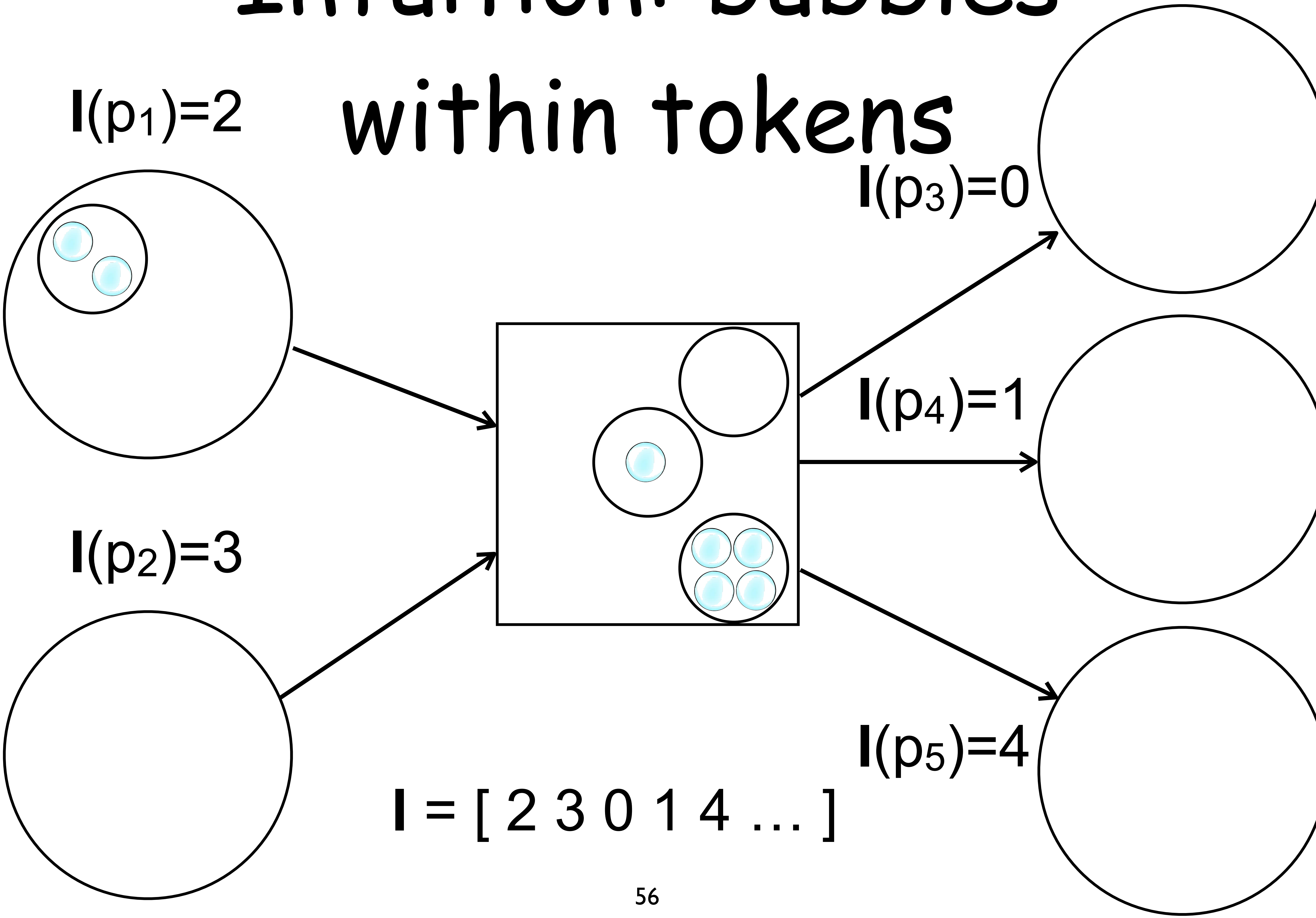

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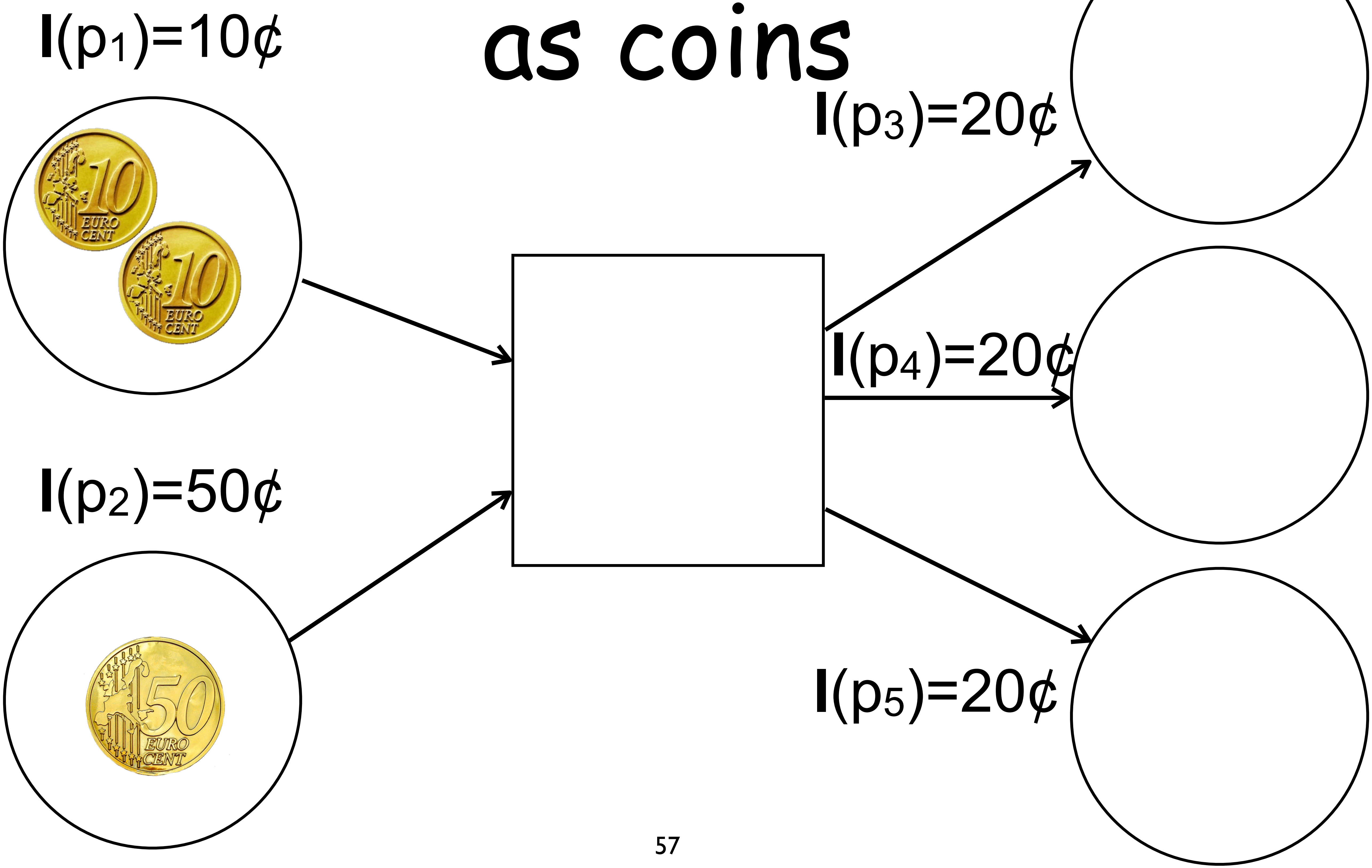
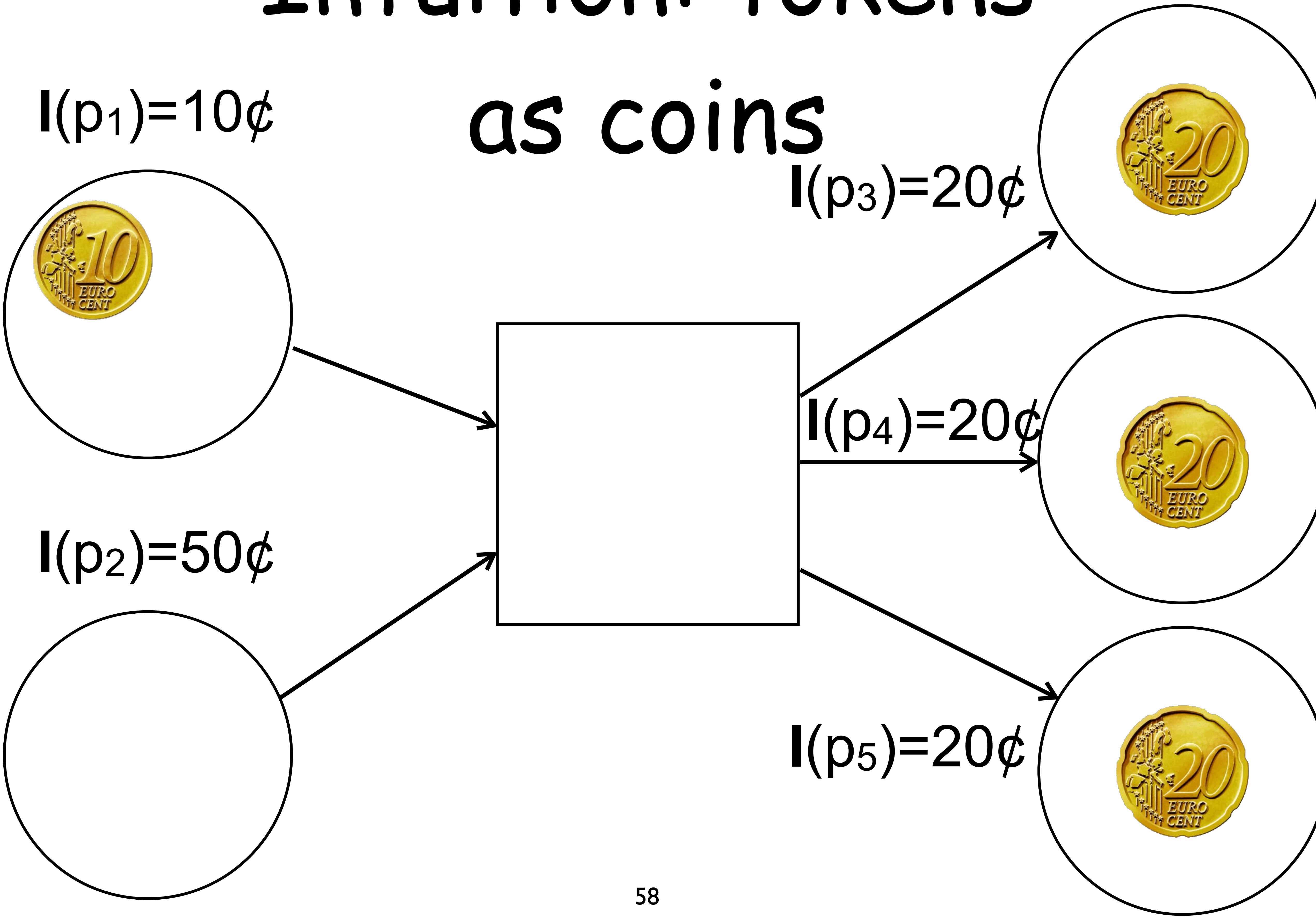
Intuition: tokens as coins



Intuition: tokens as coins



Alternative definition of S-invariant

Proposition:

A mapping $\mathbf{I} : P \to \mathbb{Q}$ is an S-invariant of N iff for any $t \in T$:

$$\sum_{p \in \bullet t} \mathbf{I}(p) = \sum_{p \in t \bullet} \mathbf{I}(p)$$

Flow-in Flow-out

Consequence of alternative definition

Very useful in proving S-invariance!

The check is possible without constructing the incidence matrix

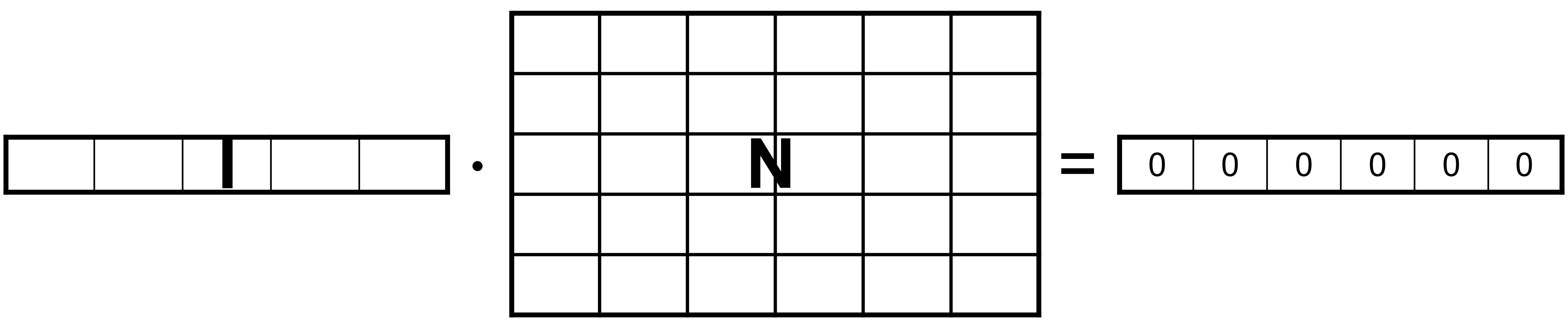
It can also help to build S-invariants directly over the picture

Exercise

Prove the proposition about the alternative characterization of S-invariants

A mapping $\mathbf{I} : P \to \mathbb{Q}$ is an S-invariant of N iff for any $t \in T$:

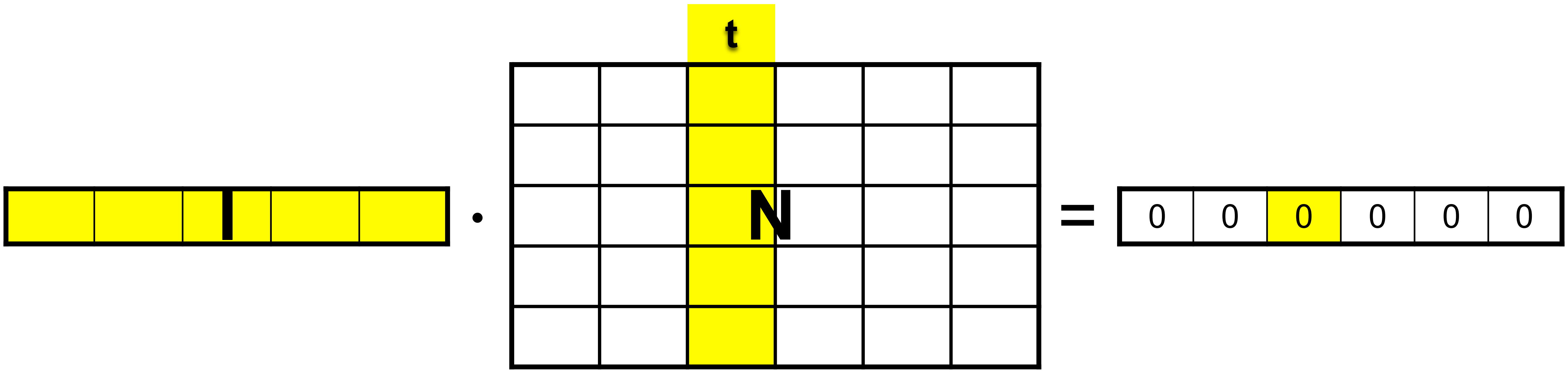
$$\sum_{p \in \bullet t} \mathbf{I}(p) = \sum_{p \in t \bullet} \mathbf{I}(p)$$



Exercise

Prove the proposition about the alternative characterization of S-invariants

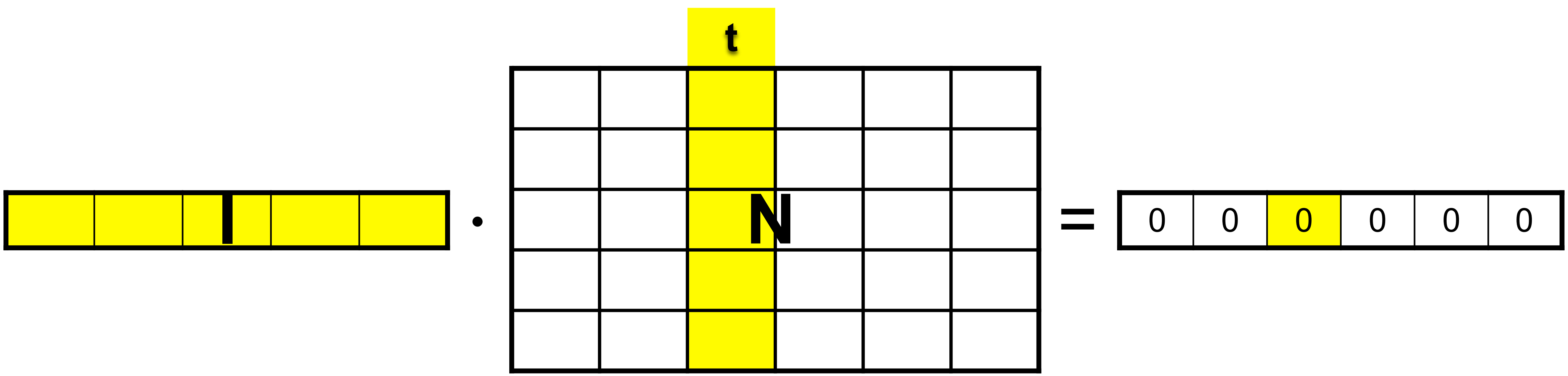
$\forall t \quad \mathbf{I} \cdot \mathbf{t} = 0$



Exercise

Prove the proposition about the alternative characterization of S-invariants

$$\forall t \quad \mathbf{I} \cdot \mathbf{t} = \sum_{p \in P} \mathbf{I}(p) \cdot \mathbf{t}(p) = 0$$



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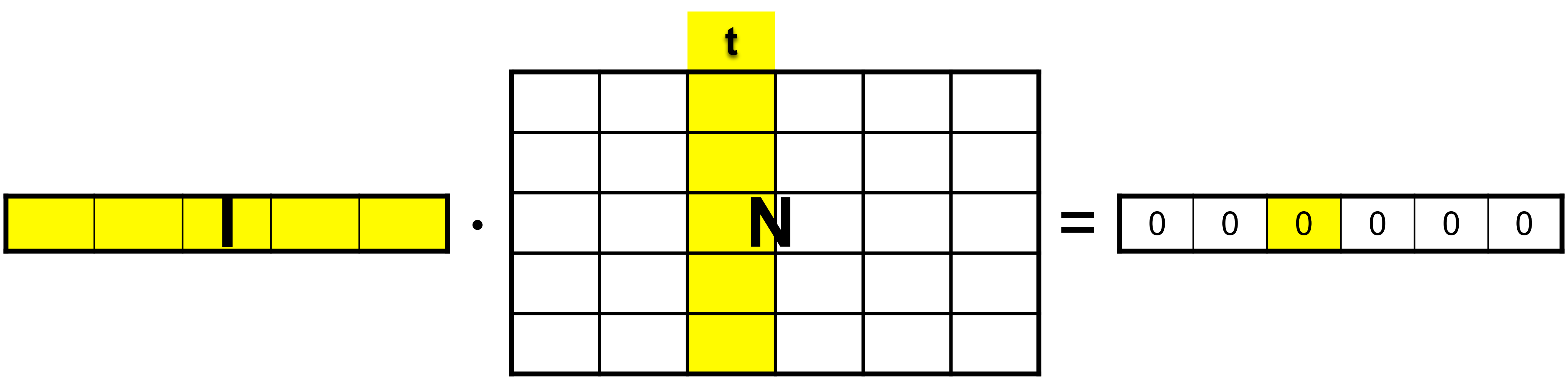
Prove the proposition about the alternative characterization of S-invariants

$$\forall t \quad \mathbf{I} \cdot \mathbf{t} = \sum_{p \in P} \mathbf{I}(p) \cdot \mathbf{t}(p) = \sum_{p \in P} \mathbf{I}(p) \cdot \mathbf{N}(p,t) = 0$$



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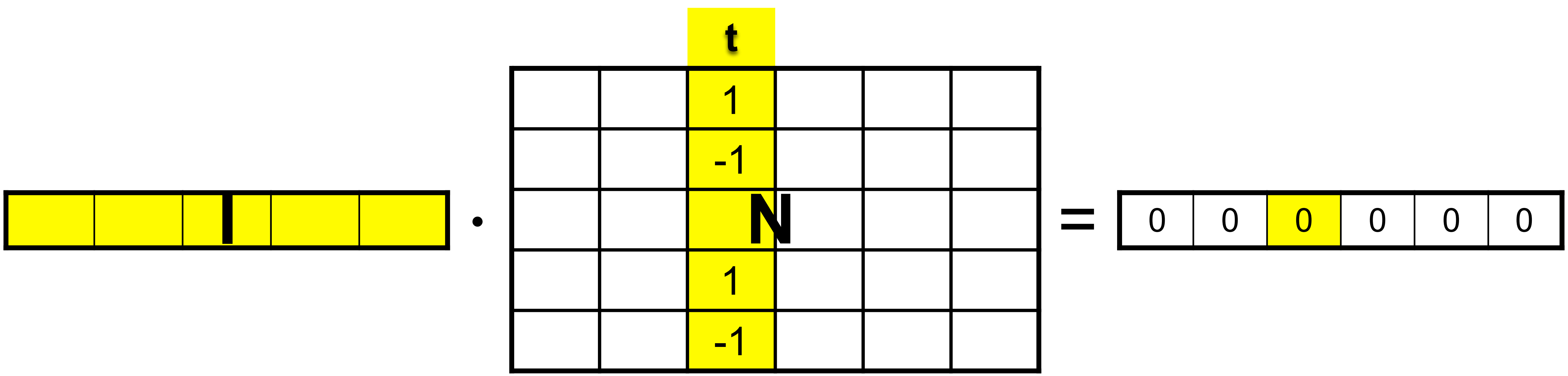
$$\forall t \quad \mathbf{I} \cdot \mathbf{t} = \sum_{p \in P} \mathbf{I}(p) \cdot \mathbf{t}(p) = \sum_{p \in P} \mathbf{I}(p) \cdot \mathbf{N}(p,t) = \sum_{p \in t\bullet} \mathbf{I}(p) - \sum_{p \in \bullet t} \mathbf{I}(p) = 0$$



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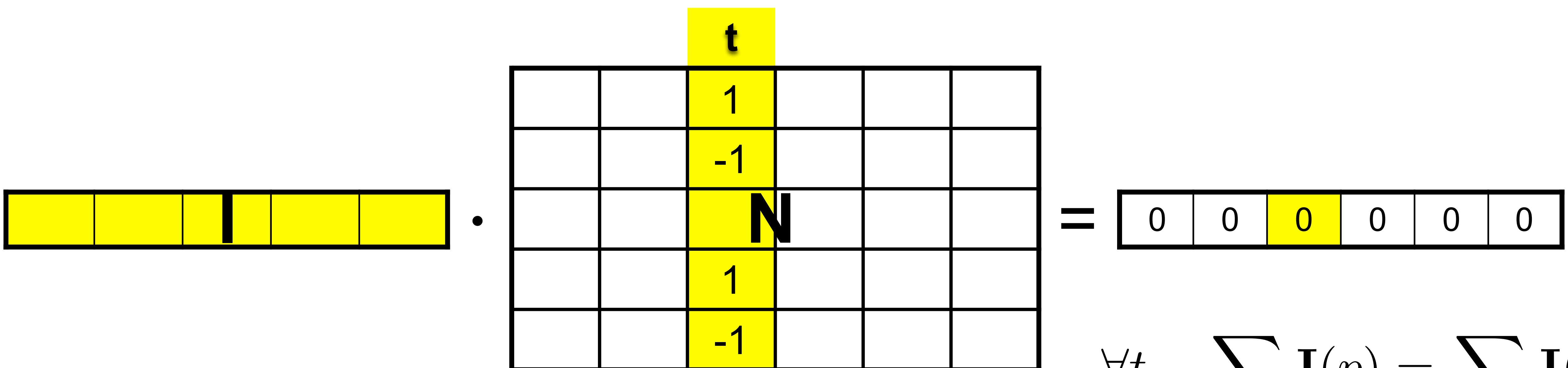
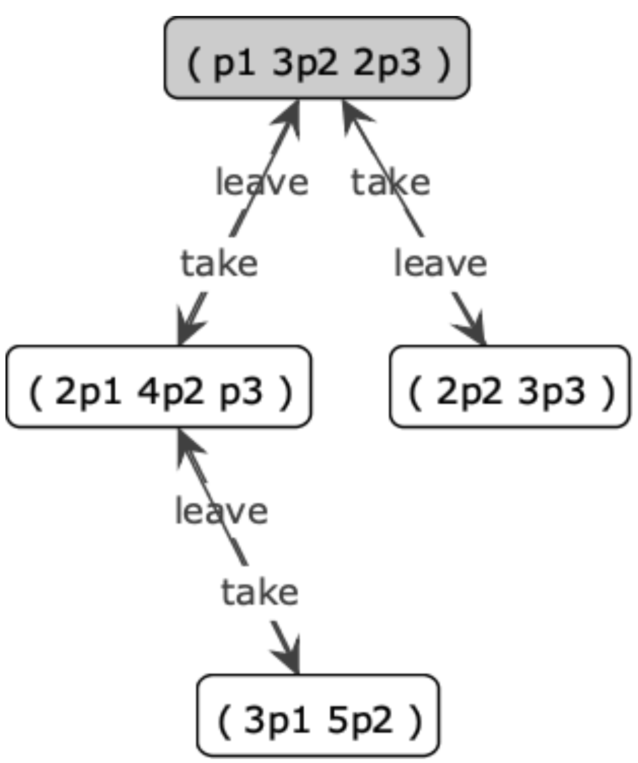


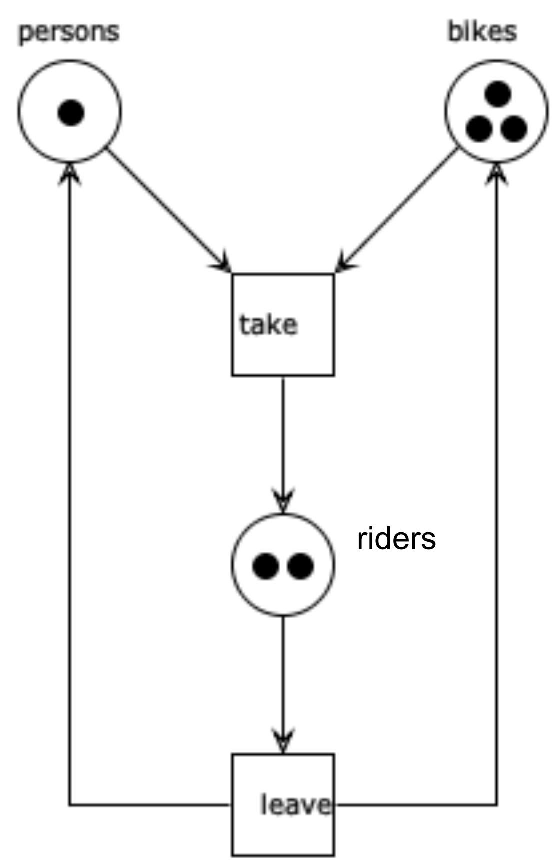
$$\forall t \qquad \sum_{p \in t\bullet} \mathbf{I}(p) = \sum_{p \in \bullet t} \mathbf{I}(p)$$

Question time

Which of the following are S-invariants?



p1 - persons
p2 - bikes
p3 - riders



p	b	r
[1	1	-1]
[1	0	1]
[0	1	1]
[1	1	1]
[1	-1	0]
[1	1	2]
[1	2	2]

$$\forall t \in T,\ \sum_{p\in \bullet t} \mathbf{I}(p) \stackrel{?}{=} \sum_{p\in t\bullet} \mathbf{I}(p)$$

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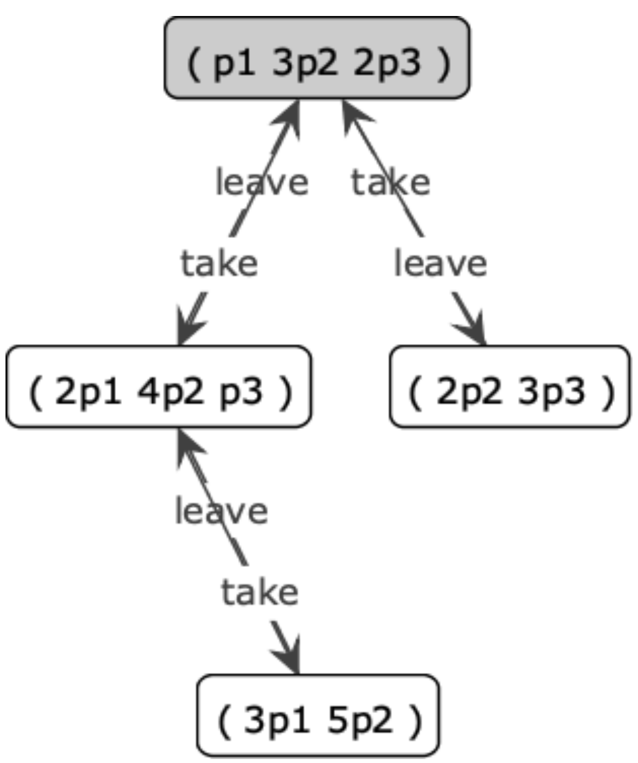

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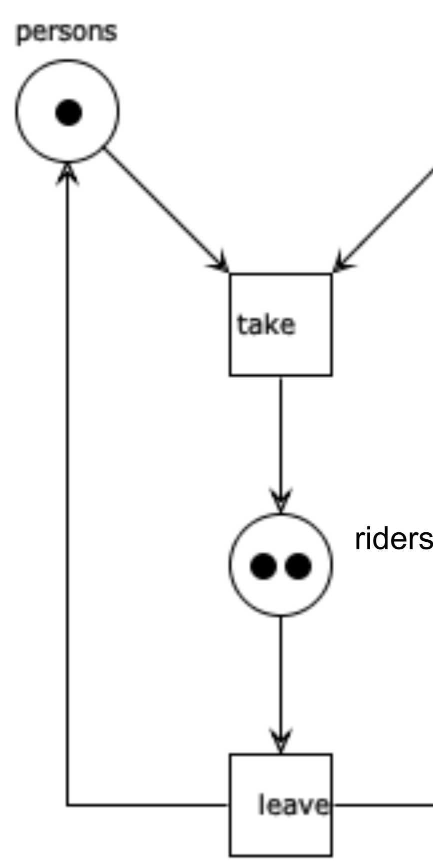

p	b	r
1	1	-1
1	0	1
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1	1	1
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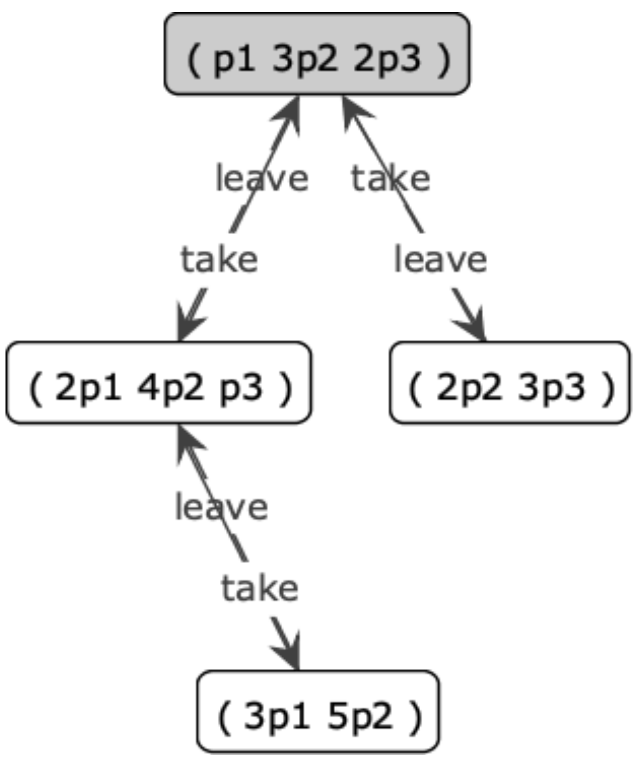

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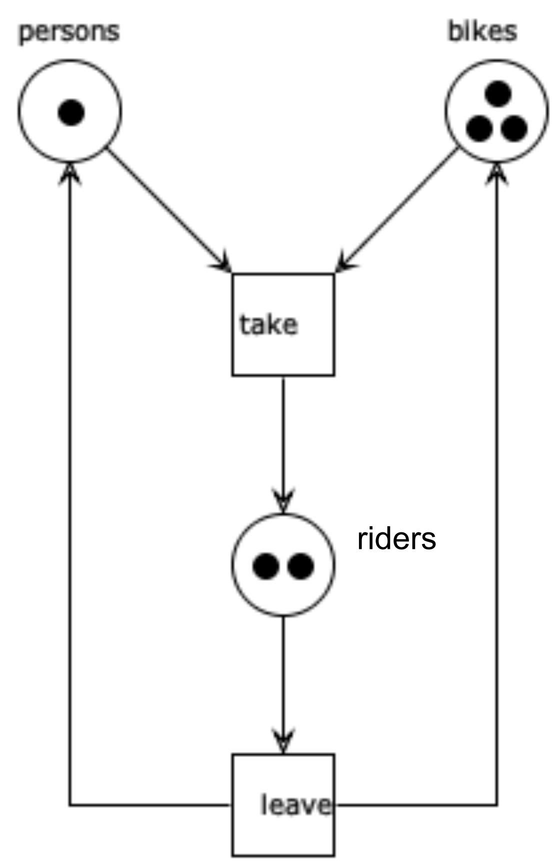

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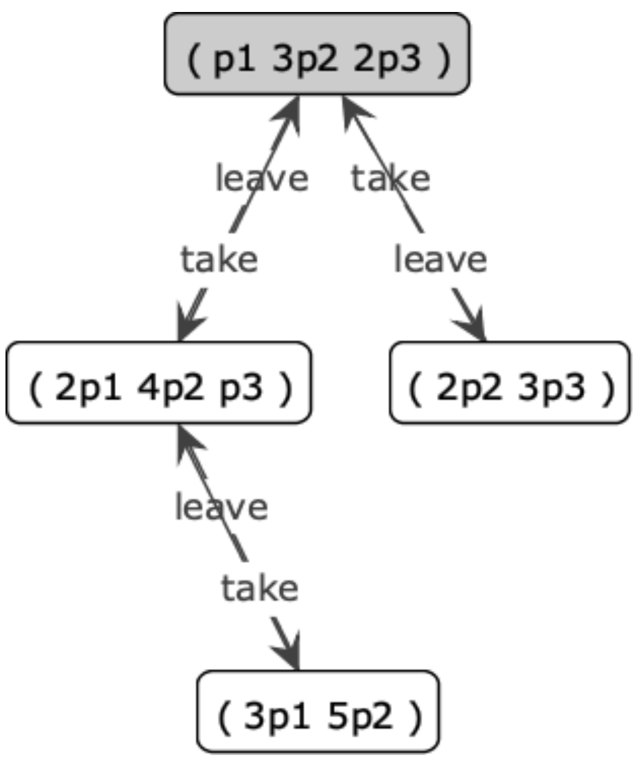

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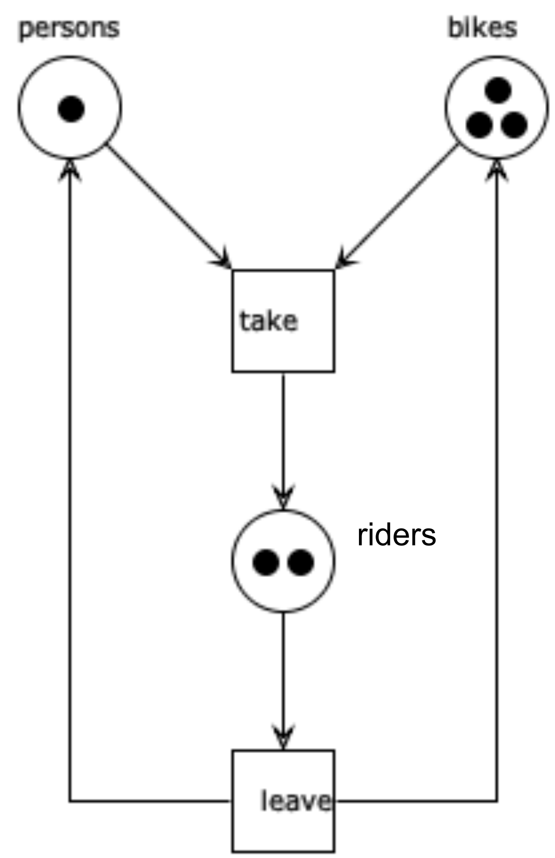

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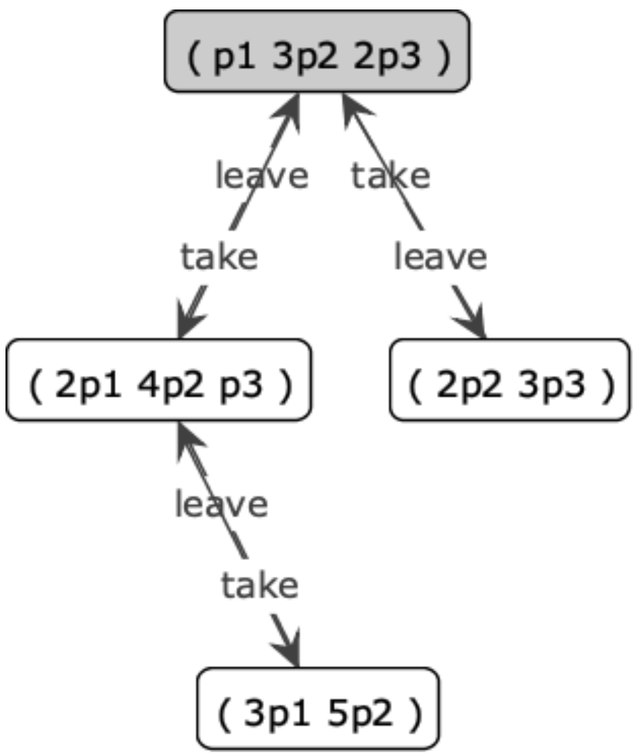

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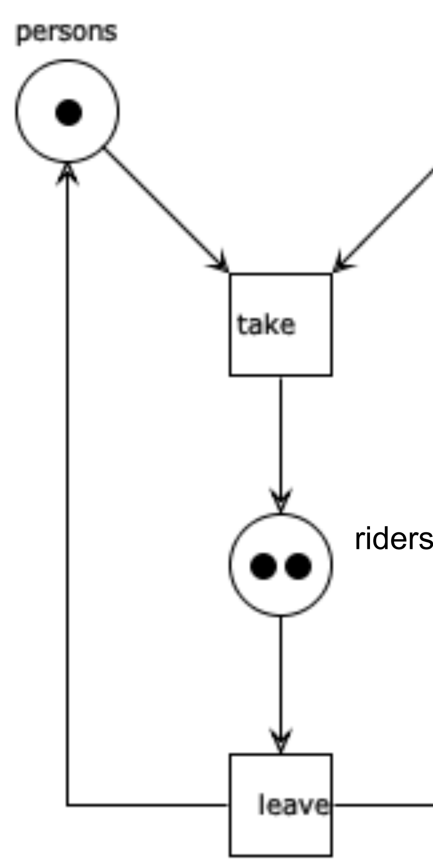

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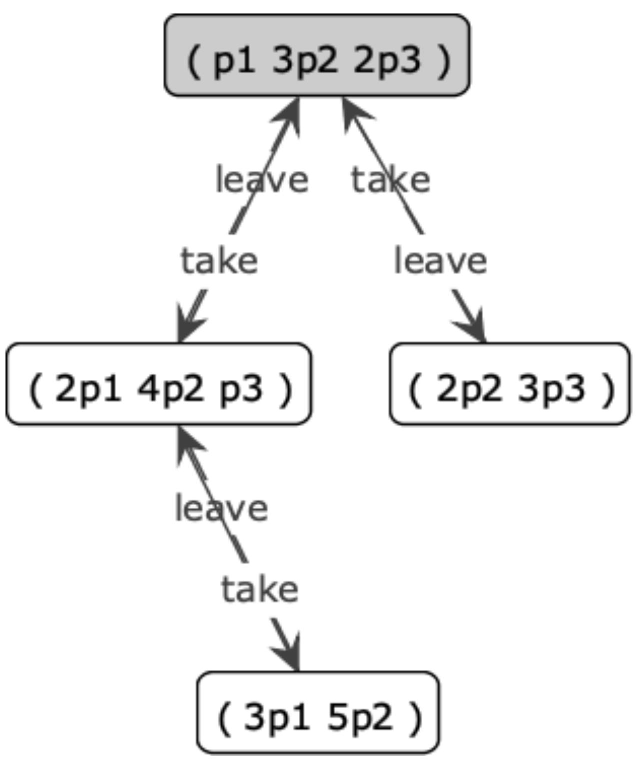

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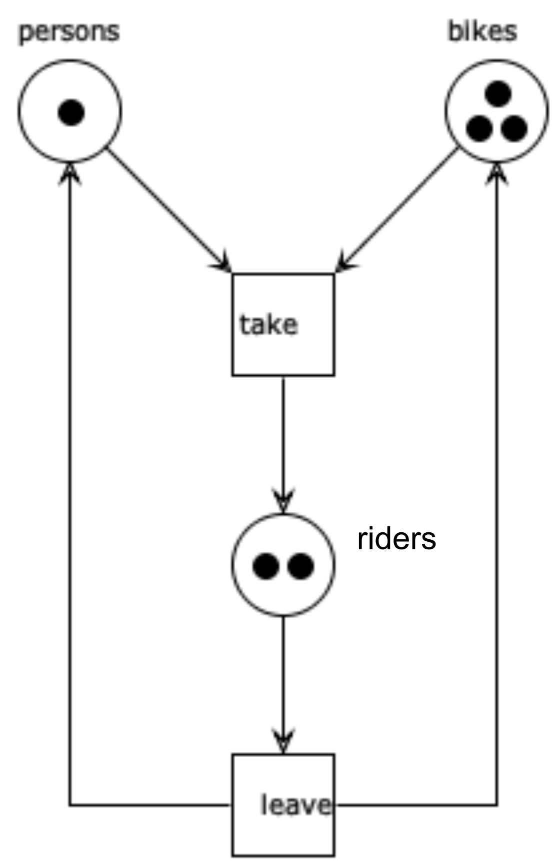

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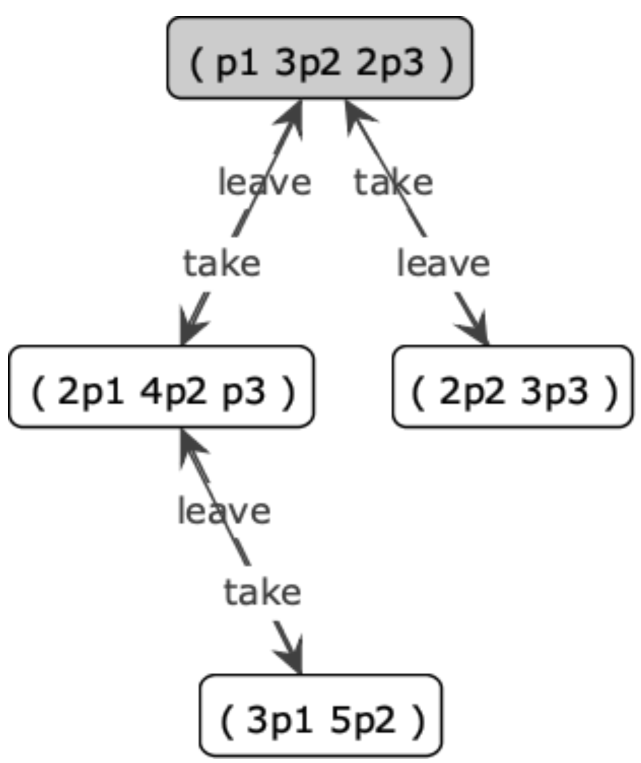

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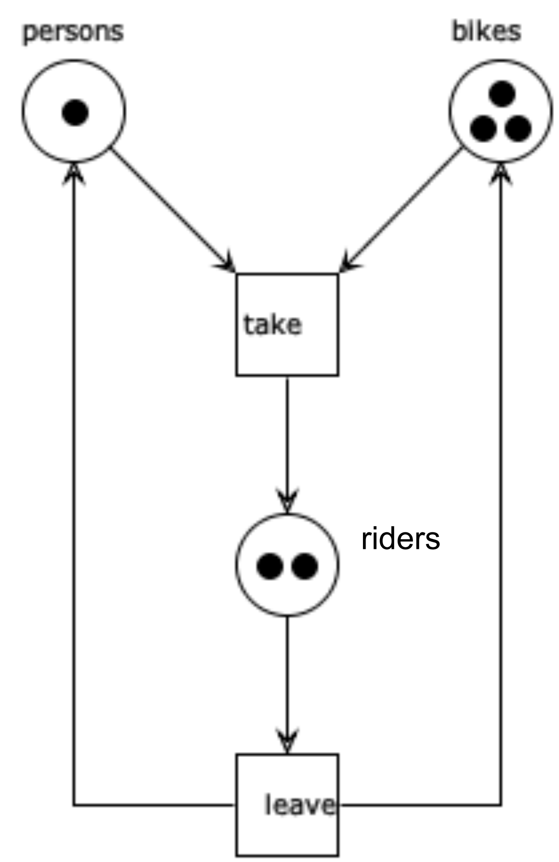

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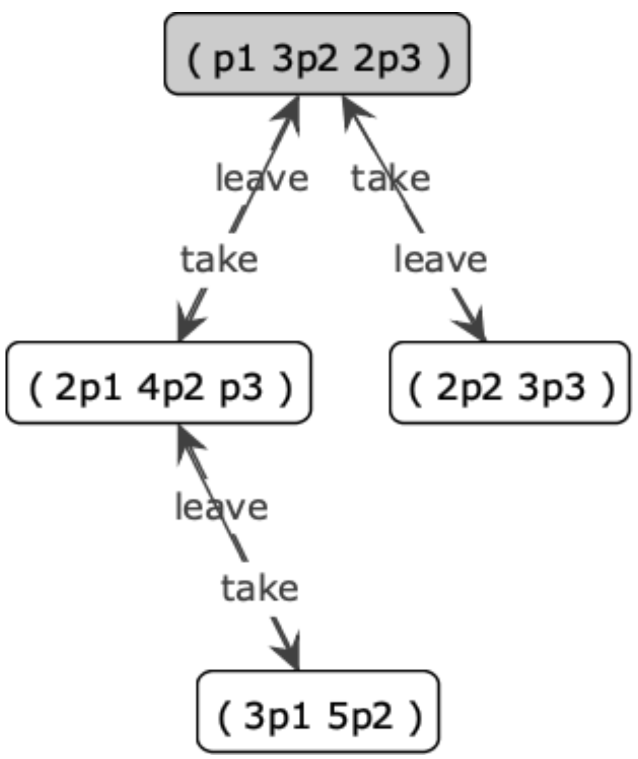

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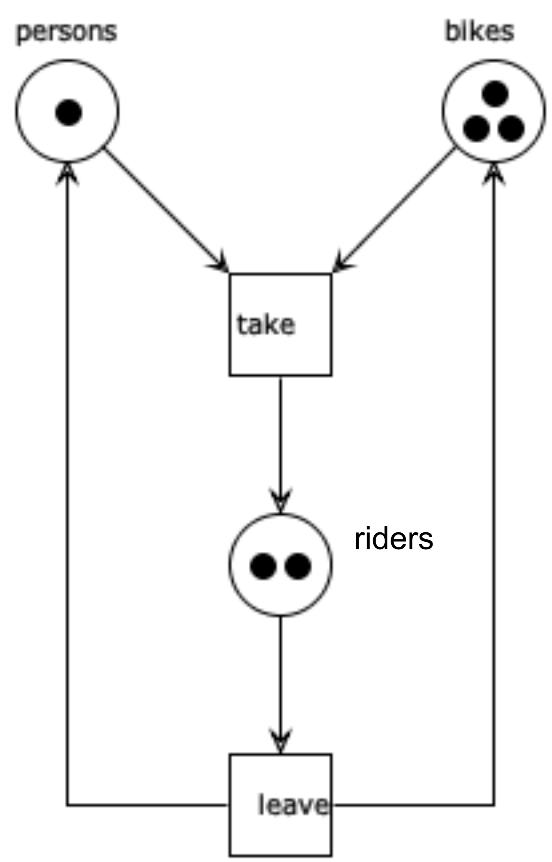

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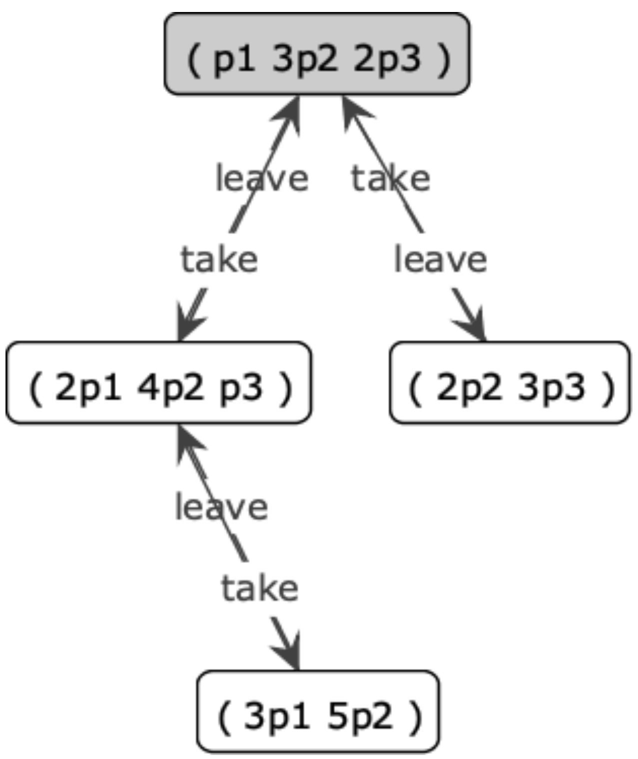

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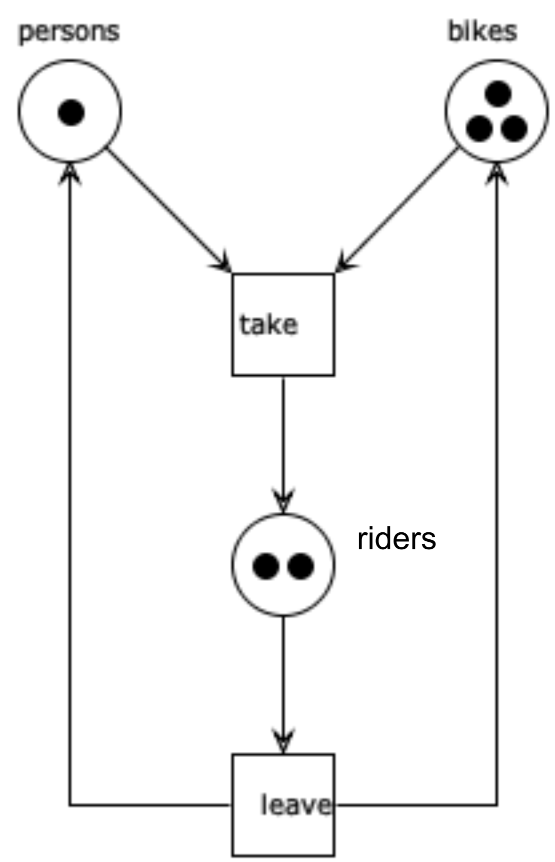

p b r

[1 1 -1]
[1 0 1]
+ [0 1 1]
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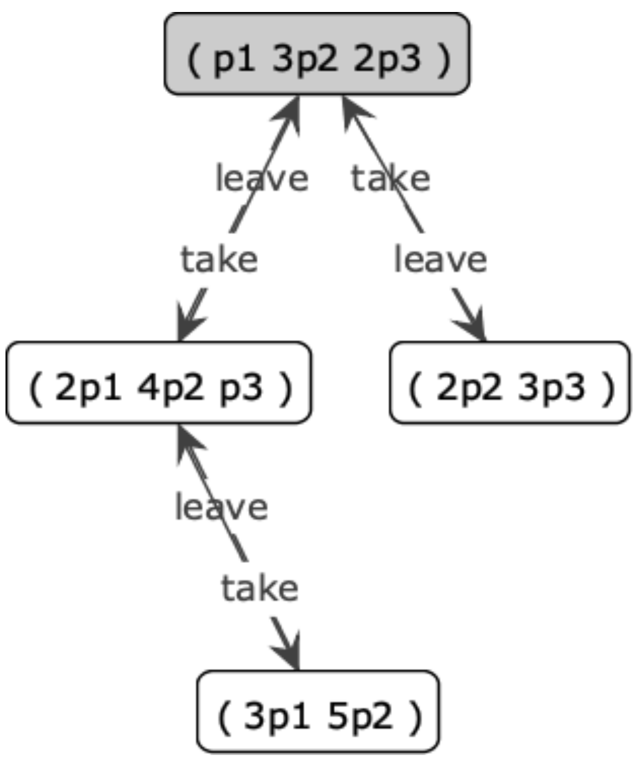

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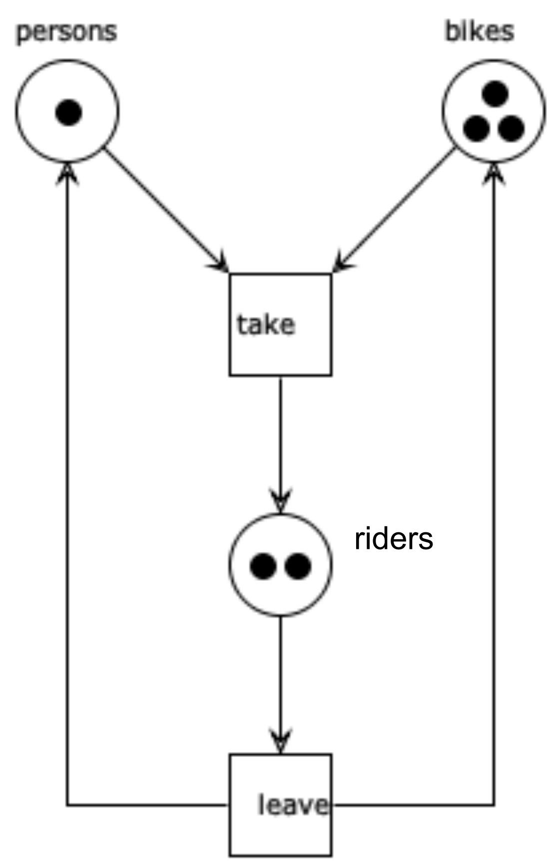

	p	b	r	
[	1	1	-1	]
[	1	0	1	]
- [	0	1	1	]
[	1	1	1	]
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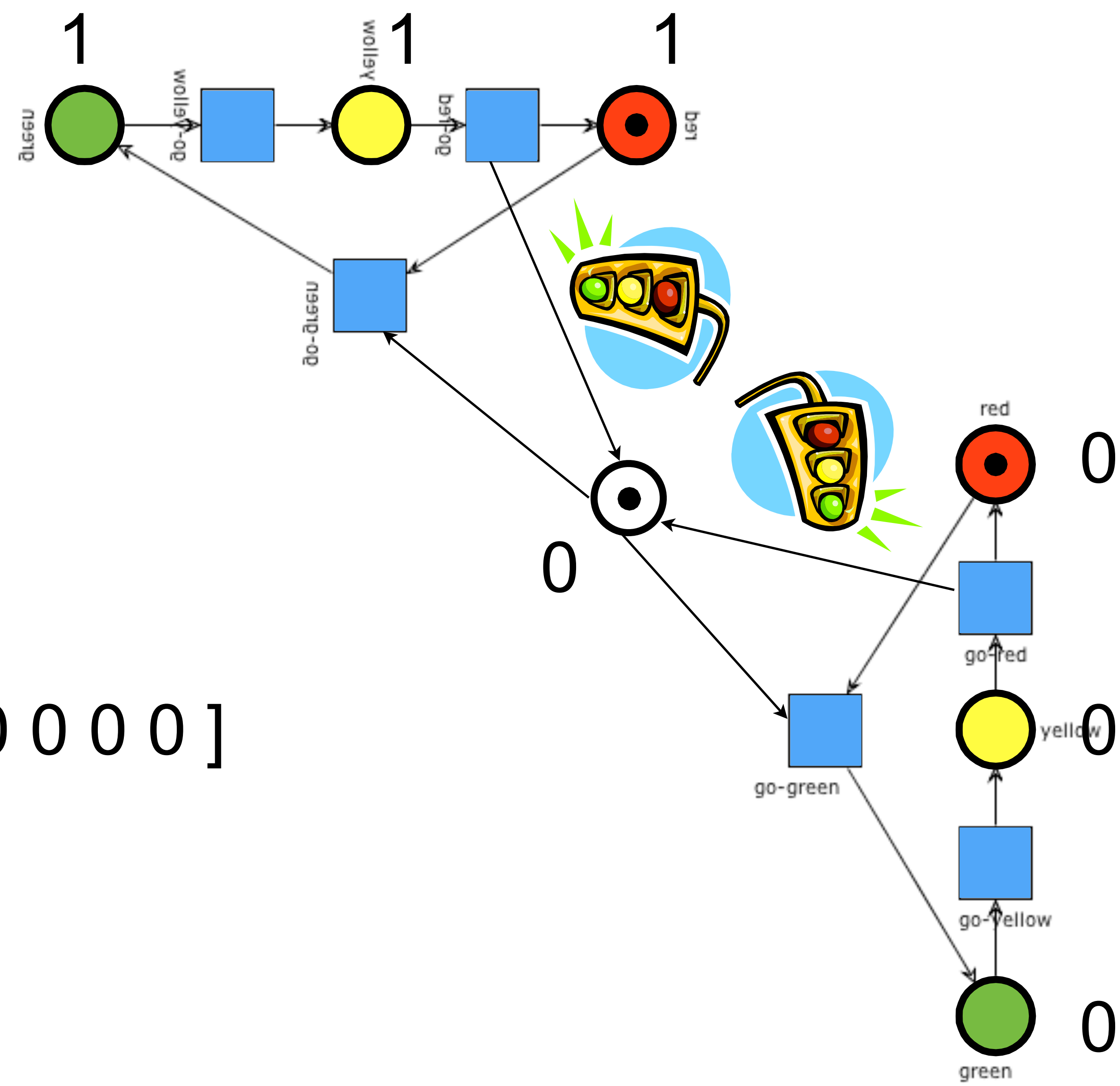
Traffic-lights example



[1 1 1 0 0 0 0]

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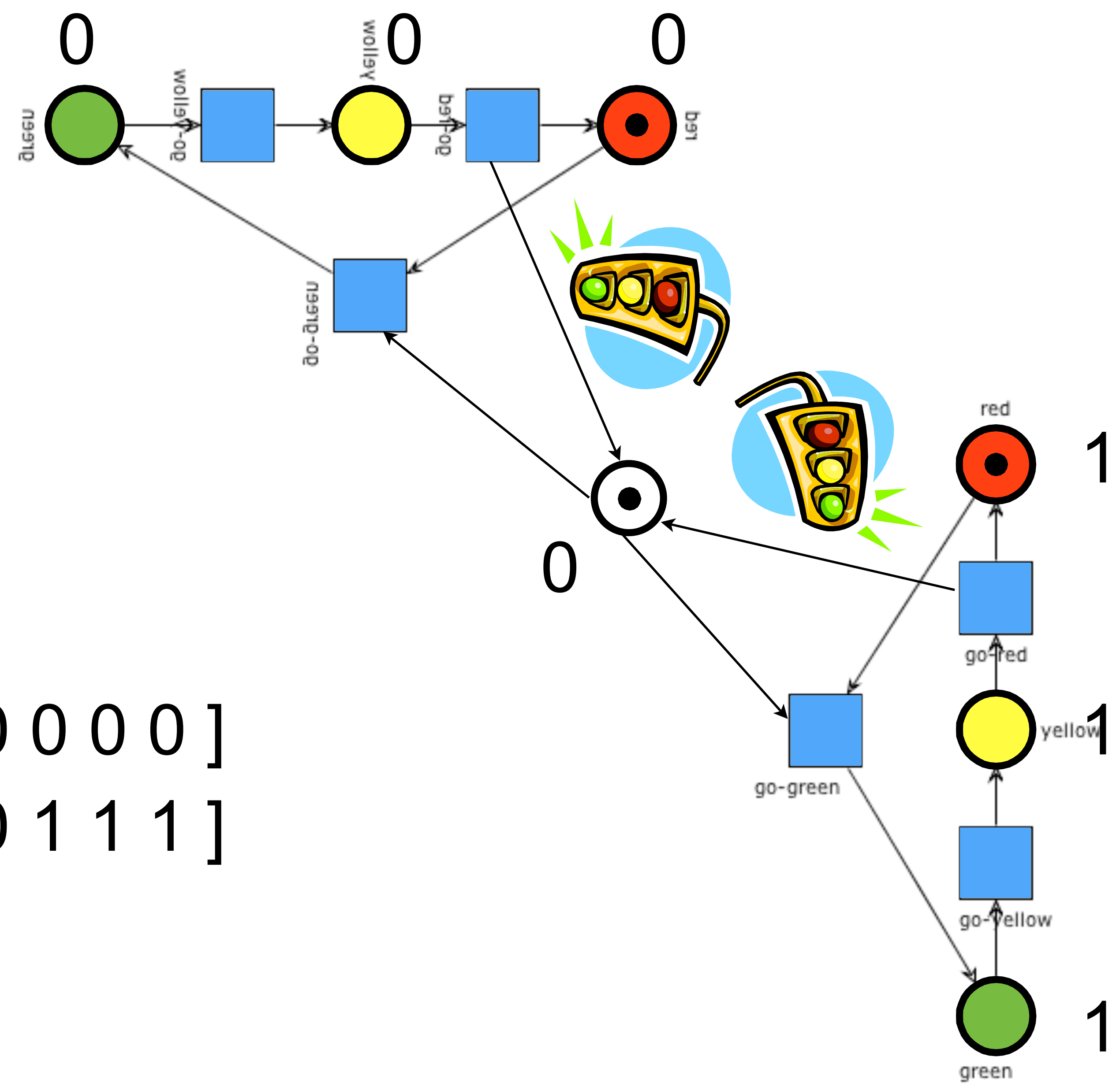

[1 1 1 0 0 0 0]
[0 0 0 0 1 1 1]

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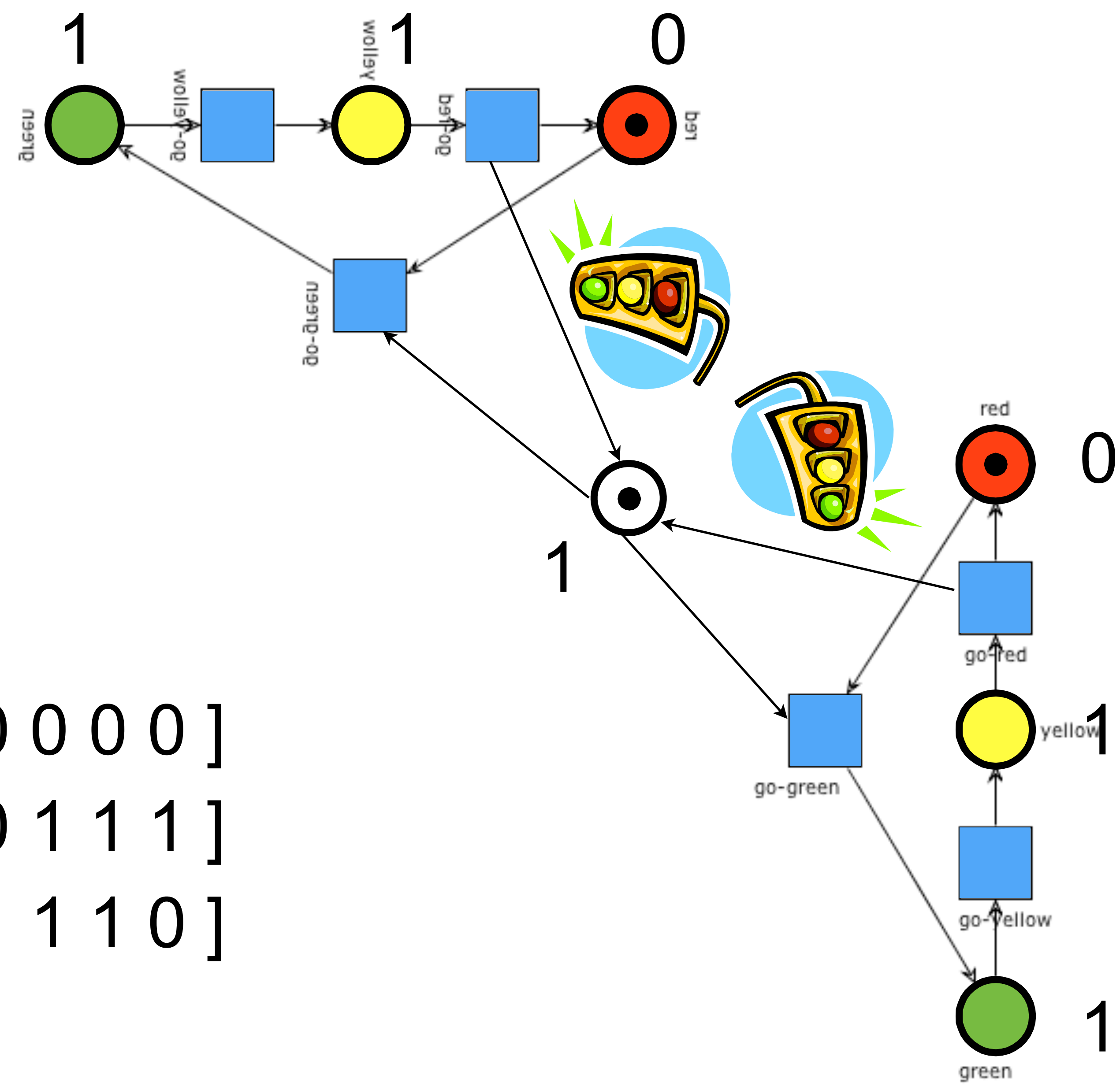

[1 1 1 0 0 0 0]
[0 0 0 0 1 1 1]
[1 1 0 1 1 1 0]

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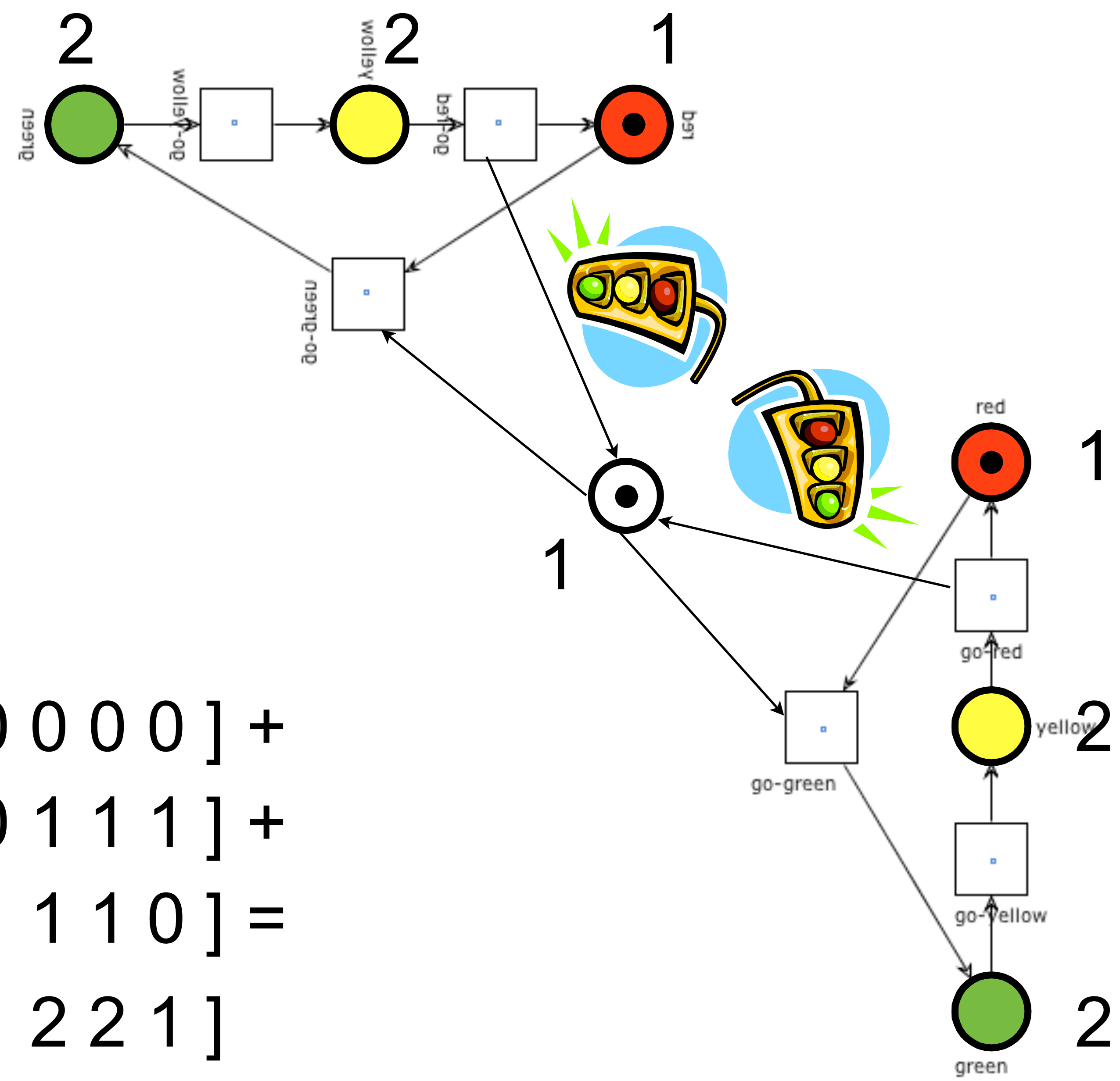
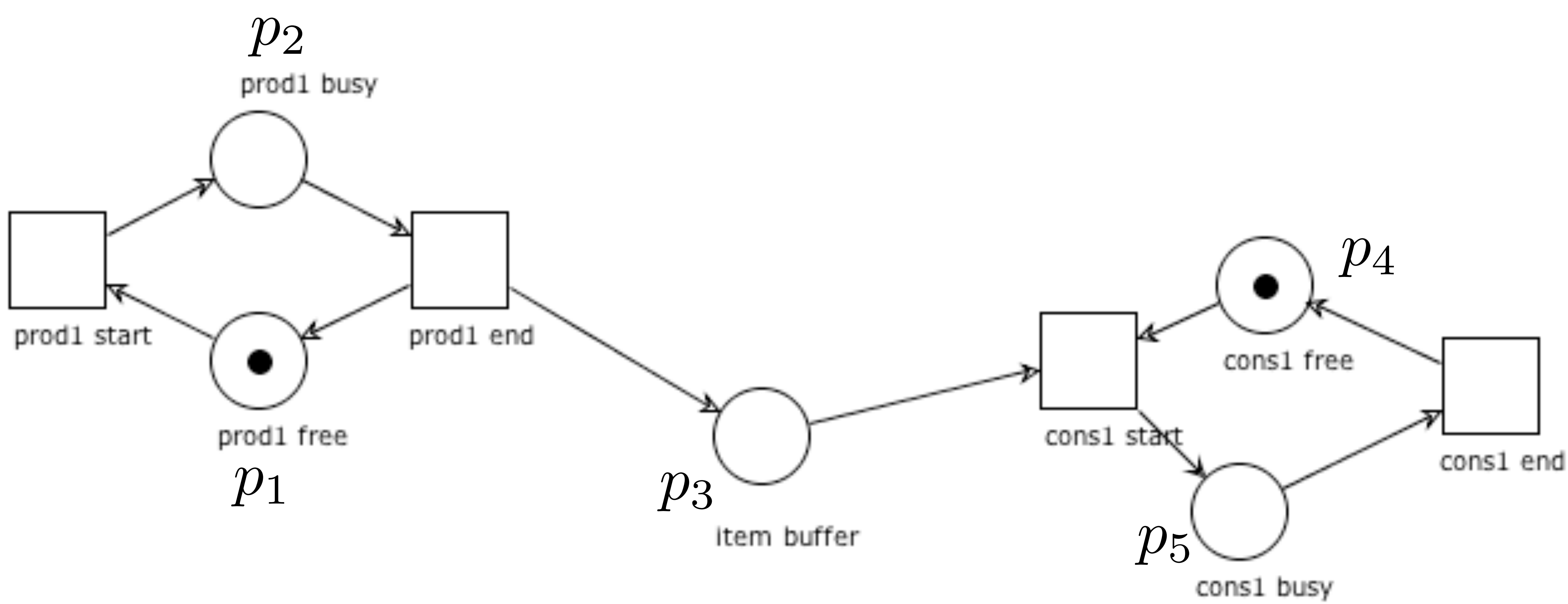


[1 1 1 0 0 0 0] +
[0 0 0 0 1 1 1] +
[1 1 0 1 1 1 0] =
[2 2 1 1 2 2 1]

Exercises

Define two (linearly independent) S-invariants

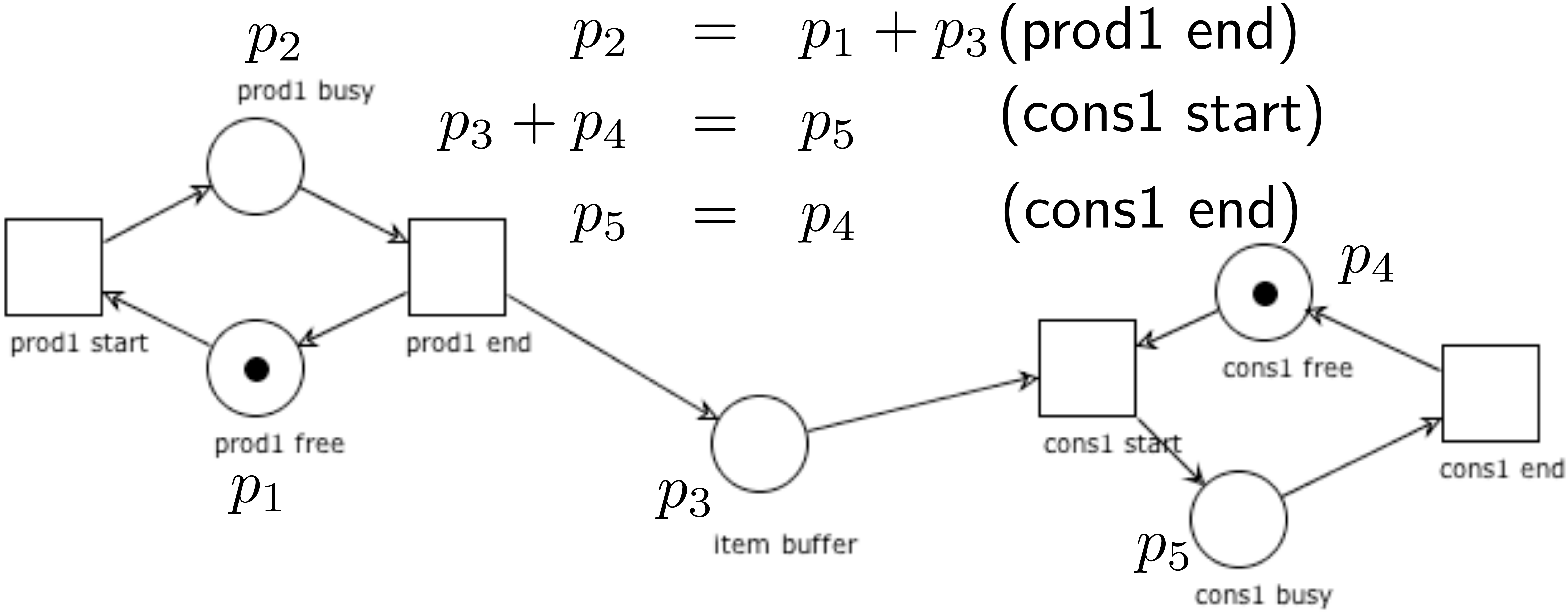


Exercises

Define two (linearly independent) S-invariants

solve a system of linear equations

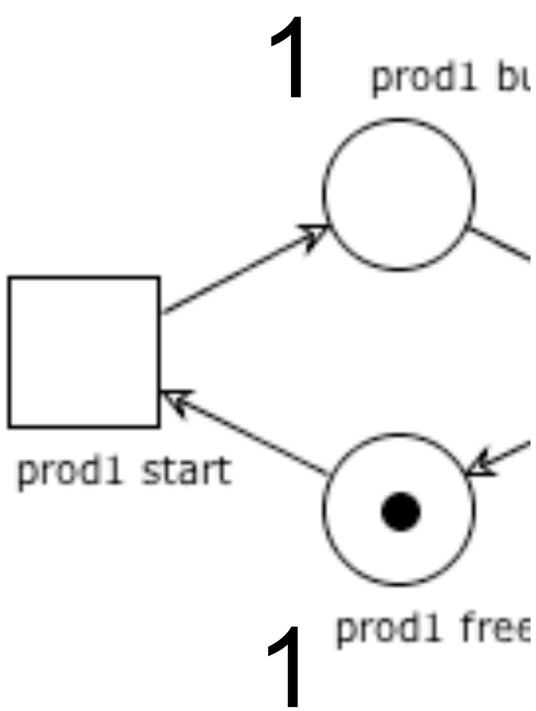
$$\begin{aligned} p_1 &= p_2 && \text{(prod1 start)} \\ p_2 &= p_1 + p_3 && \text{(prod1 end)} \\ p_3 + p_4 &= p_5 && \text{(cons1 start)} \\ p_5 &= p_4 && \text{(cons1 end)} \end{aligned}$$



Exercises

Define two (linearly independent) S-invariants

reason on the net



Exercises

Define two (linearly independent) S-invariants

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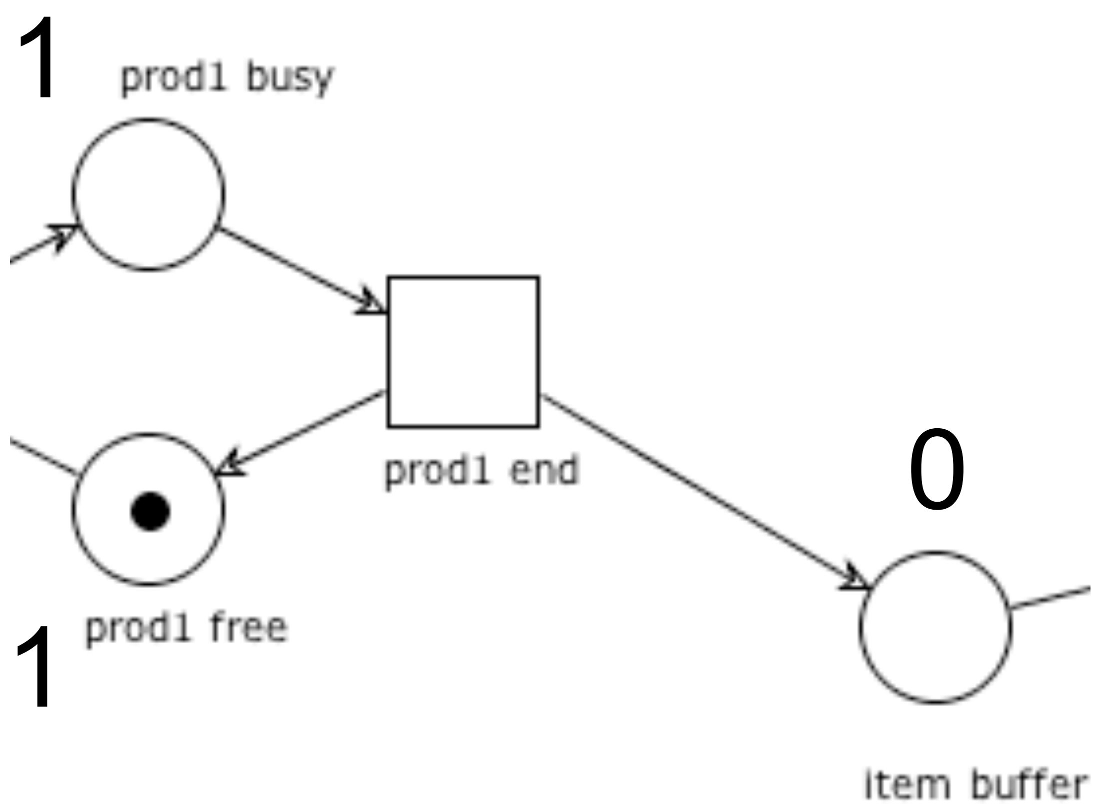
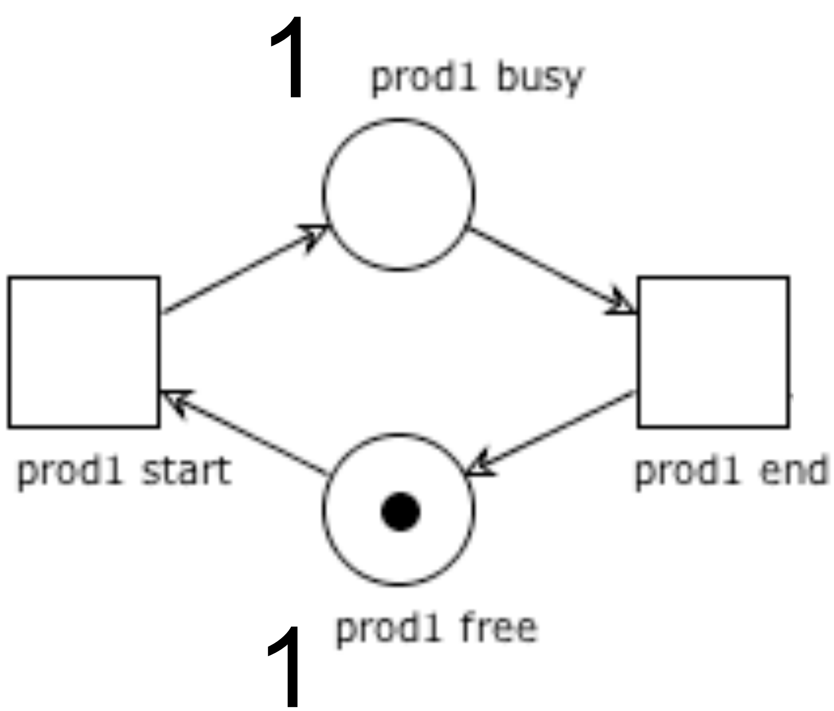
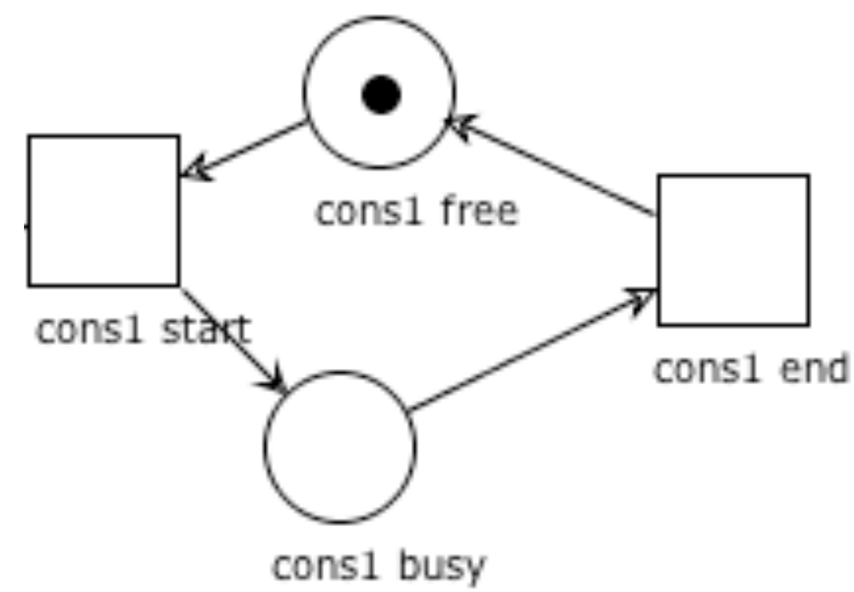


Exercises

Define two (linearly independent) S-invariants

places with weight 0 can be disregarded





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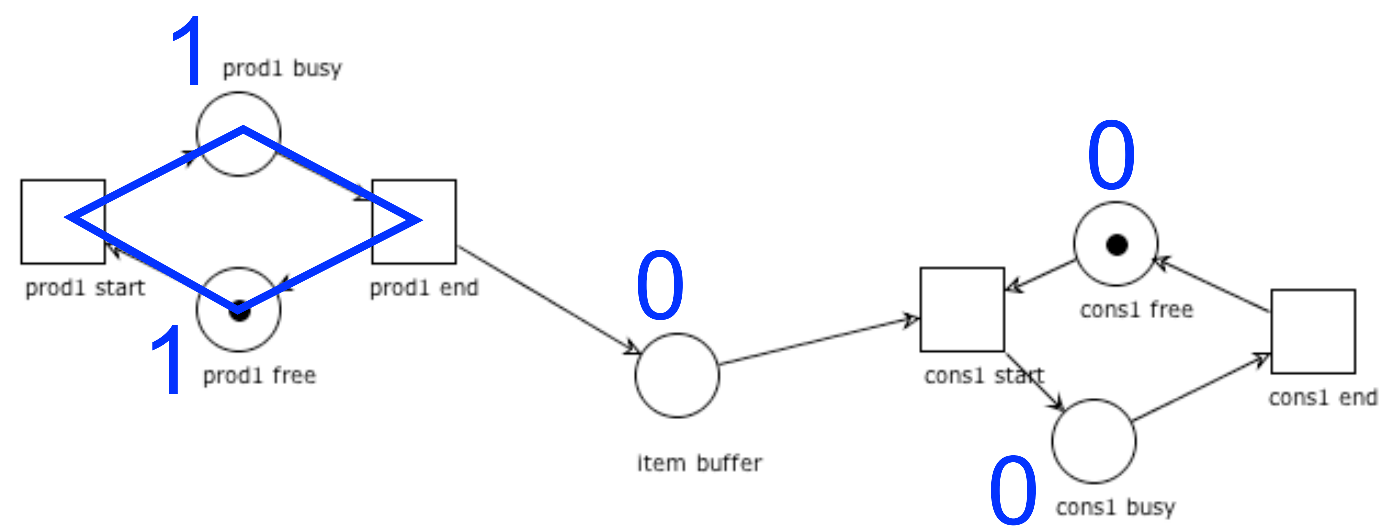
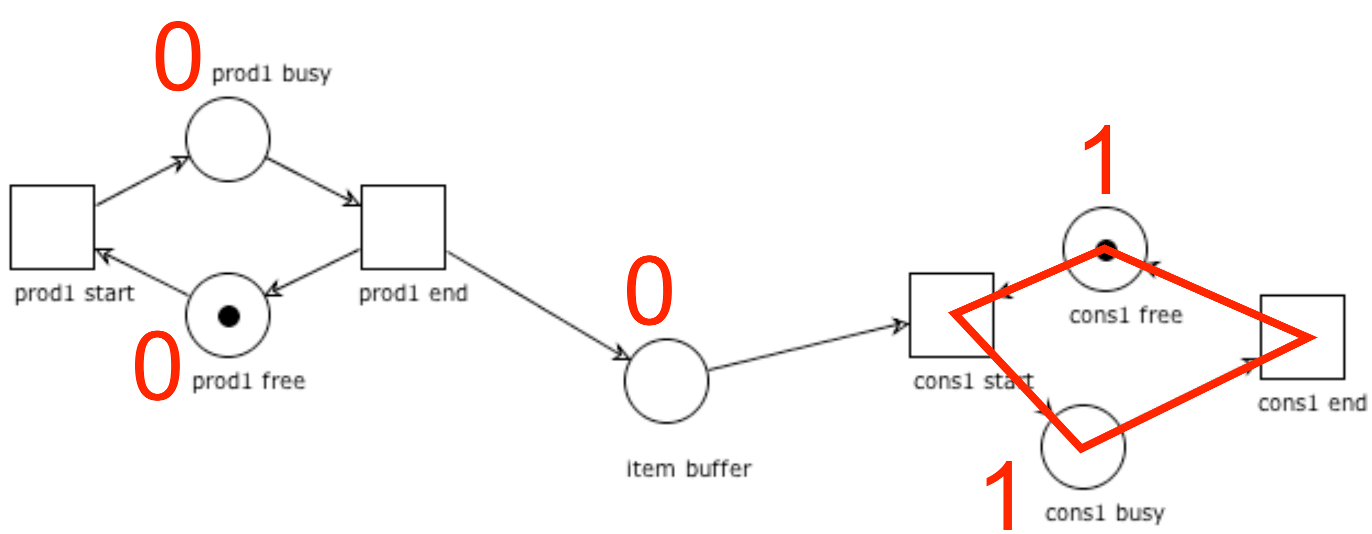
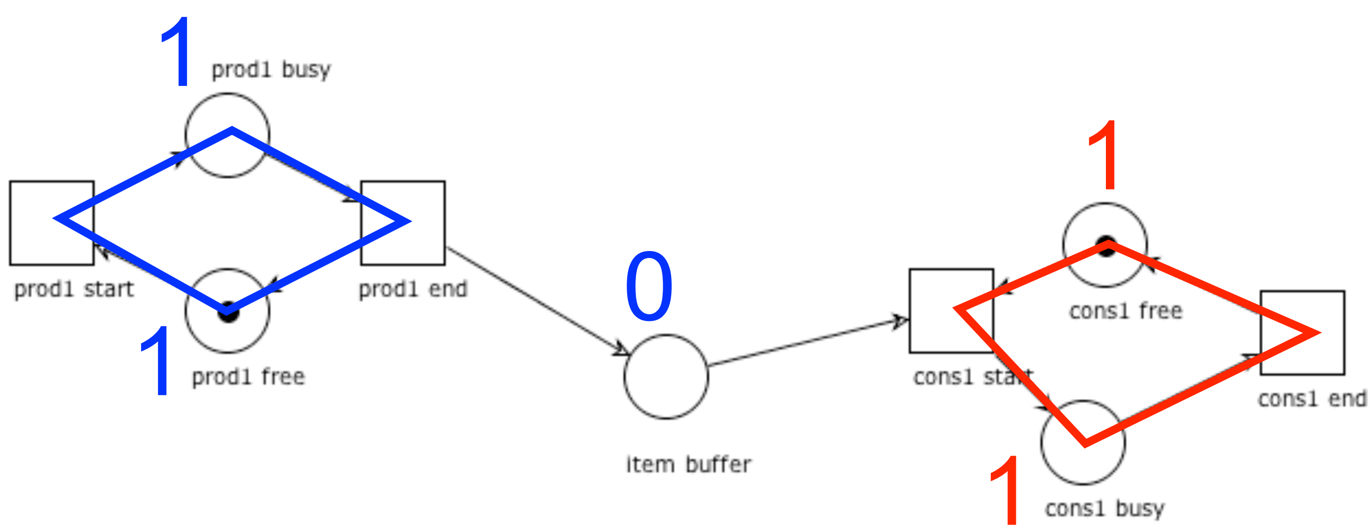




Exercises

The sum of S-invariants is still an S-invariant



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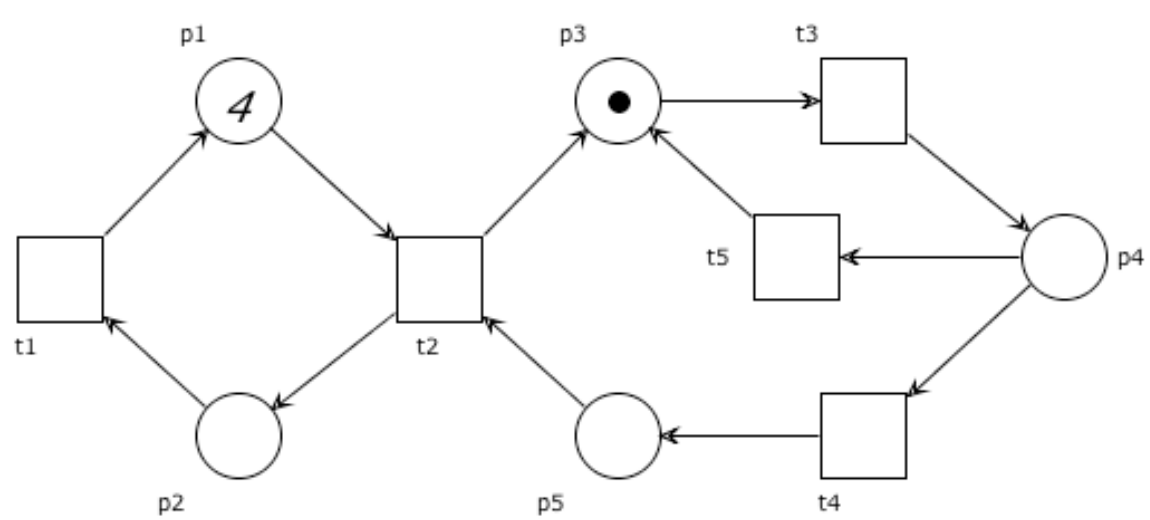

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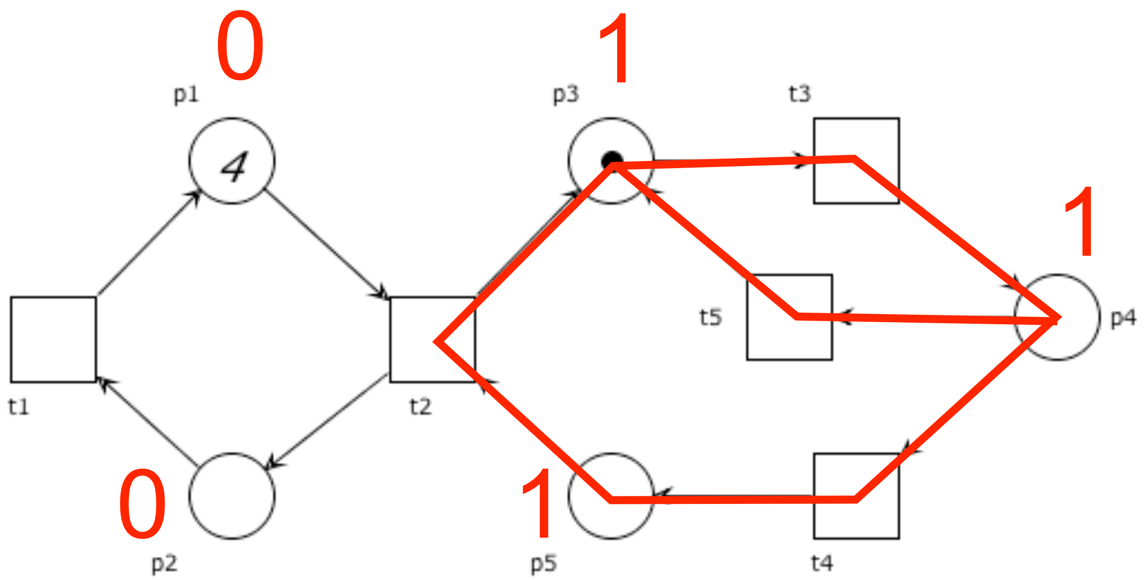

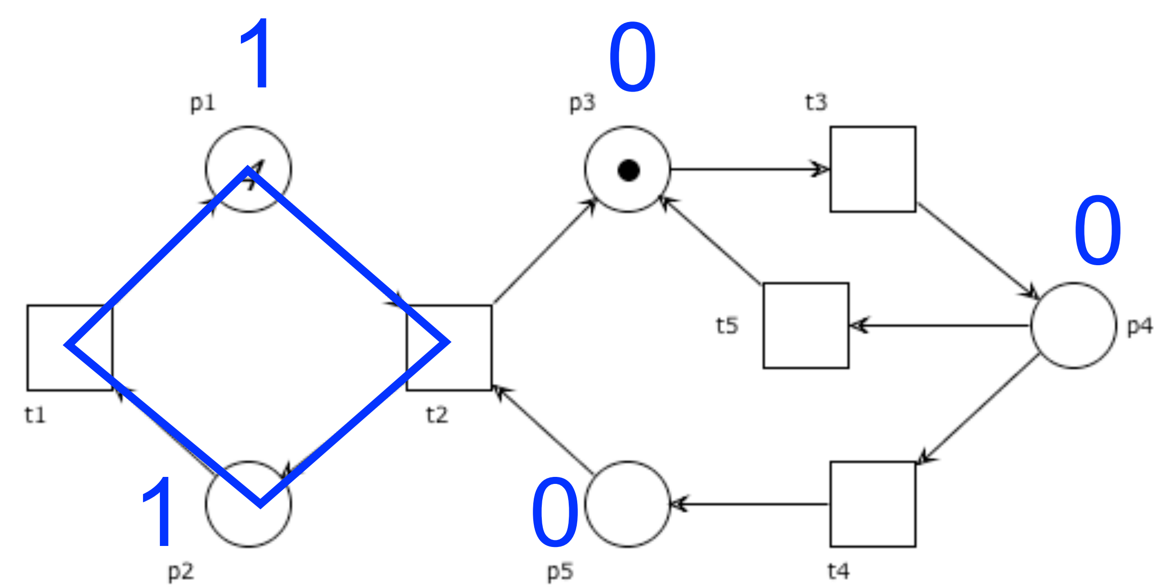

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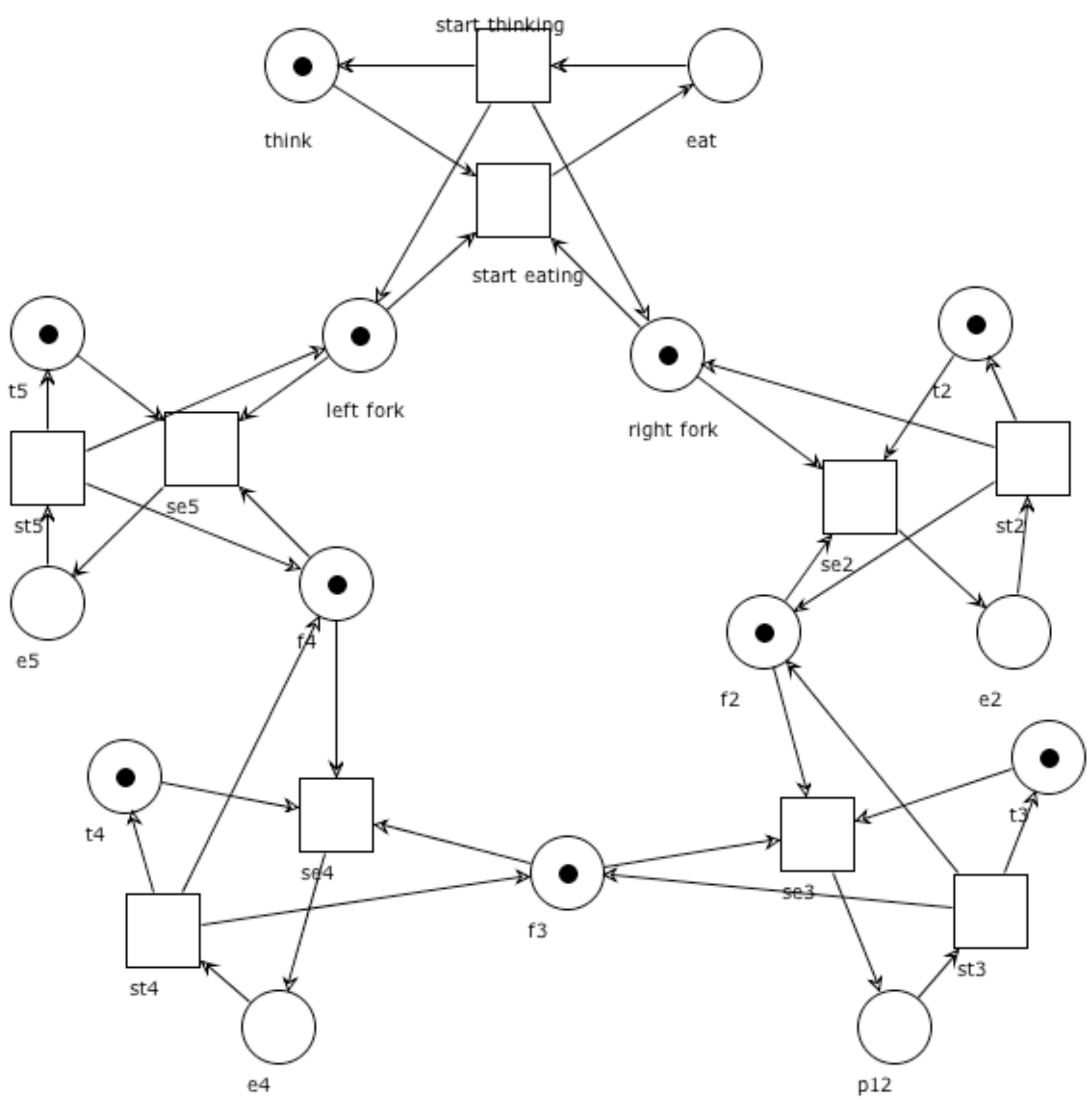

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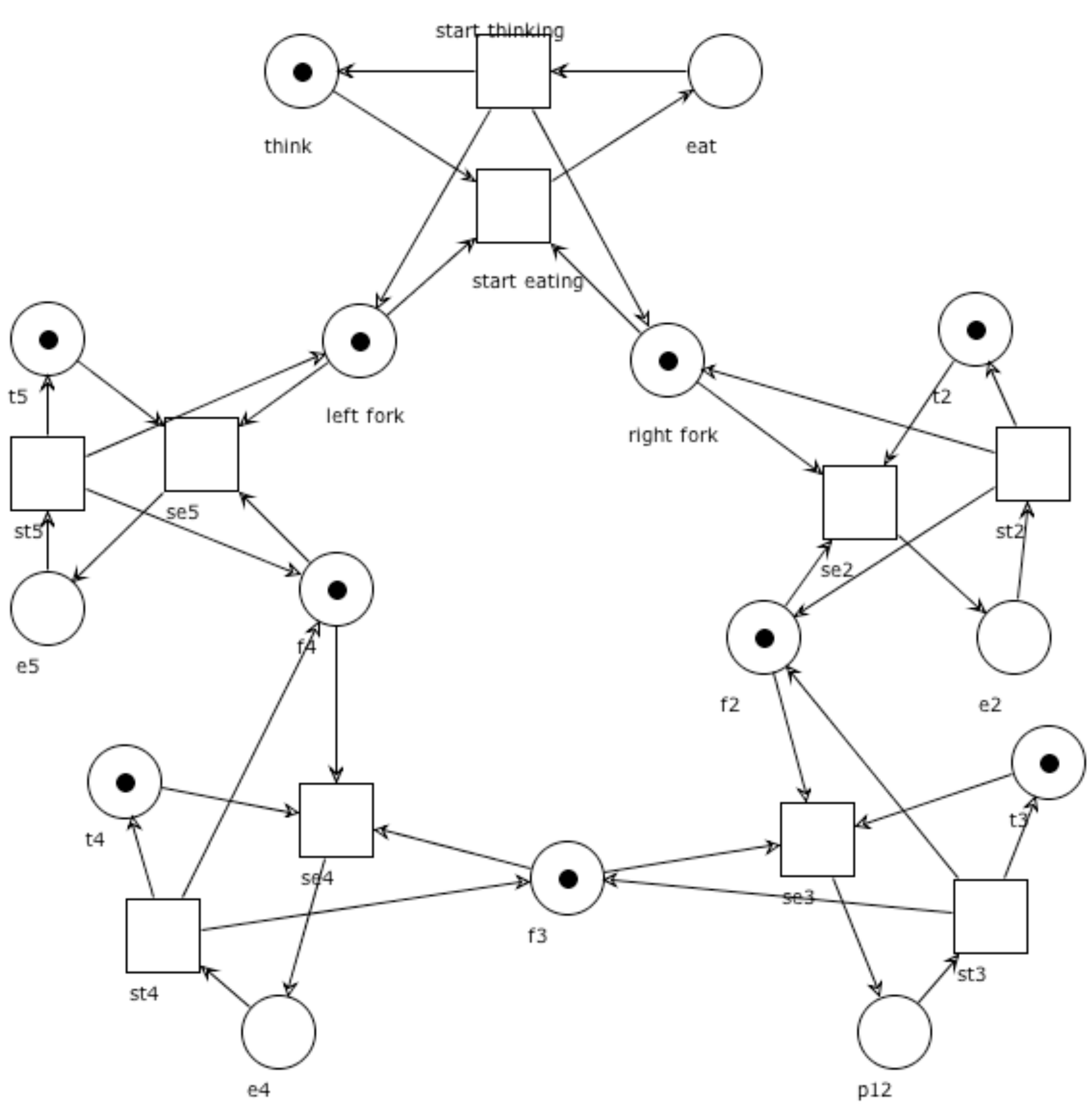

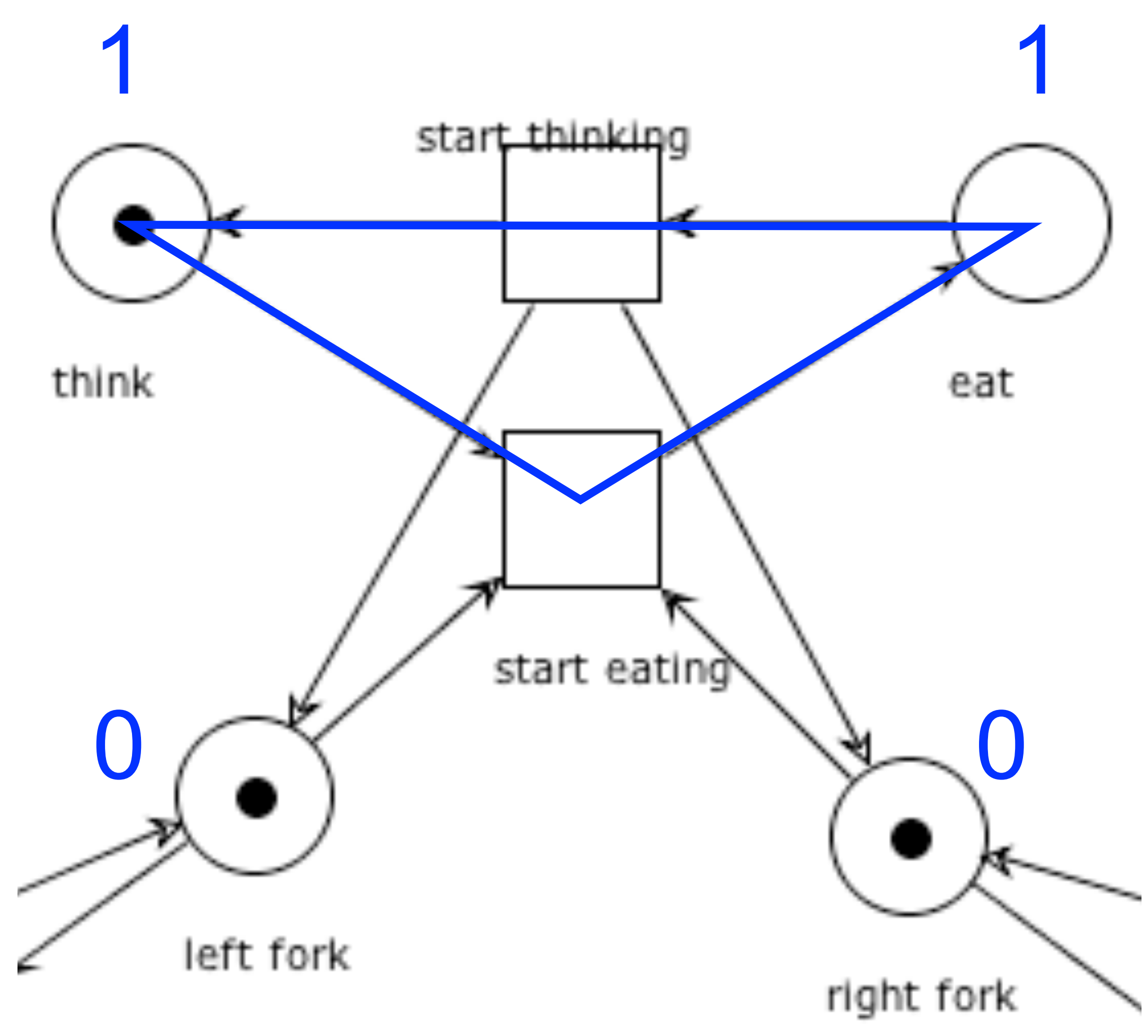

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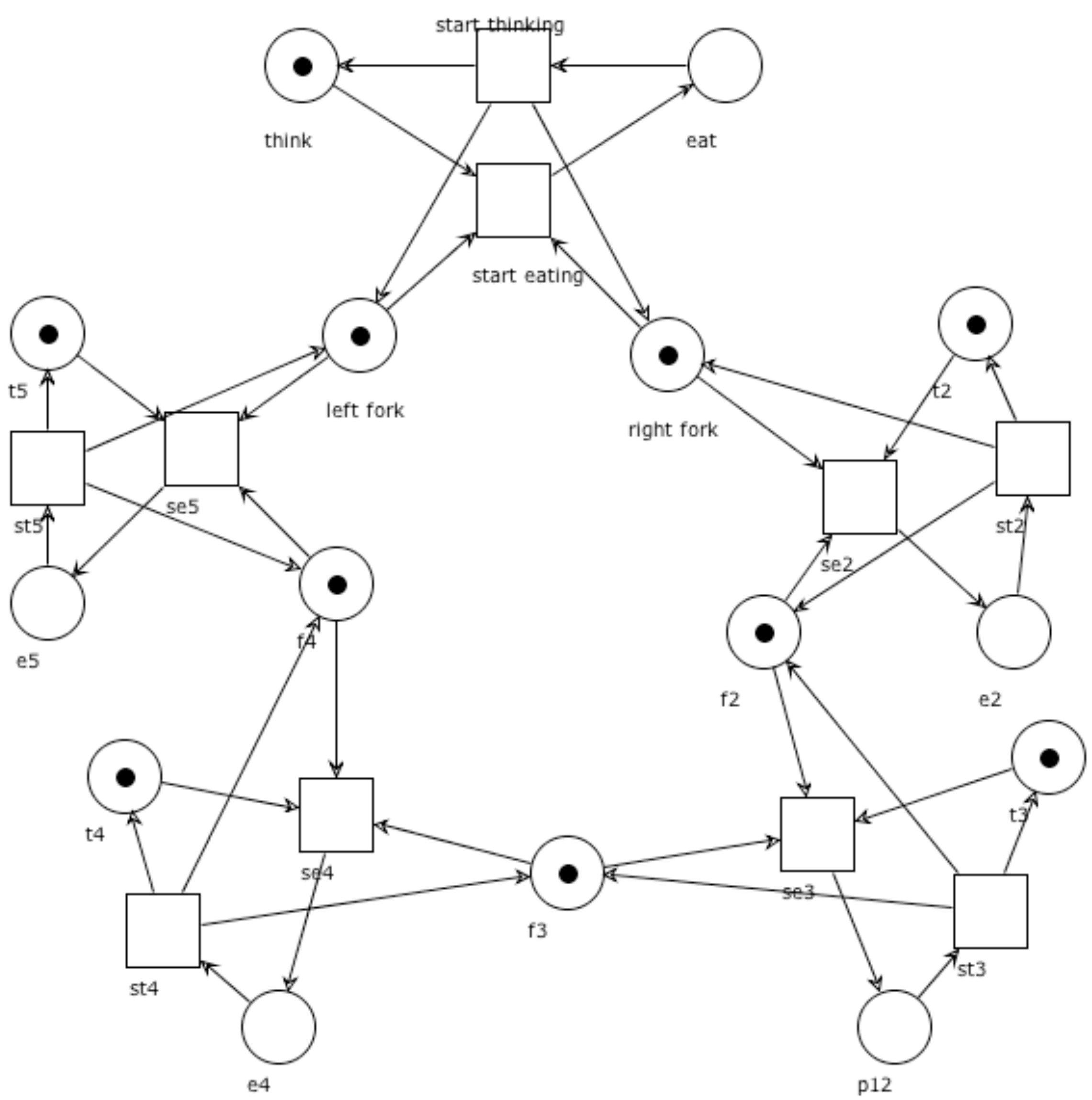

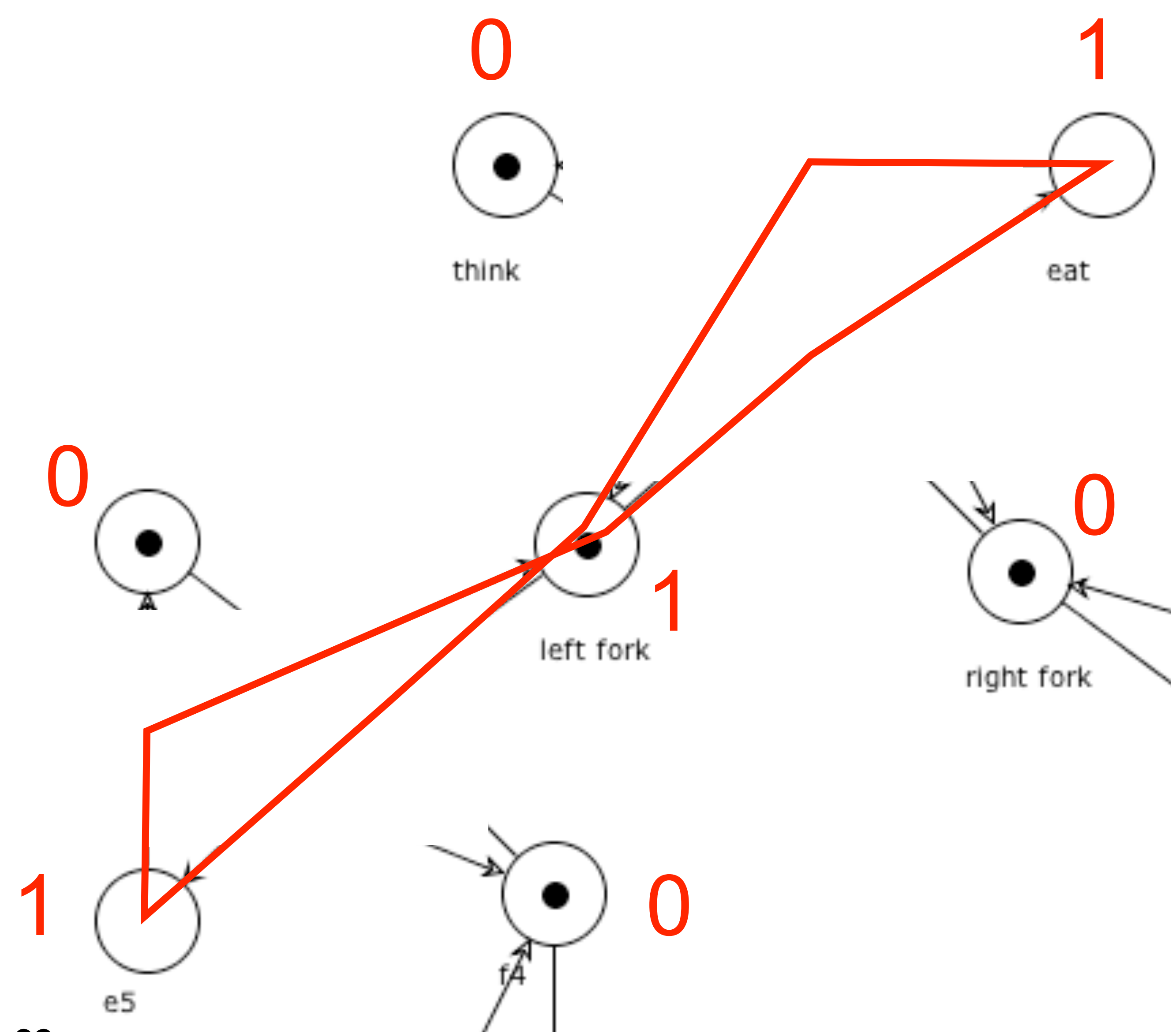

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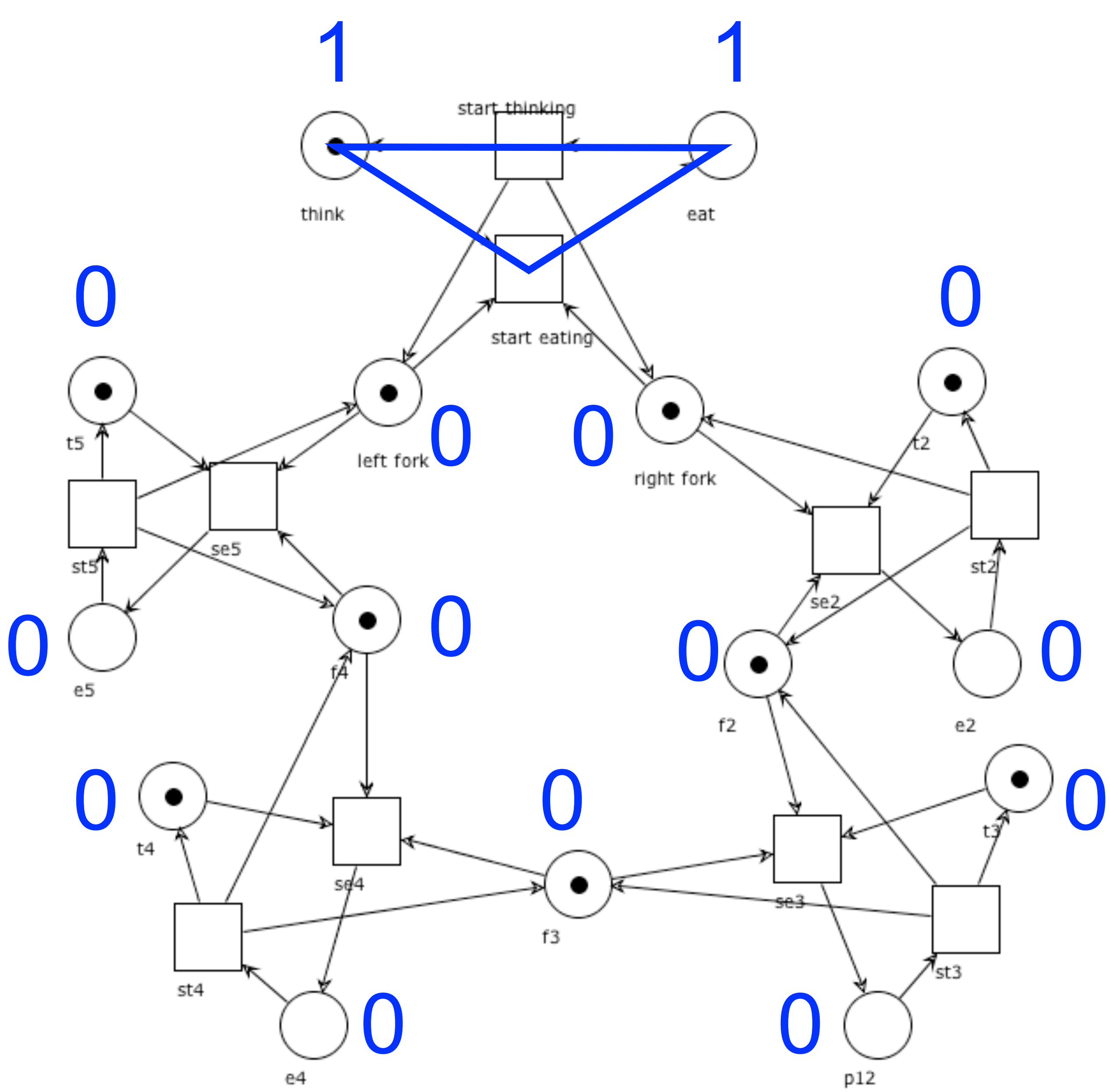
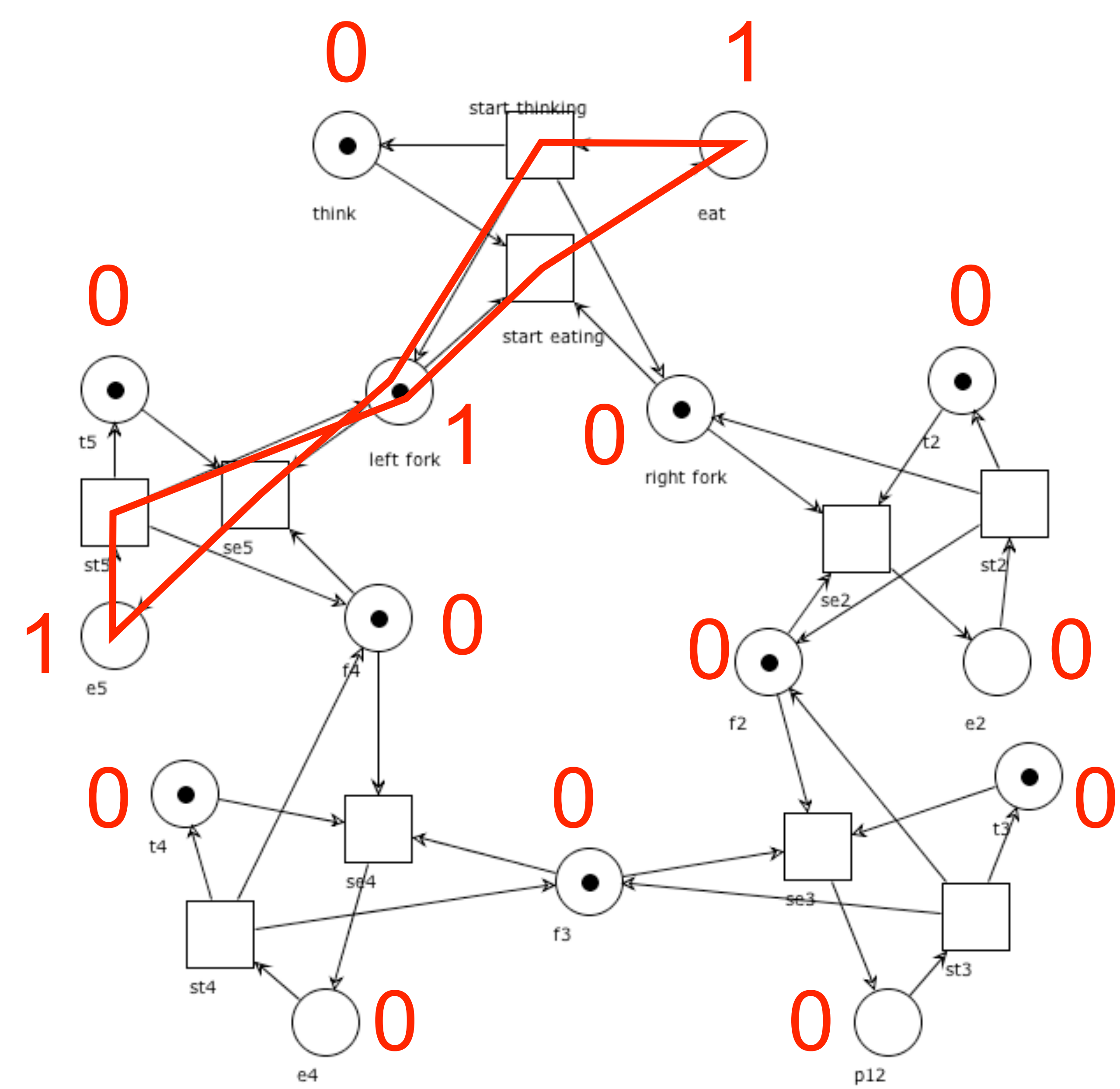
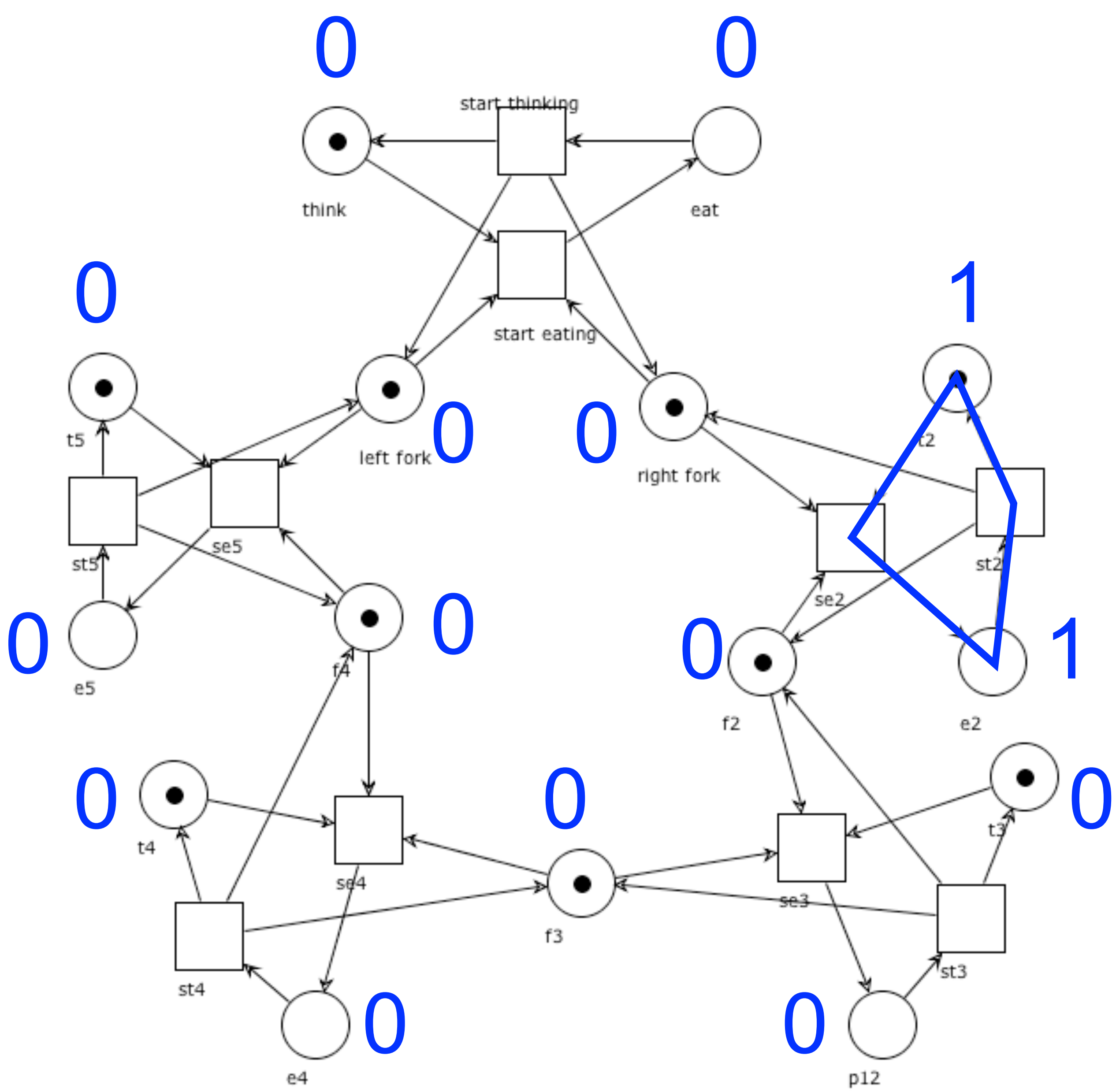
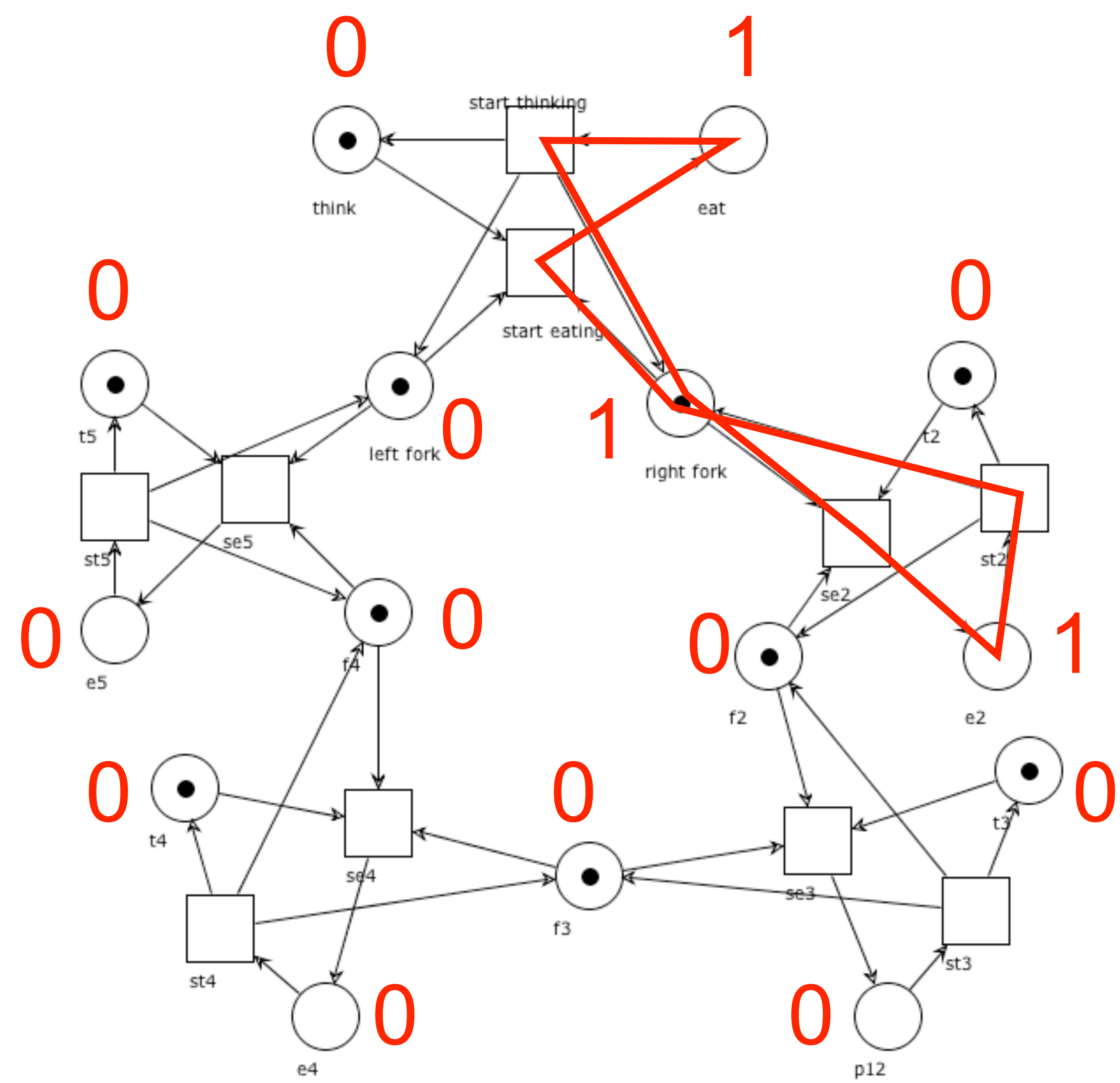




Exercises

Symmetrically





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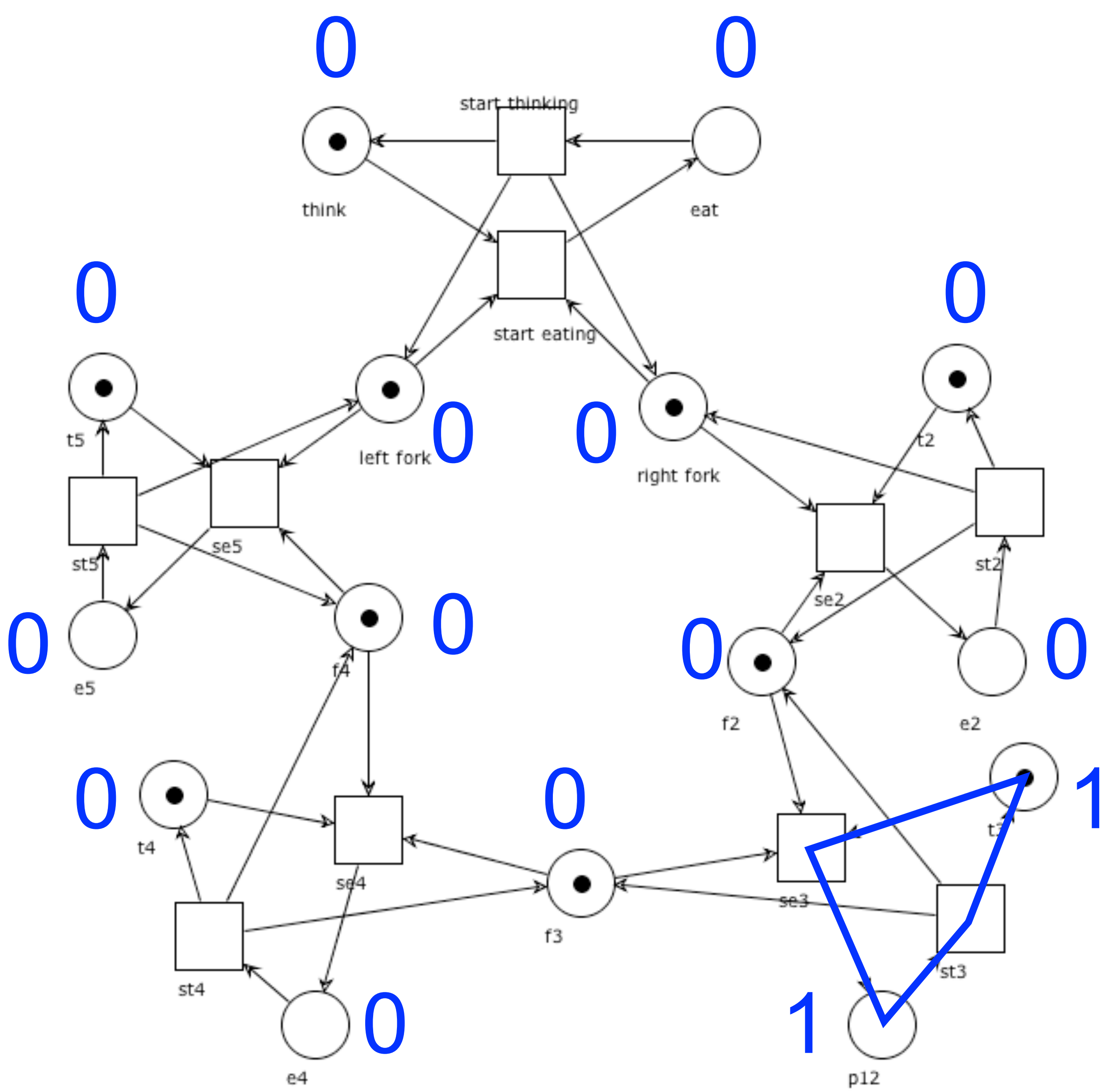
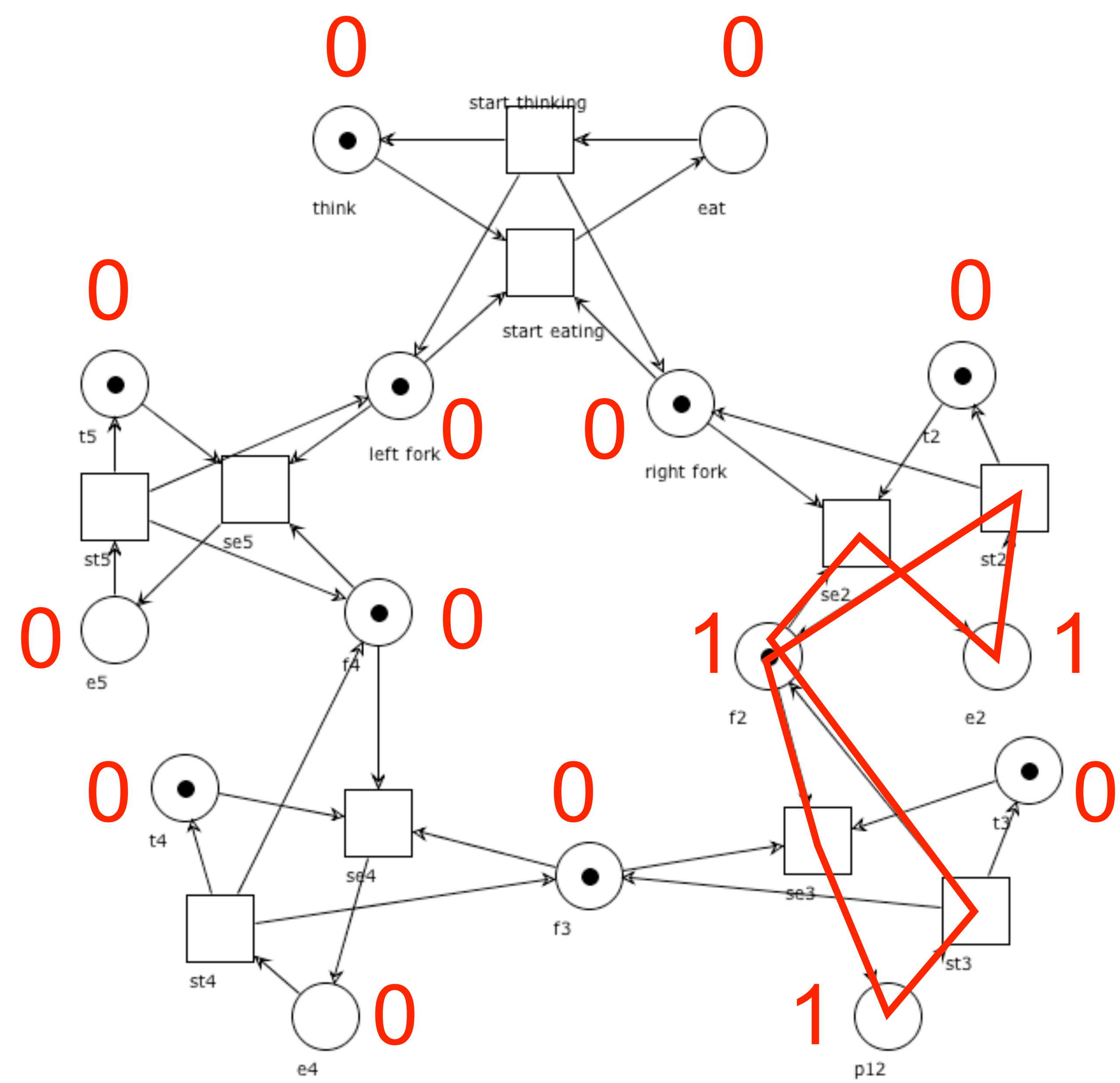
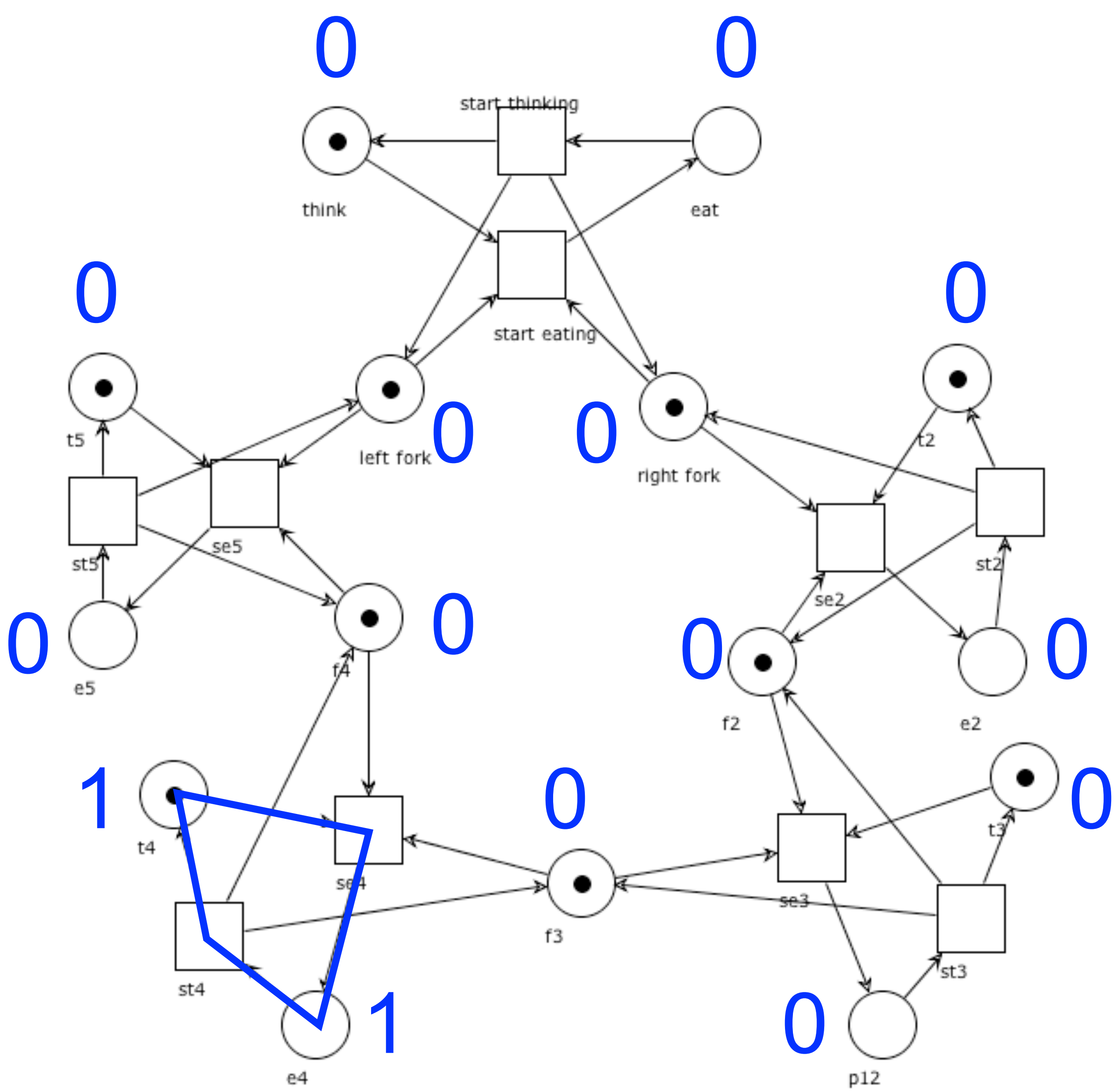
Symmetrically

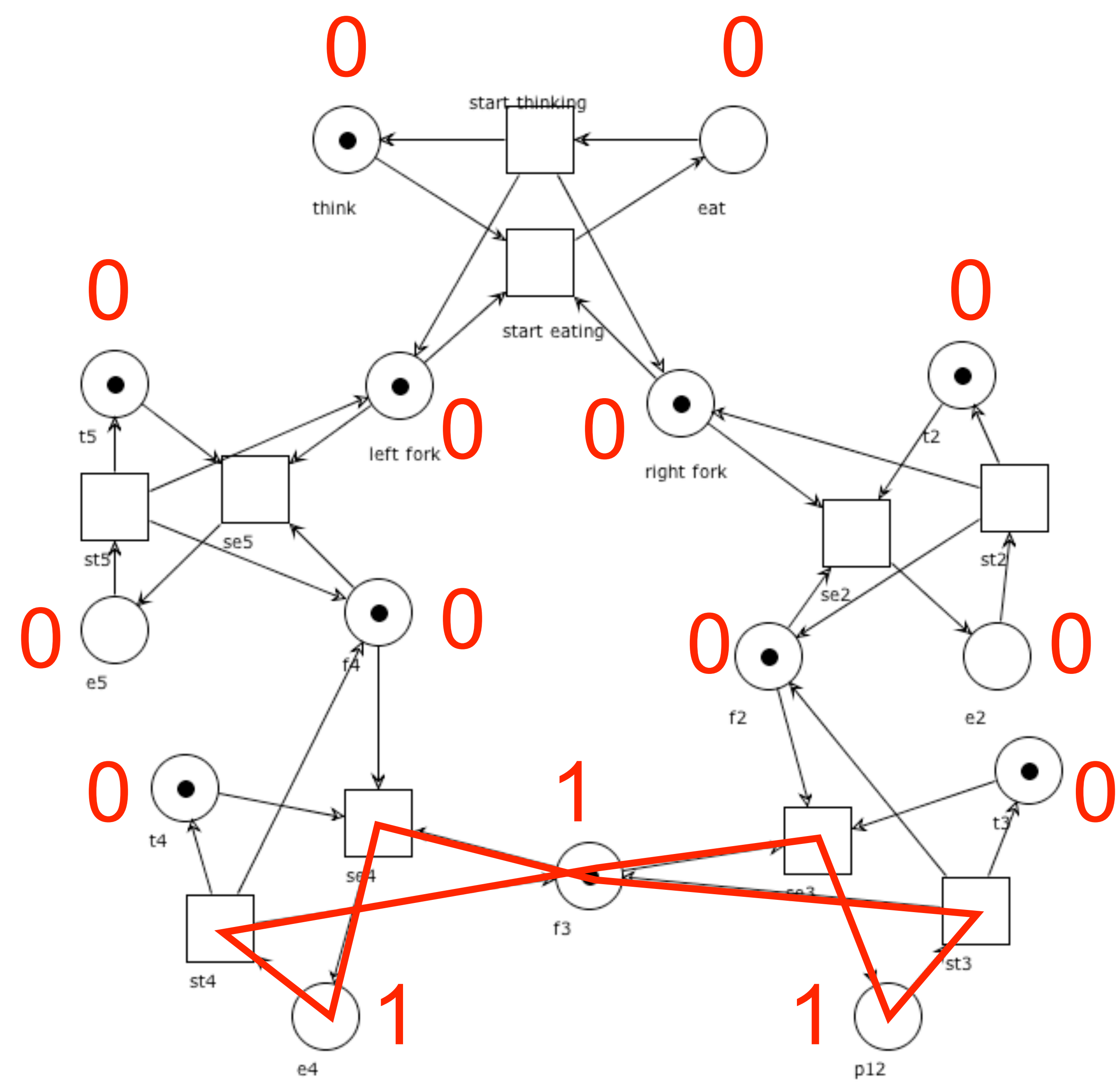




Exercises

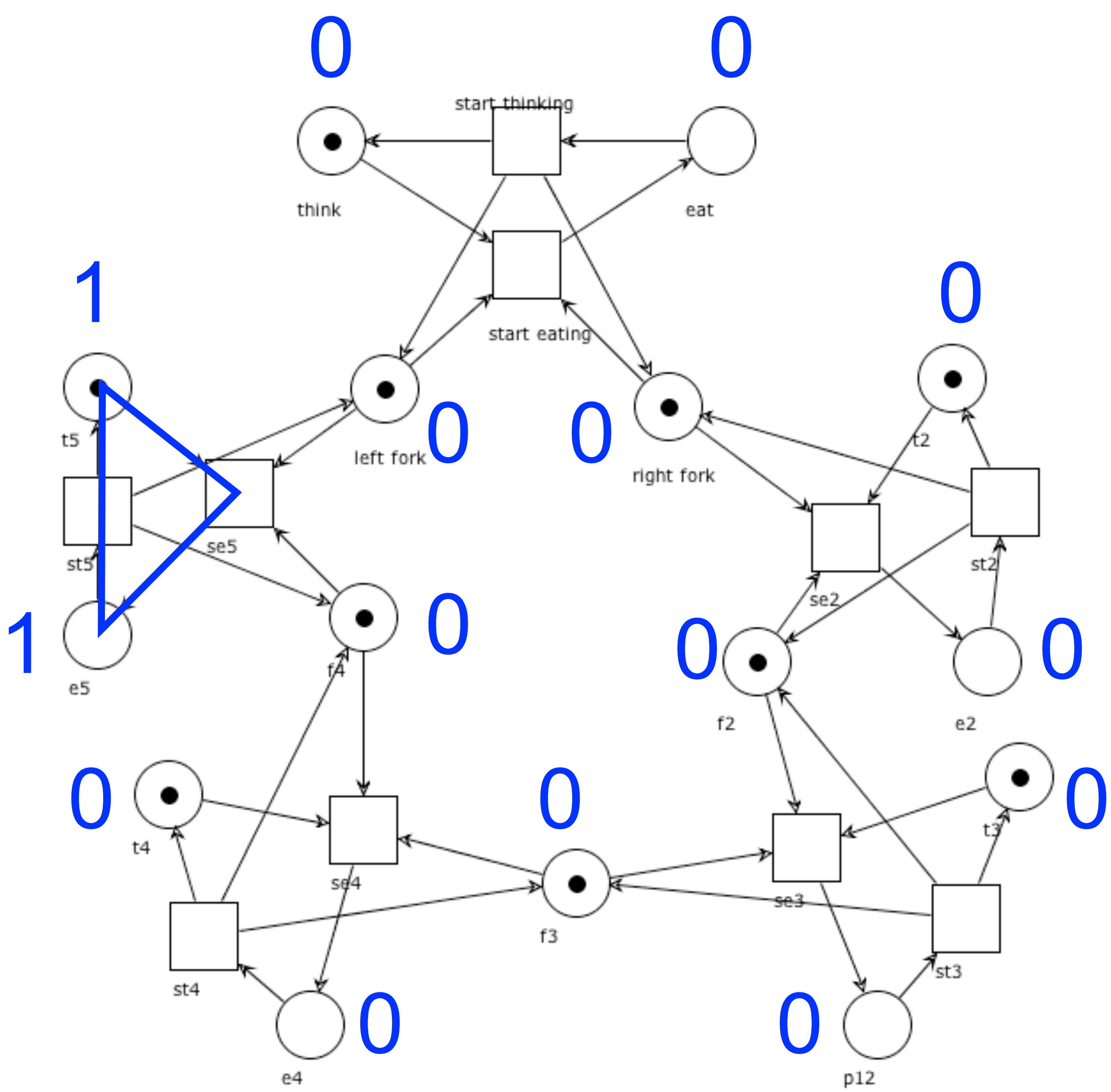
Symmetrically

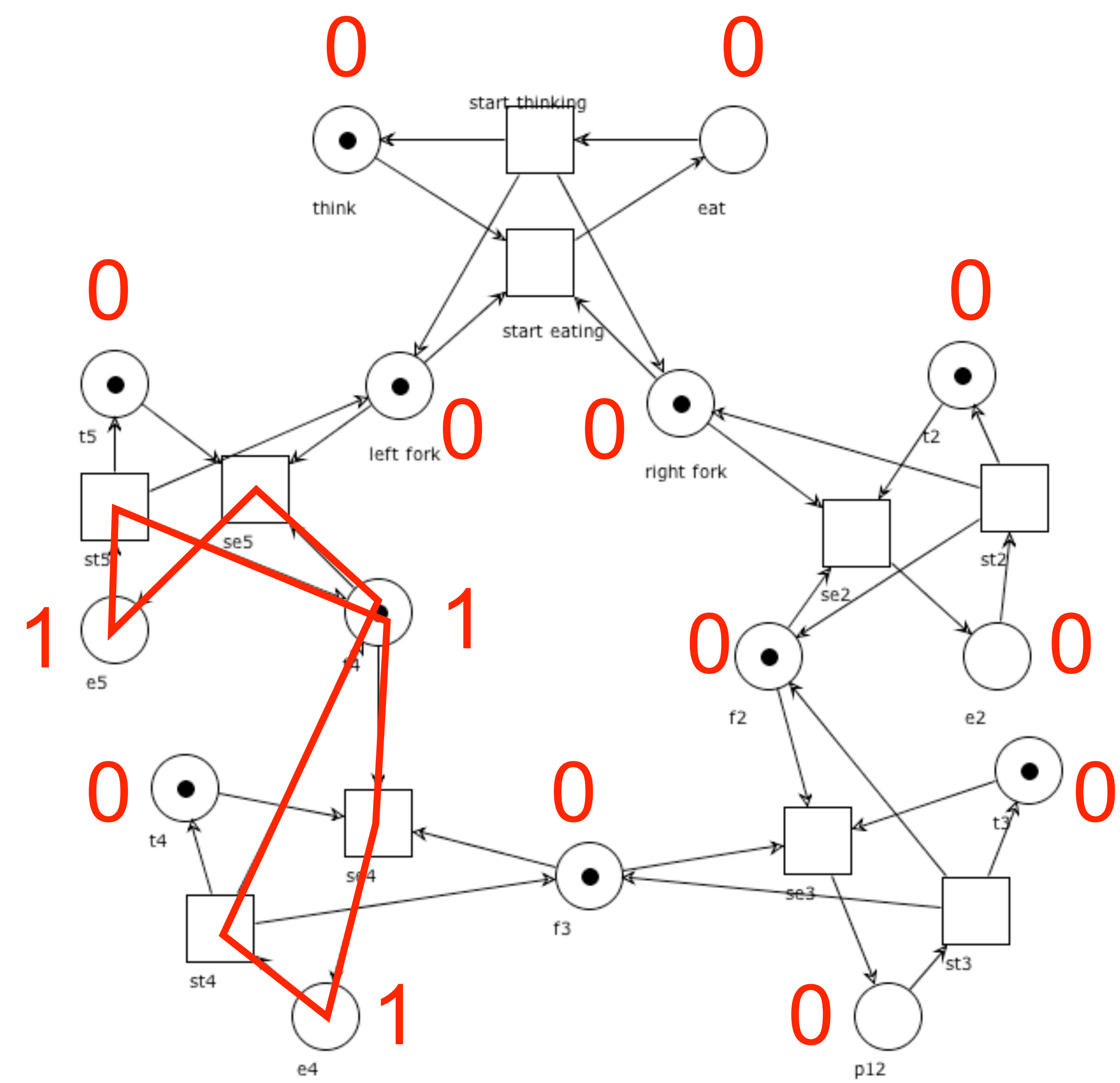




Exercises

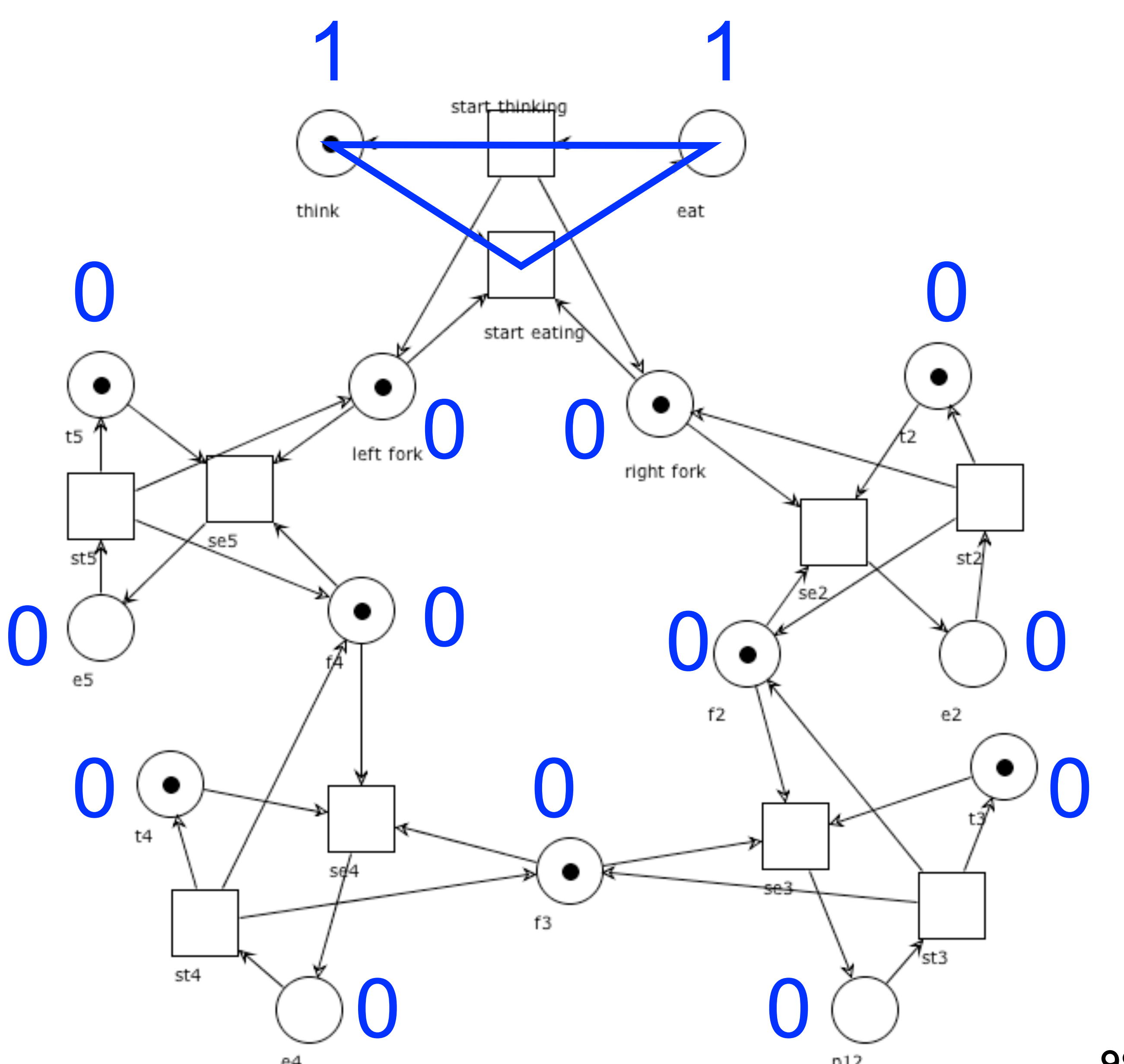
Symmetrically

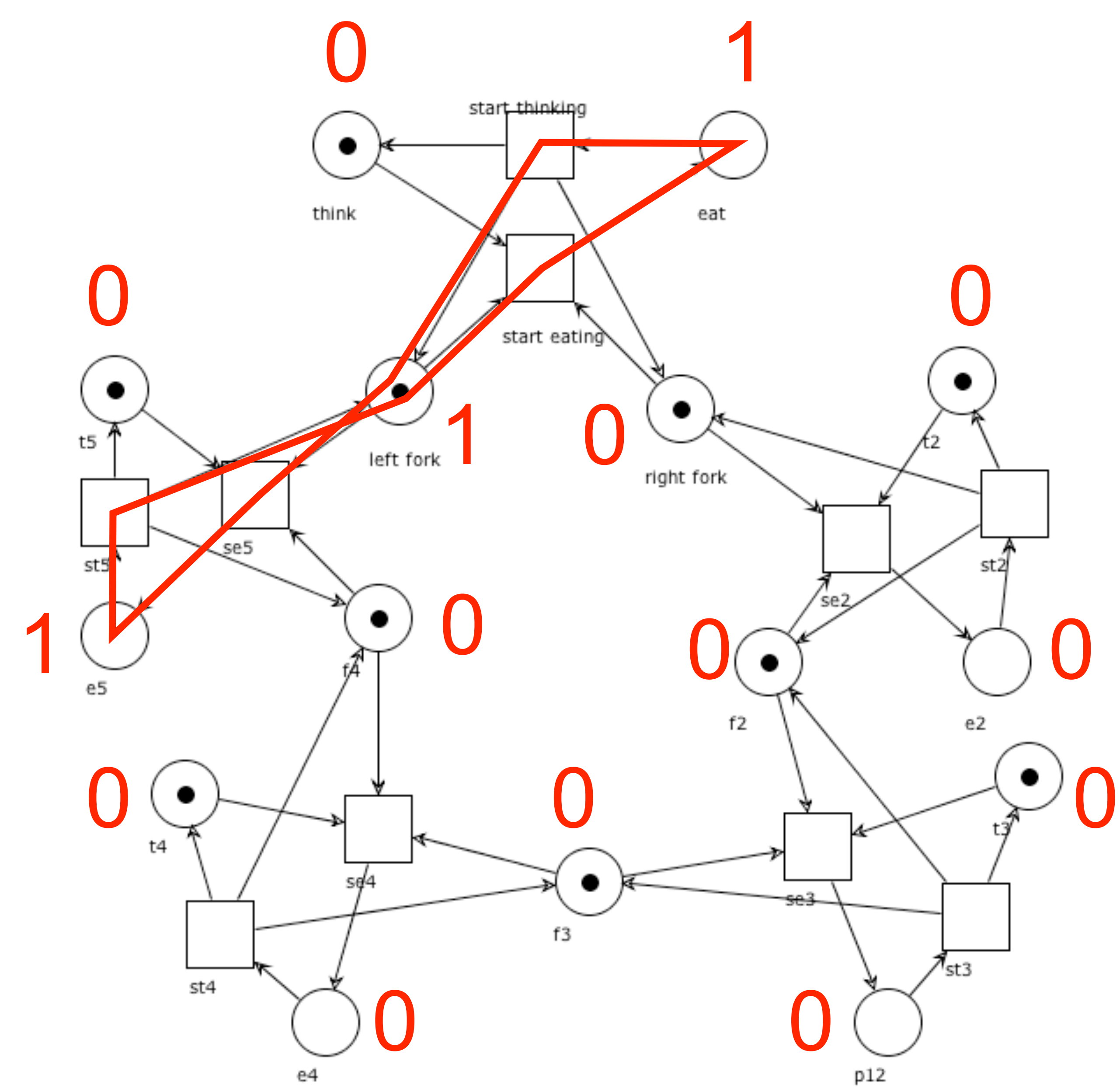




Exercises

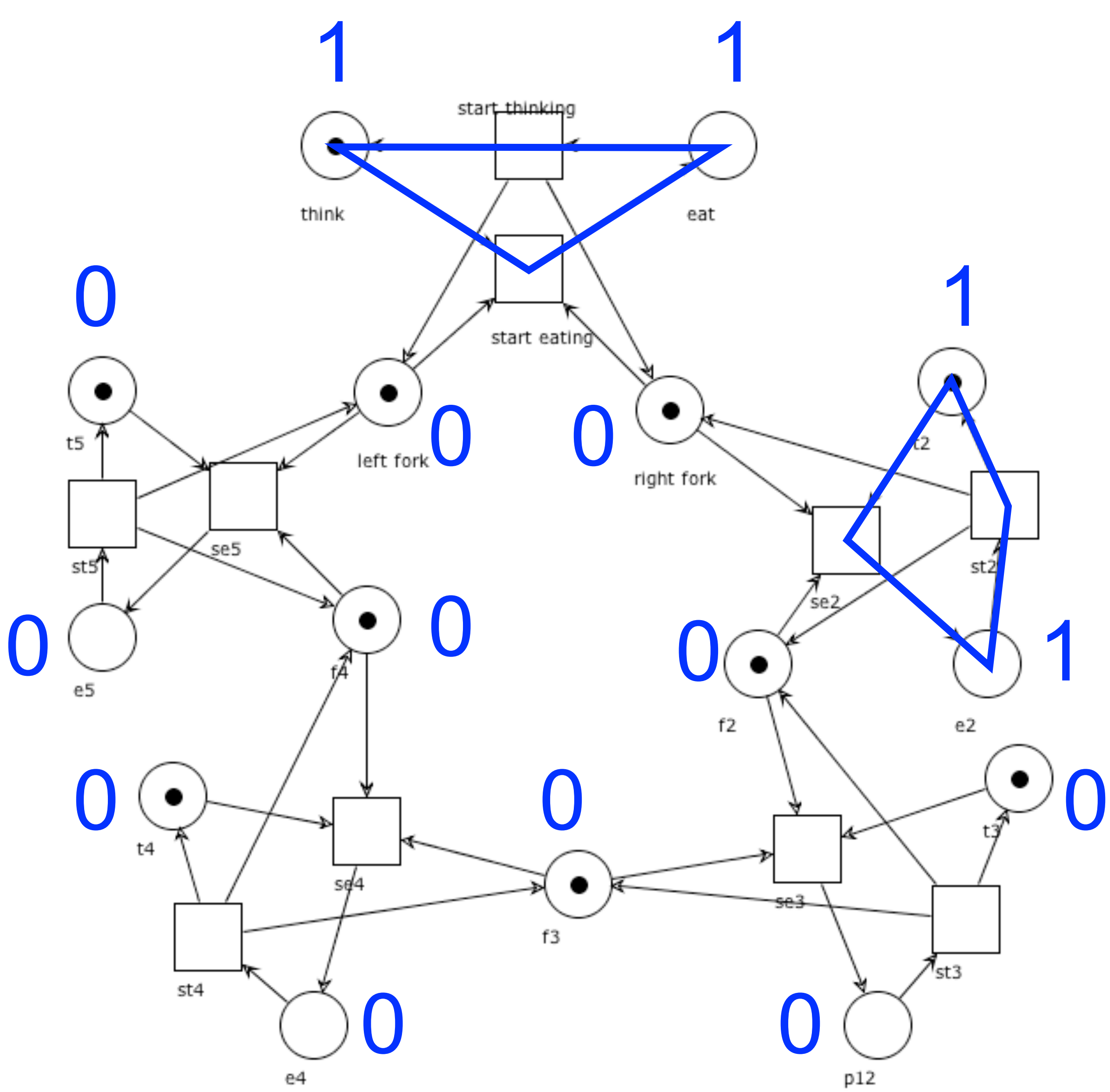
Define two (linearly independent) S-invariants

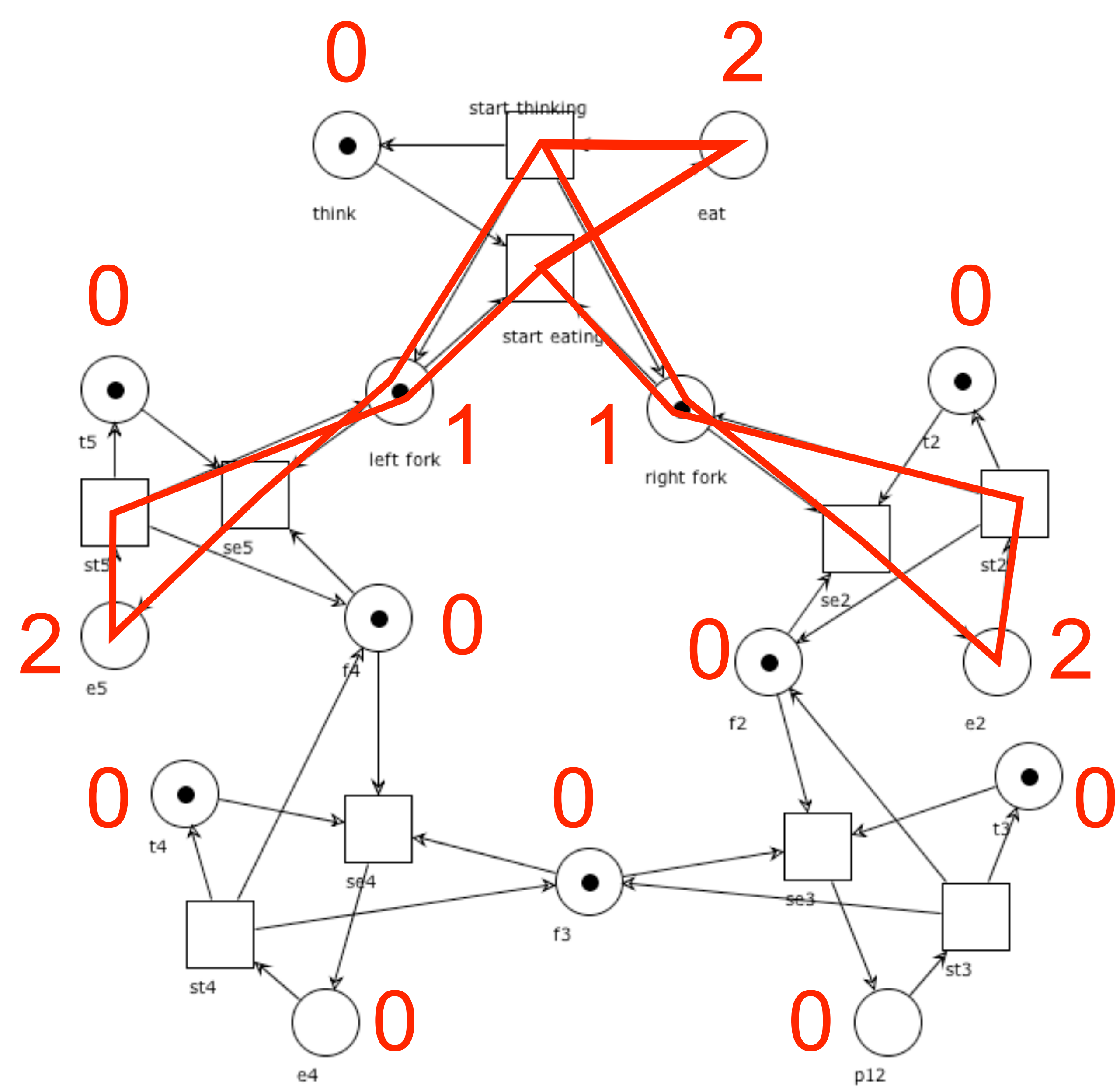




Exercises

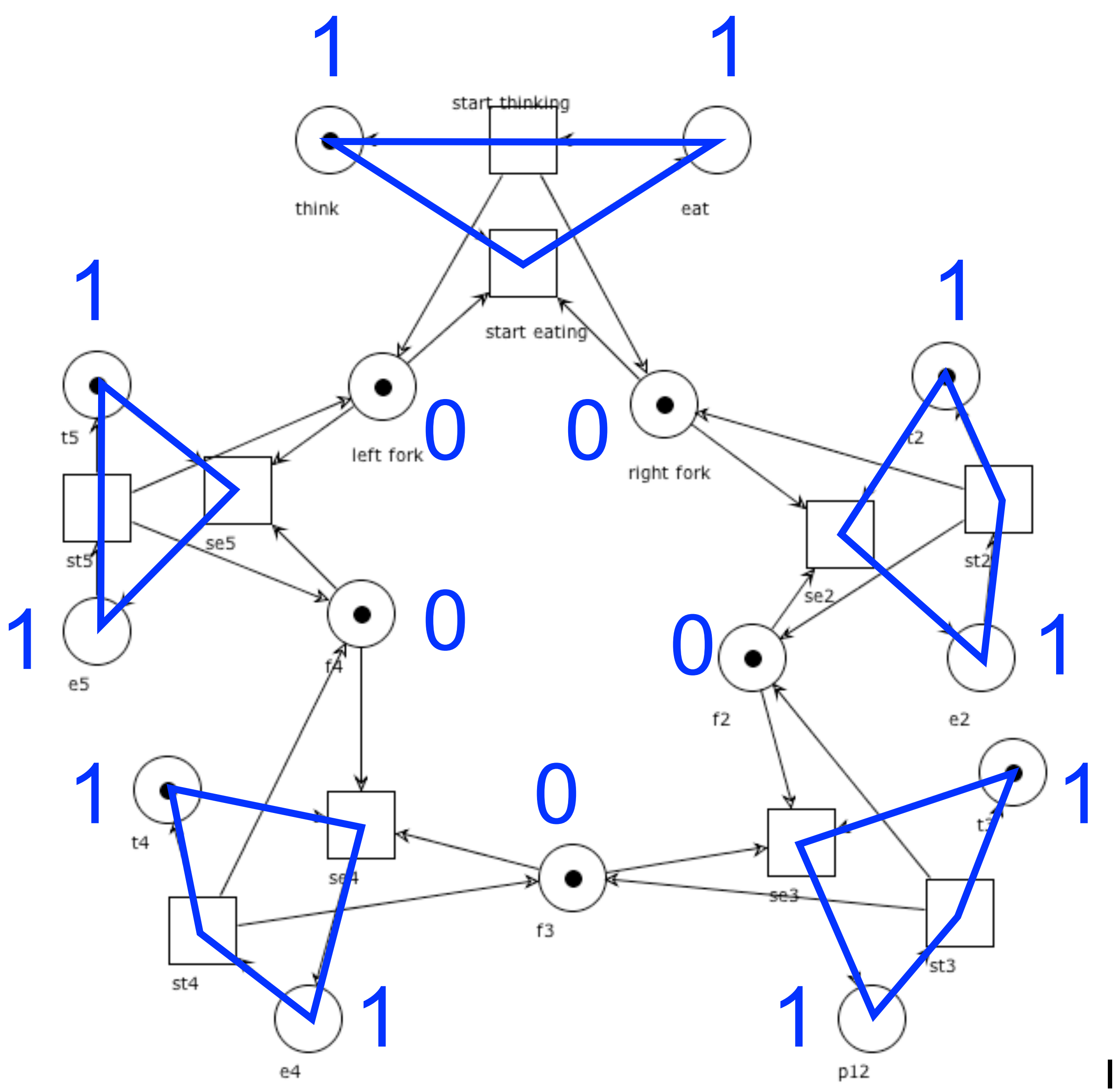
The sum of S-invariants is still an S-invariant

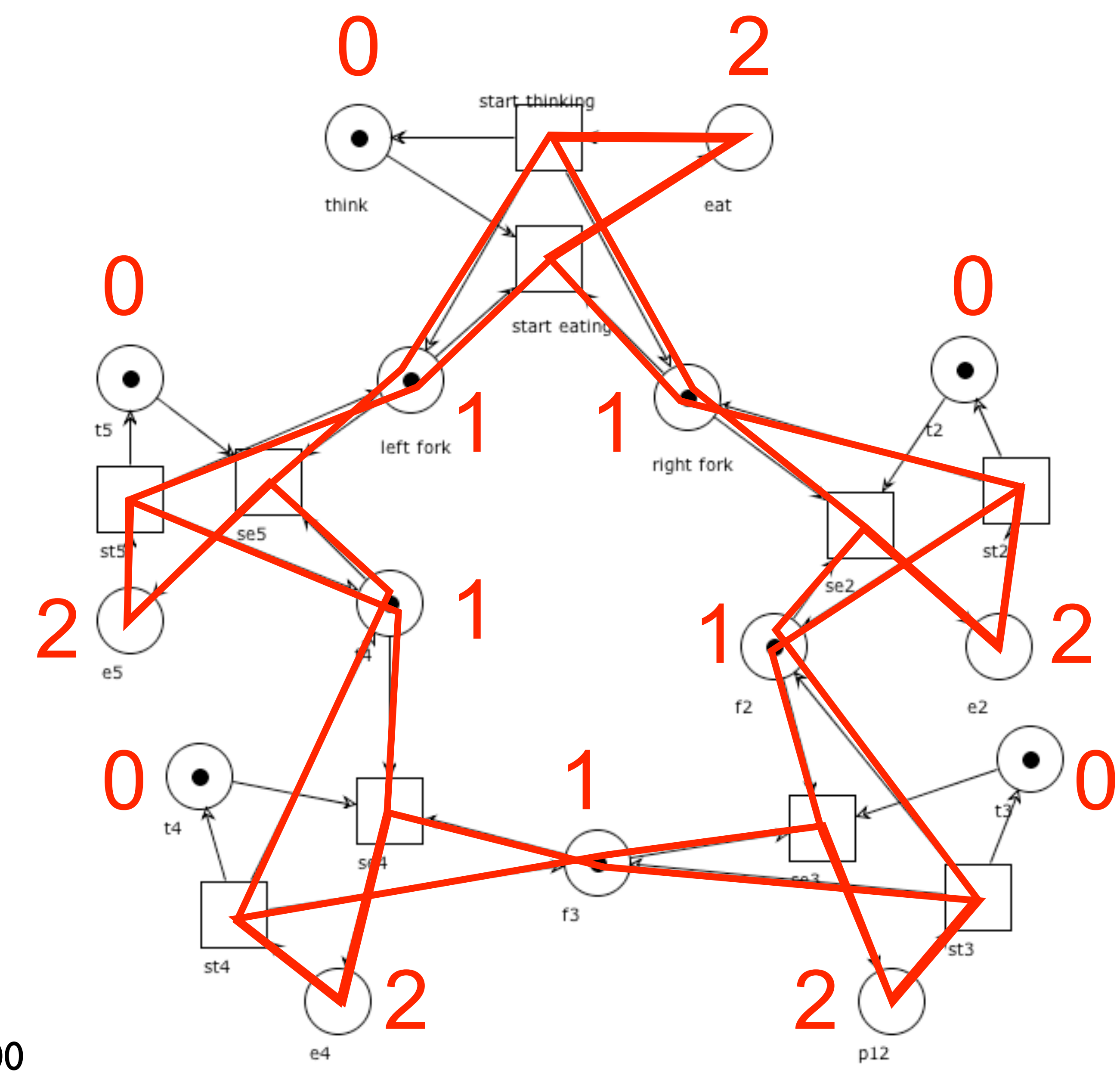




Exercises

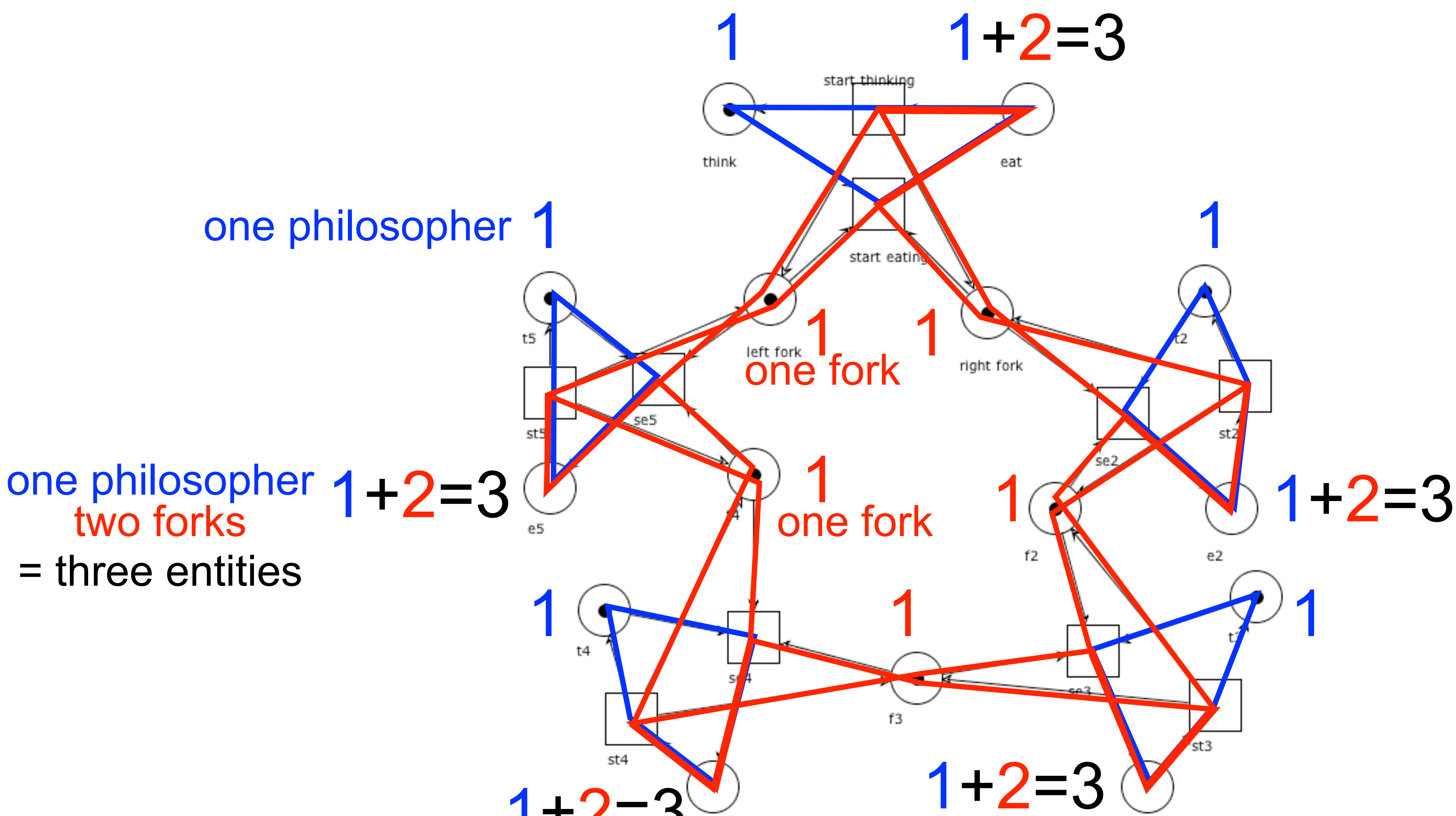
The sum of S-invariants is still an S-invariant





Exercises

The sum of S-invariants is still an S-invariant



S-invariants and system properties

(Semi-)Positive S-invariants

The S-invariant $\mathbf{I}$ is **semi-positive** if $\mathbf{I} > \mathbf{0}$
(i.e. $\mathbf{I} \geq \mathbf{0}$ and $\mathbf{I} \neq \mathbf{0}$)

all entries are non-negative and at least one is positive

The **support** of $\mathbf{I}$ is: $\langle \mathbf{I} \rangle = \{ p \mid \mathbf{I}(p) > 0 \}$

set of places with positive weights

The S-invariant $\mathbf{I}$ is **positive** if $\mathbf{I} \succ \mathbf{0}$ **all entries are positive**
(i.e. $\mathbf{I}(p) > 0$ for any place $p \in P$)
(i.e. $\langle \mathbf{I} \rangle = P$)

A (semi-positive) S-invariant whose coefficients are all 0 and 1 is called **uniform**

Note

Notation: $\bullet S = \bigcup_{s \in S} \bullet s$

Every semi-positive invariant
satisfies the equation

transitions that produce tokens in some places of the support $\bullet\langle \mathbf{I} \rangle = \langle \mathbf{I} \rangle\bullet$ transitions that consume tokens from some places of the support

pre-sets of support equal post-sets of support

(the result holds for both S-invariants and T-invariants)

A sufficient condition for boundedness

Theorem:
If (P, T, F, M_0) has a positive S-invariant then it is bounded

Let $M \in [\, M_0 \,\rangle$ and let $\mathbf{I}$ be a positive S-invariant.

Let $p \in P$. Then $\mathbf{I}(p)M(p) \leq \mathbf{I} \cdot M = \mathbf{I} \cdot M_0$

Since $\mathbf{I}$ is positive, we can divide by $\mathbf{I}(p)$:
$M(p) \leq (\mathbf{I} \cdot M_0)/\mathbf{I}(p)$

$$\mathbf{I} \cdot M = \sum_{q \in P} \mathbf{I}(q)M(q)$$

Consequences of previous theorem

By exhibiting a positive S-invariant we can prove that the system is **bounded for any initial marking**

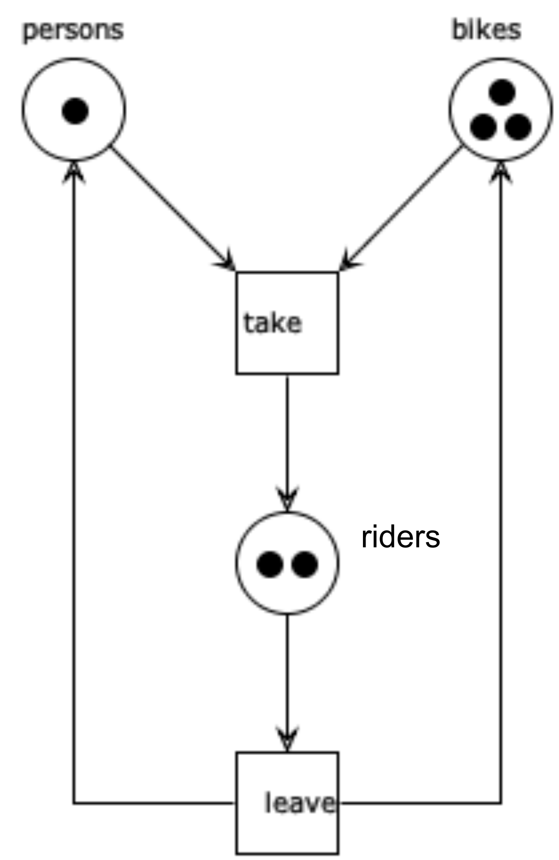
Note that all places in the support of a semi-positive S-invariant are **bounded for any initial marking**

$$M(p) \leq \frac{\mathbf{I} \cdot M_0}{\mathbf{I}(p)}$$

this value is independent from the reachable marking M

Example

To prove that the system is bounded we can just exhibit a positive S-invariant

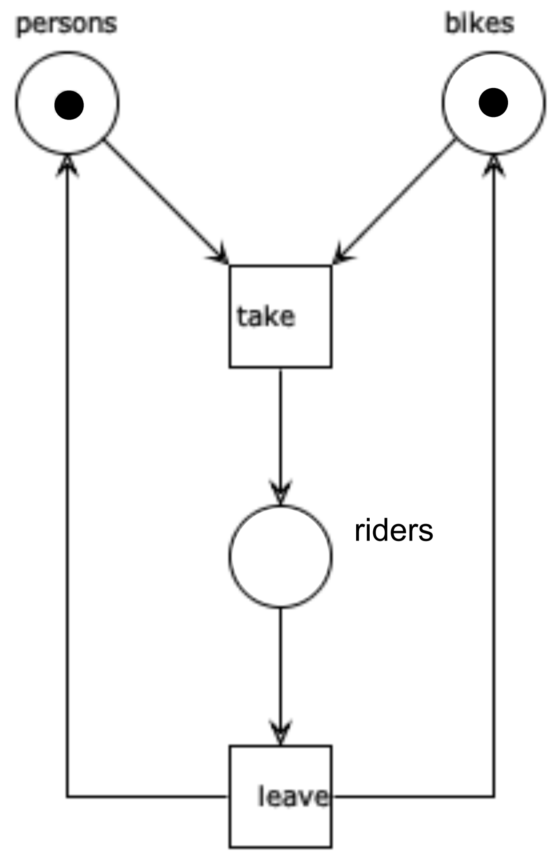


I = [1 1 2]

p1 - persons
p2 - bikes
p3 - riders

Example

How many tokens are at most in p_3?



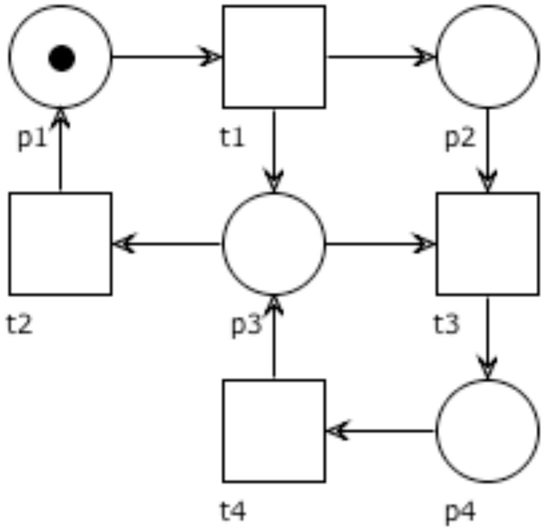
p1 - persons
p2 - bikes
p3 - riders

I = [1 1 2]

$$\frac{\mathbf{I} \cdot M_0}{\mathbf{I}(p_3)} = \frac{2}{2} = 1$$

Example

How many tokens are at most in p_3?

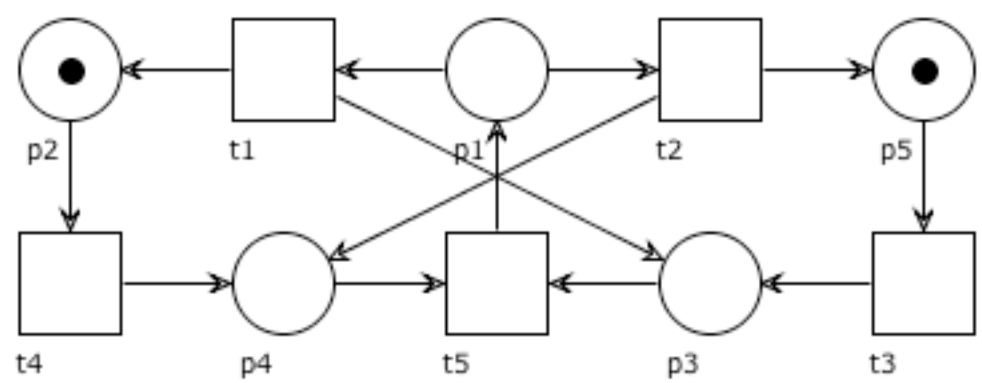


I = [1 0 1 1]

$$\frac{\mathbf{I} \cdot M_0}{\mathbf{I}(p_3)} = \frac{1}{1} = 1$$

Question time

live, **deadlock-free**, **bounded**, **safe**, cyclic
Prove boundedness by exhibiting an S-invariant



Question time

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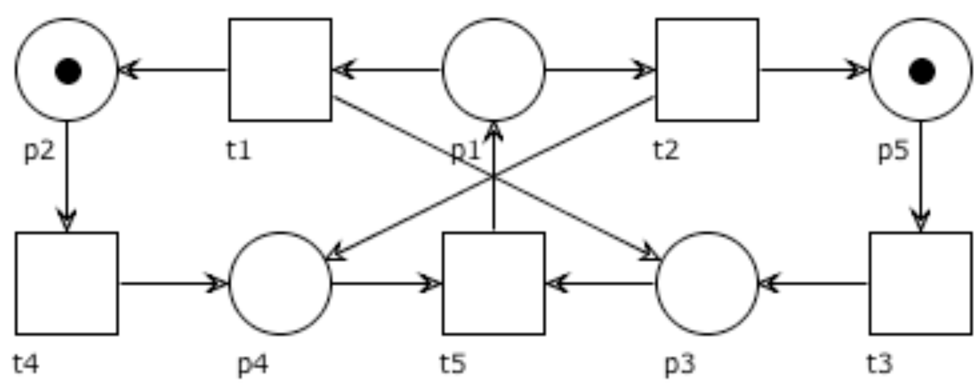
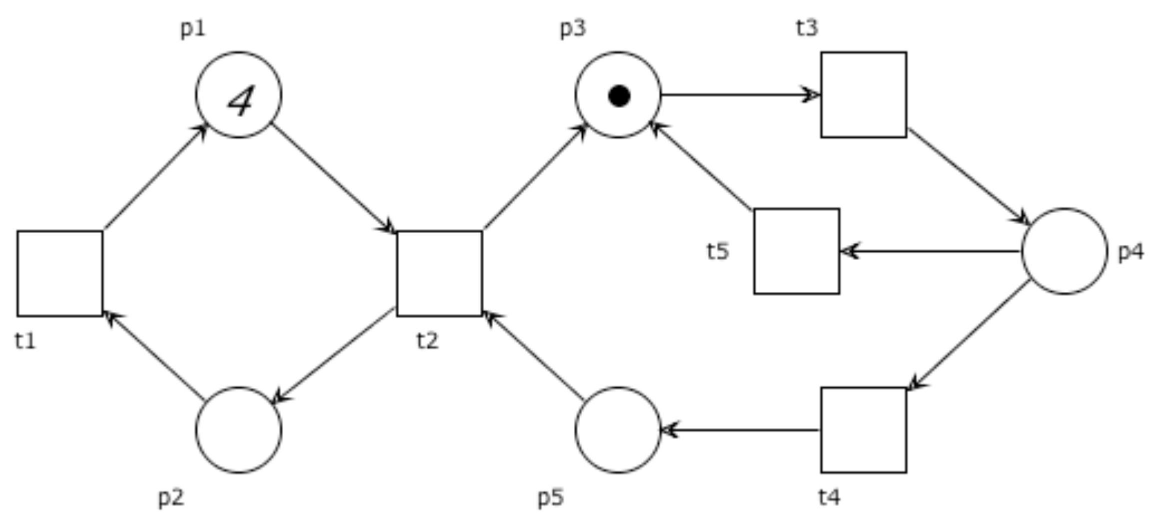


I = [2 1 1 1 1]

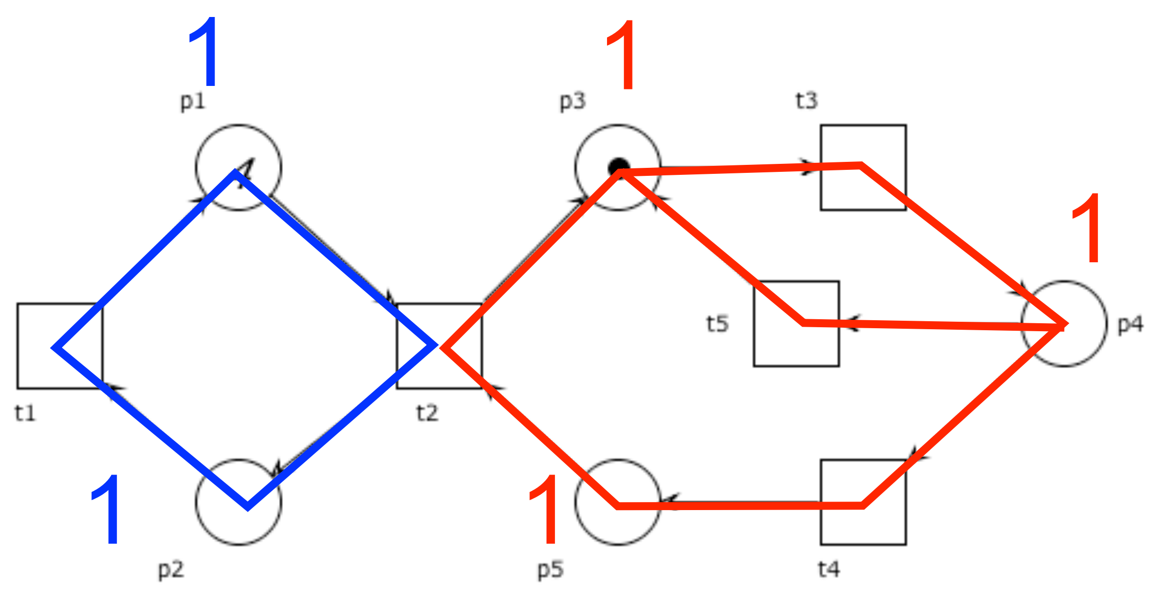
Exercises

Find a positive S-invariant for the net below



Exercises

Find a positive S-invariant for the net below



A necessary condition for liveness

Theorem:
If (P, T, F, M_0) is live then for every semi-positive invariant $\mathbf{I}$:

$$\mathbf{I} \cdot M_0 > 0$$

Let $p \in \langle \mathbf{I} \rangle$ and take any $t \in \bullet p \cup p \bullet$.

By liveness, there are $M, M' \in [\, M_0 \,\rangle$ with $M \xrightarrow{t} M'$

Then, $M(p) > 0$ (if $t \in p\bullet$) or $M'(p) > 0$ (if $t \in \bullet p$)

If $M(p) > 0$, then $\mathbf{I} \cdot M \geq \mathbf{I}(p) M(p) > 0$
If $M'(p) > 0$, then $\mathbf{I} \cdot M' \geq \mathbf{I}(p) M'(p) > 0$

In any case, $\mathbf{I} \cdot M_0 = \mathbf{I} \cdot M = \mathbf{I} \cdot M' > 0$

$$\mathbf{I} \cdot M = \sum_{q \in P} \mathbf{I}(q) M(q)$$

Consequence of previous theorem

If we find a semi-positive invariant such that

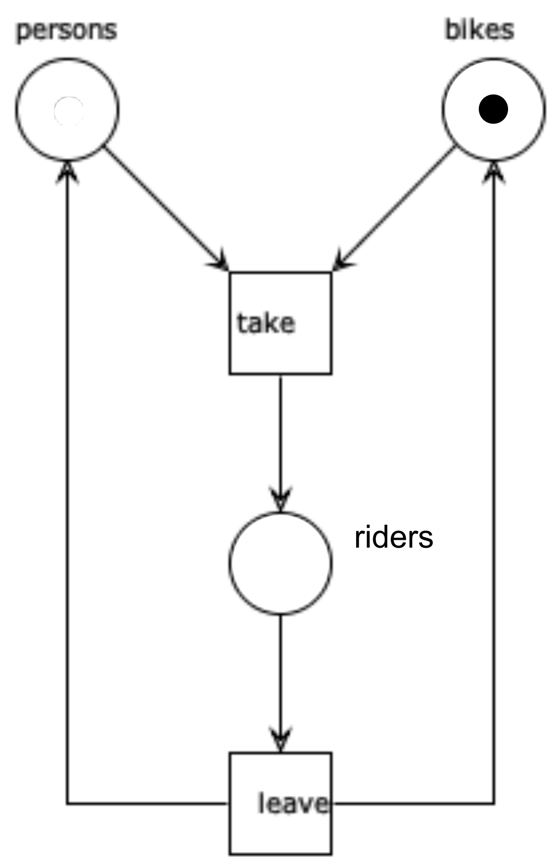
$$\mathbf{I} \cdot M_0 = 0$$

Then we can conclude that the system **is not live**

Example

the system is not live

It is immediate to check the counter-example



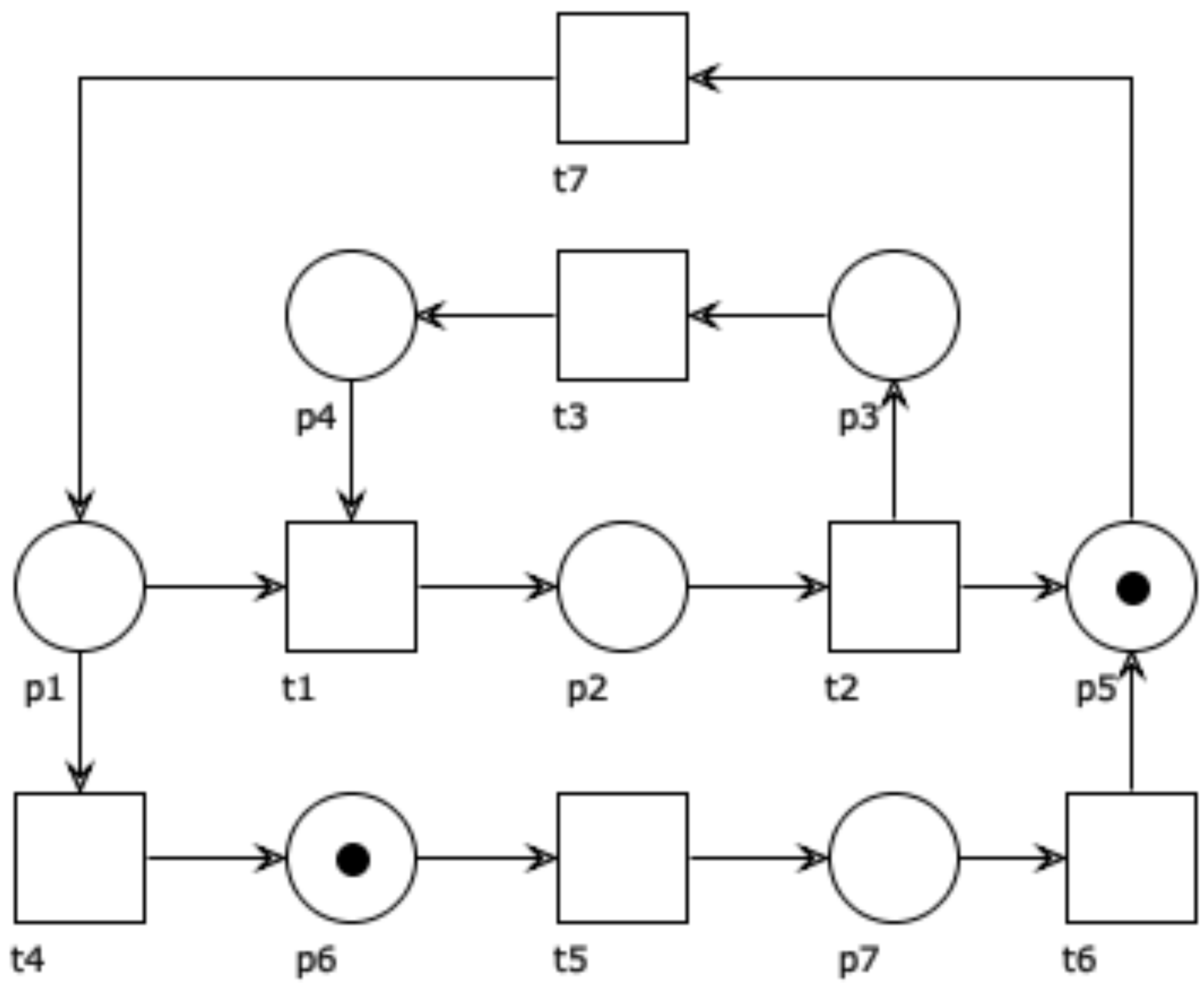
p1 - persons
p2 - bikes
p3 - riders

$$\mathbf{I} = [\,1 \;\; 0 \;\; 1\,]$$

$$\underbrace{[\,1\;0\;1\,]}_{\mathbf{I}} \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{M_0} = 0$$

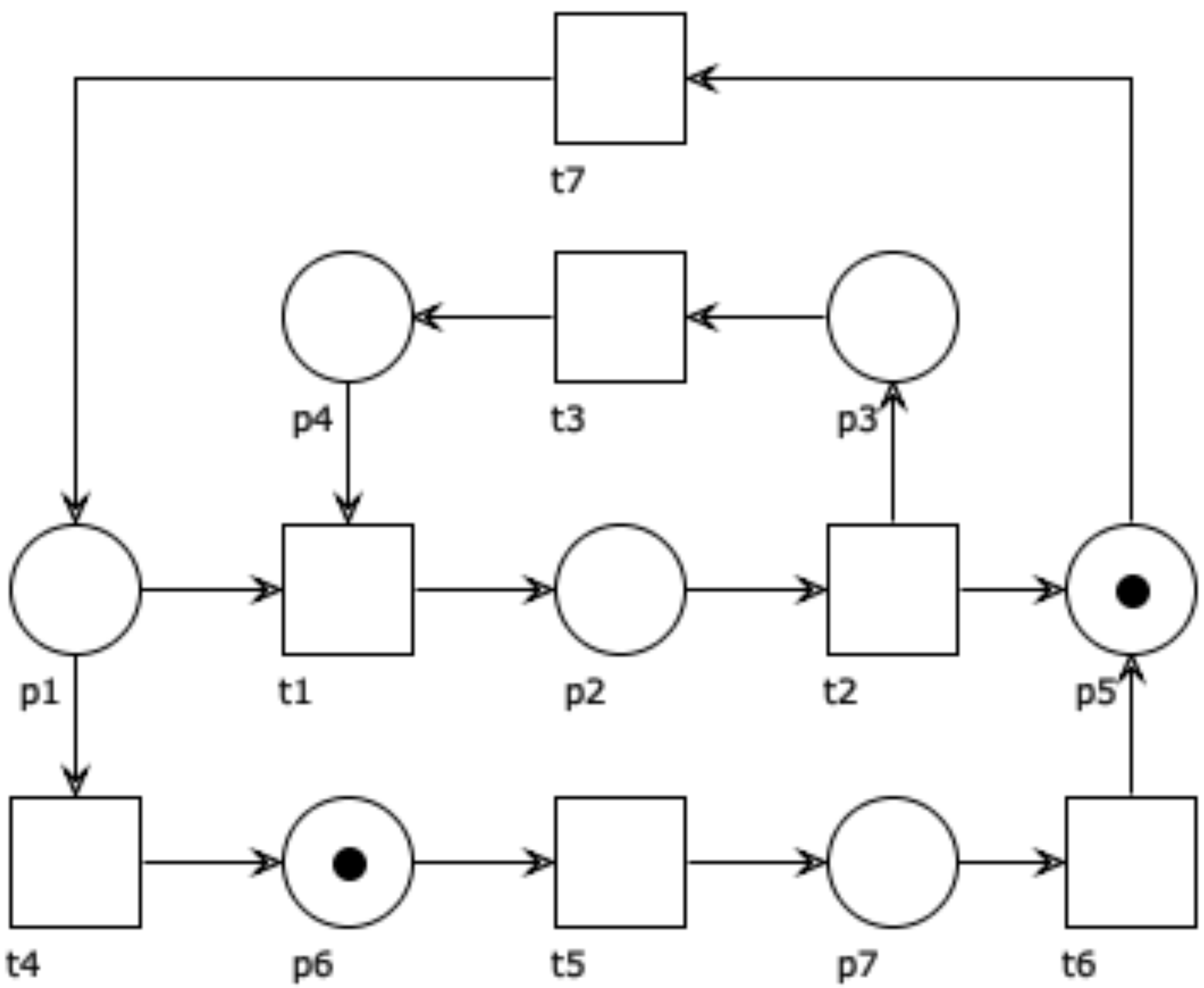
Exercises

Find an S-invariant that proves the net non-live



Exercises

Find an S-invariant that proves the net non-live

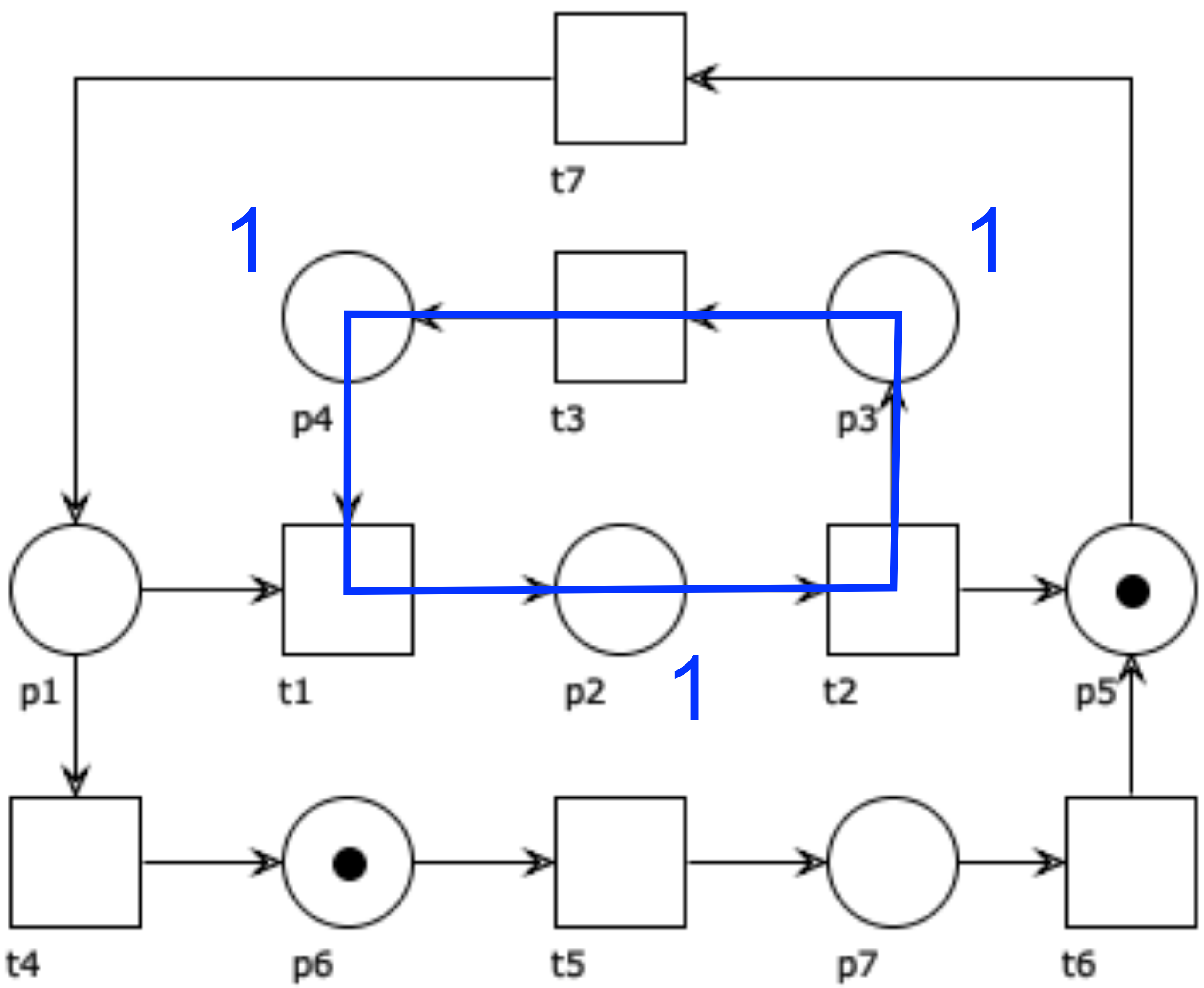


I = [? ? ? ? ? ? ?]

$$\mathbf{I} \cdot M_0 = 0$$

Exercises

Find an S-invariant that proves the net non-live



I = [0 1 1 1 0 0 0]

$$\mathbf{I} \cdot M_0 = 0$$

Markings that agree on all S-invariant

Definition: M and M' **agree on all S-invariants** if for every S-invariant **I** we have $\mathbf{I} \cdot M = \mathbf{I} \cdot M'$

Note: by properties of linear algebra,
this corresponds to require that the equation on **y**
$\mathbf{N} \cdot \mathbf{y} = M' - M$ has some rational-valued solution

Remark: In general, there can exist M and M' that agree on all S-invariants but such that none of them is reachable from the other

S-invariants: recap

Positive S-invariant => boundedness
Unboundedness => no positive S-invariant

Semi-positive S-invariant **I** and liveness => $\mathbf{I} \cdot M_0 > 0$
Semi-positive S-invariant **I** and $\mathbf{I} \cdot M_0 = 0$ => non-live

S-invariant **I** and M reachable => $\mathbf{I} \cdot M = \mathbf{I} \cdot M_0$
S-invariant **I** and $\mathbf{I} \cdot M \neq \mathbf{I} \cdot M_0$ => M not reachable

S-invariants: pay attention to implication

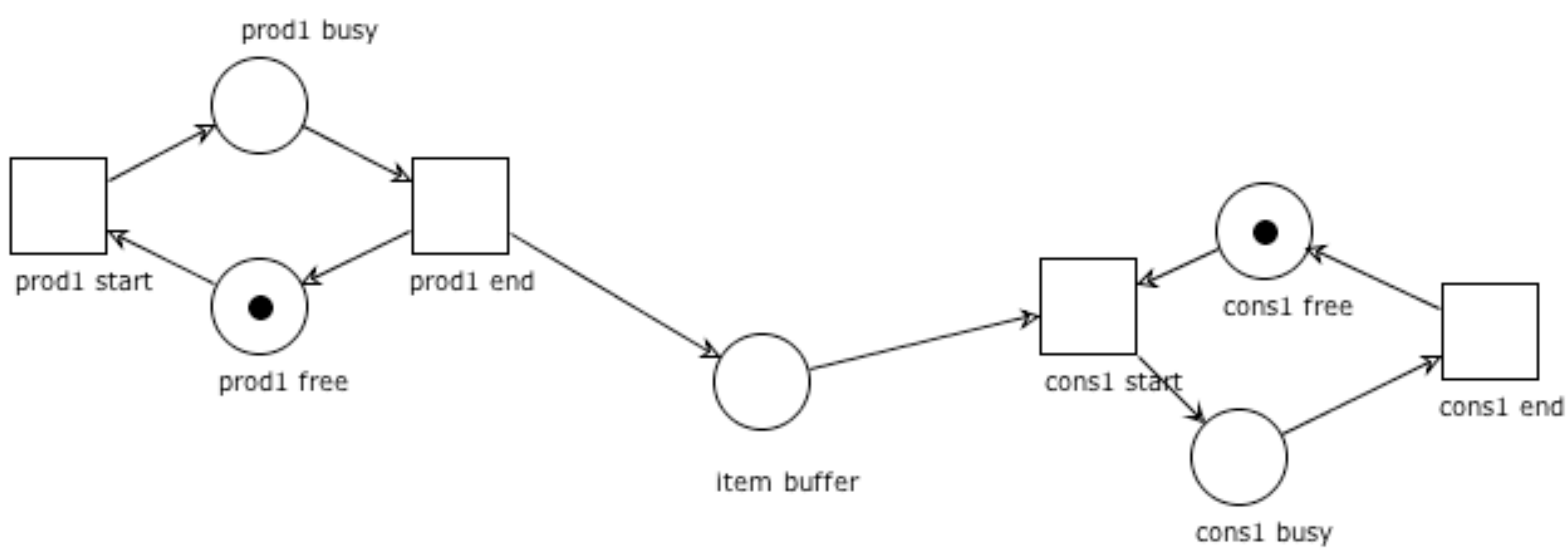
No positive S-invariant => **maybe** unbounded

Semi-positive S-invariant **I** and **I**$\cdot M_0 > 0$ => **maybe** live

S-invariant **I** and **I**$\cdot M =$ **I**$\cdot M_0$ => **maybe** M reachable

Exercises

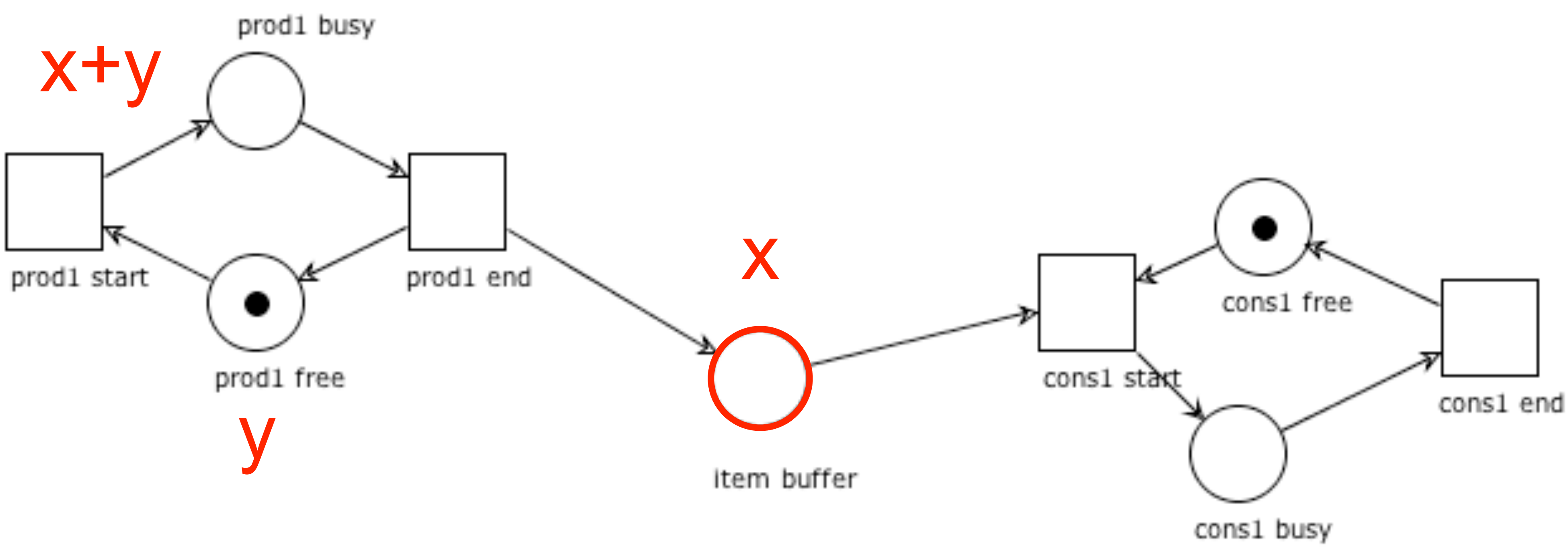
Can you find a positive S-invariant?



Exercises

Can you find a positive S-invariant?

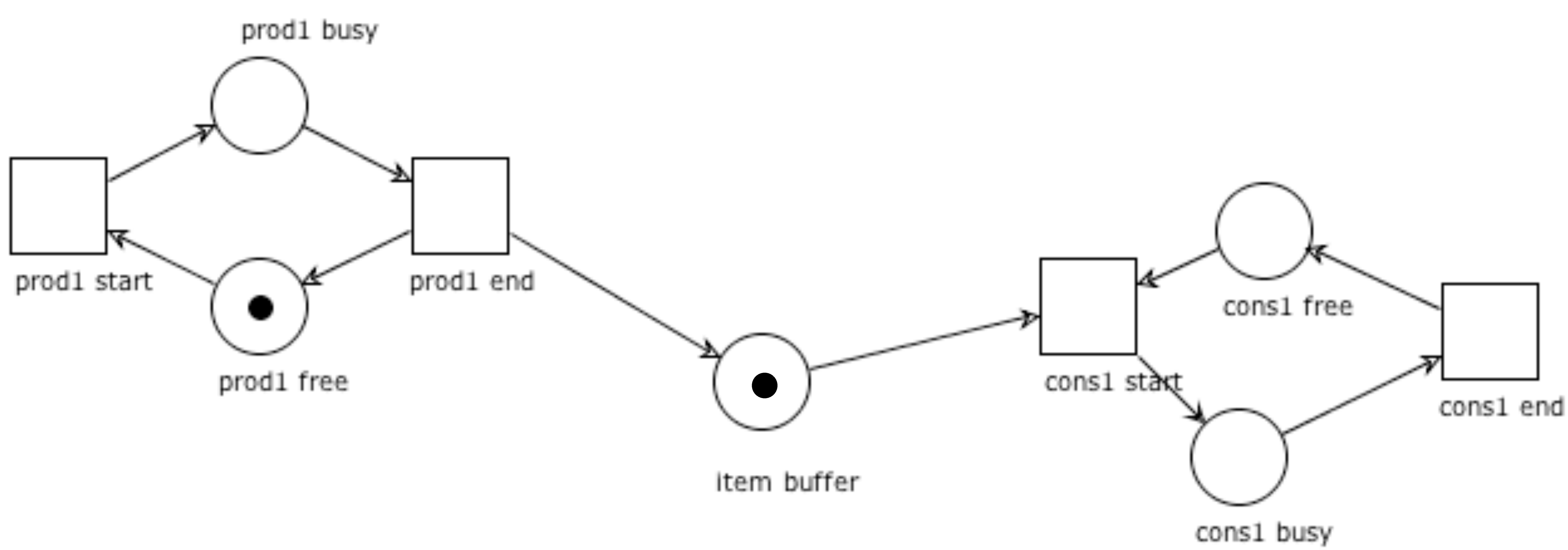
No, the existence of a positive S-invariant implies that the net is bounded and this net has one unbounded place



$x+y = y \Rightarrow x = 0$

Exercises

Prove that the system is not live by exhibiting a suitable S-invariant



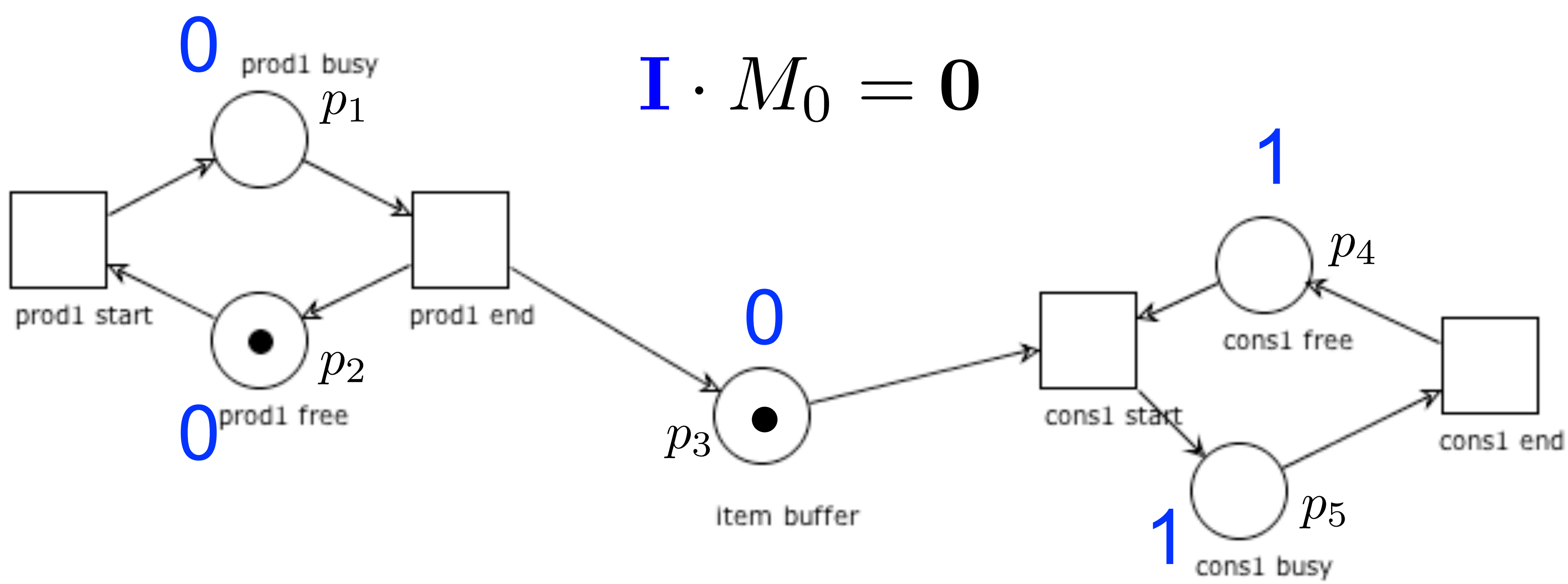
Exercises

Prove that the system is not live by exhibiting a suitable S-invariant

I = [? ? ? ? ? ?]

$$M_0 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{I} \cdot M_0 = \mathbf{0}$$



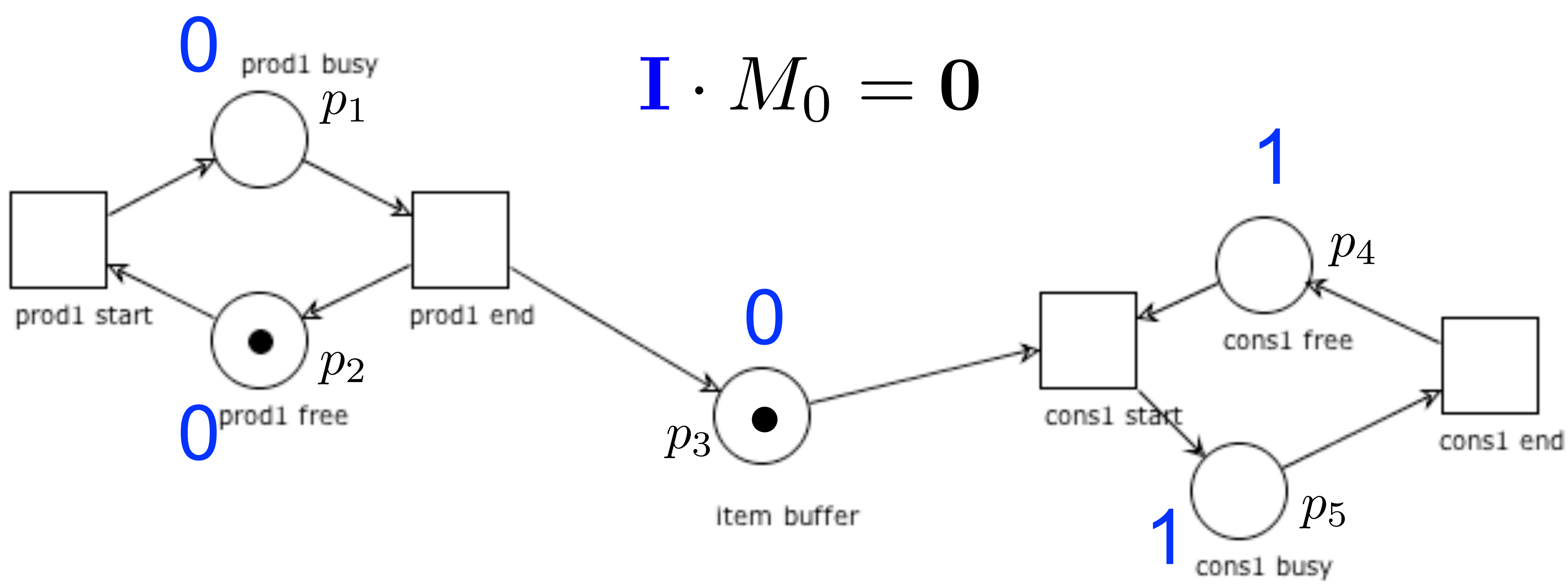
Exercises

Prove that the system is not live by exhibiting a suitable S-invariant

$$\mathbf{I} = [0 \quad 0 \quad 0 \quad 1 \quad 1]$$

$$M_0 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{I} \cdot M_0 = \mathbf{0}$$



Terminology

The S-invariant $\mathbf{I}$ is **semi-positive** if $\mathbf{I} > \mathbf{0}$
(i.e. $\mathbf{I} \geq \mathbf{0}$ and $\mathbf{I} \neq \mathbf{0}$)

all entries are non-negative and at least one is positive

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(i.e. $\langle \mathbf{I} \rangle = P$)

A (semi-positive) S-invariant whose coefficients are all 0 and 1 is called **uniform**

Recap: S-invariants can be used to

Prove boundedness
(by finding a positive S-invariant)

Disprove liveness
(by finding a semi-positive S-invariant $\mathbf{I}$ such that $\mathbf{I} \cdot M_0 = \mathbf{0}$)

Disprove M is reachable
(by finding an S-invariant $\mathbf{I}$ such that $\mathbf{I} \cdot M \neq \mathbf{I} \cdot M_0$)

T-invariants

Dual reasoning

The S-invariants of a net N are vectors satisfying the equation

$$\mathbf{x} \cdot \mathbf{N} = \mathbf{0}$$

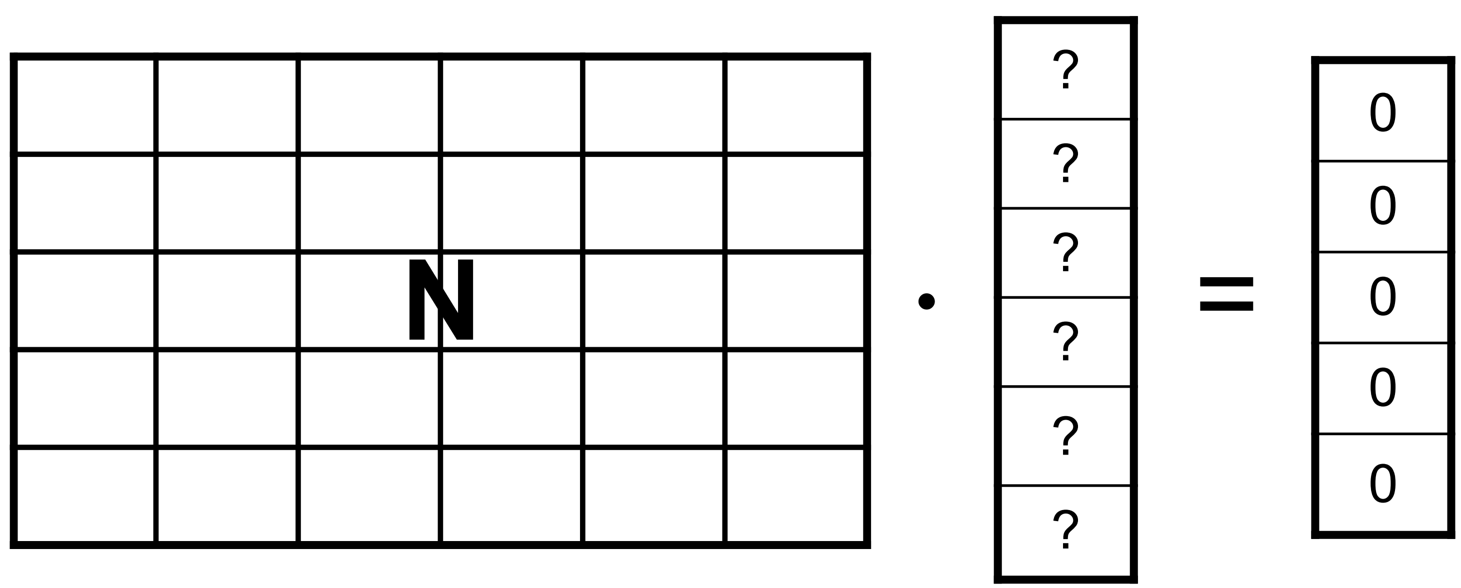
It seems natural to ask if we can find some interesting properties also for the vectors satisfying the equation

$$\mathbf{N} \cdot \mathbf{y} = \mathbf{0}$$

T-invariant
(aka transition-invariant)

Definition: A **T-invariant** of a net N=(P,T,F) is a rational-valued solution **y** of the equation

$$\mathbf{N} \cdot \mathbf{y} = \mathbf{0}$$



Fundamental property of T-invariants

Proposition: Let $M \xrightarrow{\sigma} M'$.

The Parikh vector $\vec{\sigma}$ is a T-invariant iff $M' = M$

$\Rightarrow$) By the marking equation lemma $M' = M + \mathbf{N} \cdot \vec{\sigma}$
Since $\vec{\sigma}$ is a T-invariant $\mathbf{N} \cdot \vec{\sigma} = \mathbf{0}$, thus $M' = M$.

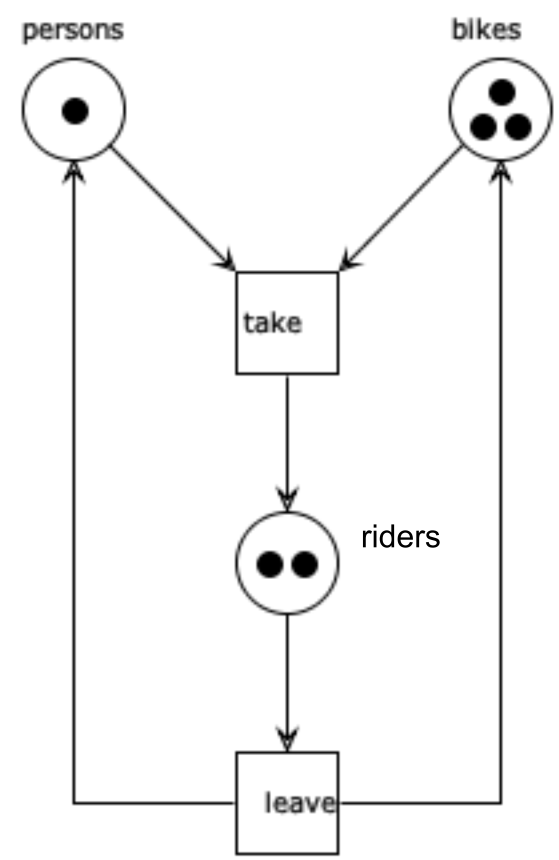
$\Leftarrow$) If $M \xrightarrow{\sigma} M$, by the marking equation lemma $M = M + \mathbf{N} \cdot \vec{\sigma}$
Thus $\mathbf{N} \cdot \vec{\sigma} = M - M = \mathbf{0}$ and $\vec{\sigma}$ is a T-invariant

Transition-invariant, intuitively

A transition-invariant assigns a **number of occurrences to each transition** such that any occurrence sequence comprising exactly those transitions leads to the same marking where it started (independently from the order of execution)

Example

An easy-to-be-found T-invariant



p1 - persons
p2 - bikes
p3 - riders

t1 - take
t2 - leave

t	l
[1	1]

Alternative definition of T-invariant

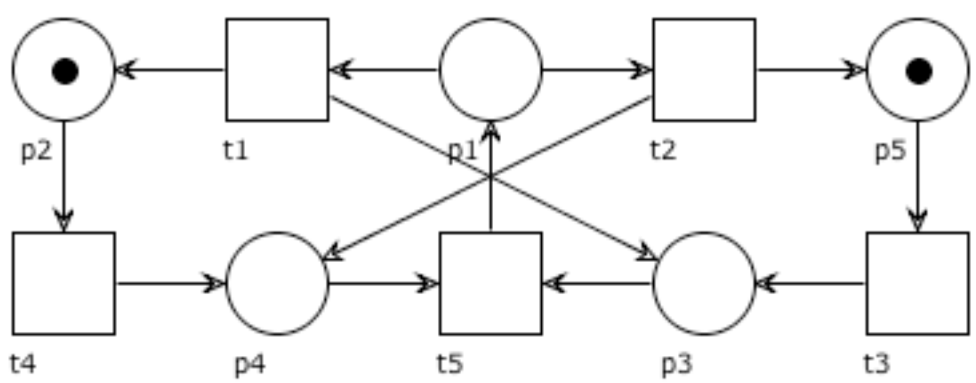
Proposition:

A mapping $\mathbf{J} : T \to \mathbb{Q}$ is a T-invariant of N iff for any $p \in P$:

$$\sum_{t \in \bullet p} \mathbf{J}(t) = \sum_{t \in p \bullet} \mathbf{J}(t)$$

Question time

Which of the following are T-invariants?

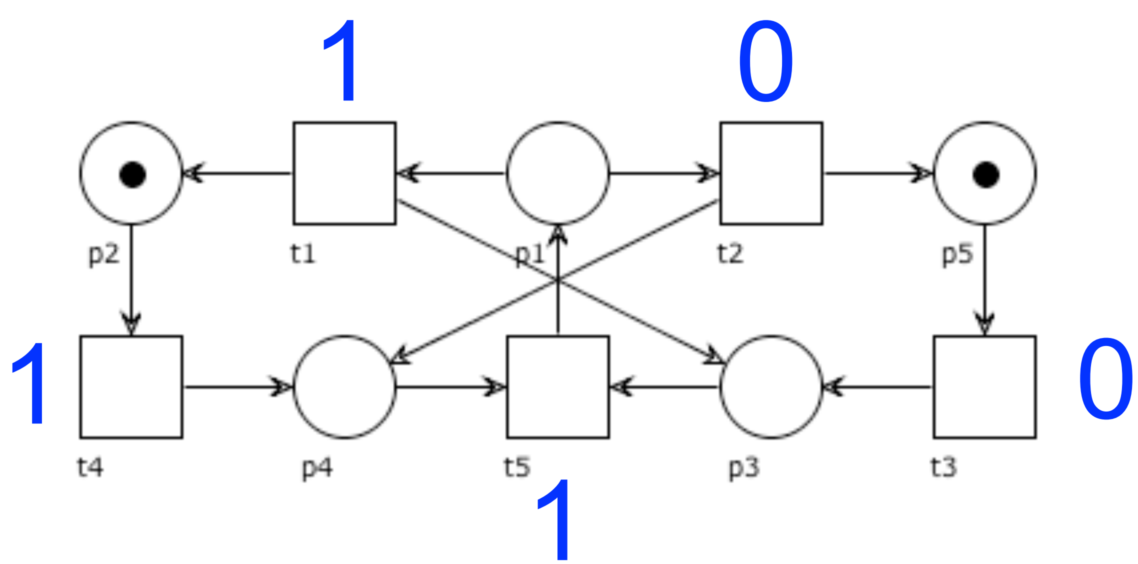


t_1	t_2	t_3	t_4	t_5
[1	0	0	1	1]
[1	1	2	1	2]
[1	1	1	0	2]
[1	1	1	1	2]
[0	1	1	0	1]

$$\forall p \in P, \ \sum_{t \in \bullet p} \mathbf{J}(t) \stackrel{?}{=} \sum_{t \in p \bullet} \mathbf{J}(t)$$

Question time

Which of the following are T-invariants?



t_1	t_2	t_3	t_4	t_5
[1	0	0	1	1]
[1	1	2	1	2]
[1	1	1	0	2]
[1	1	1	1	2]
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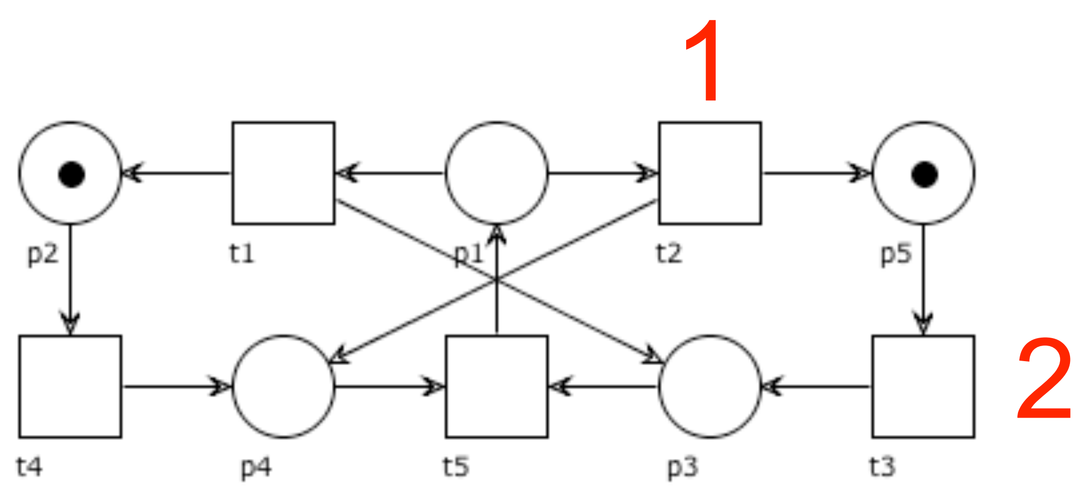

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Question time

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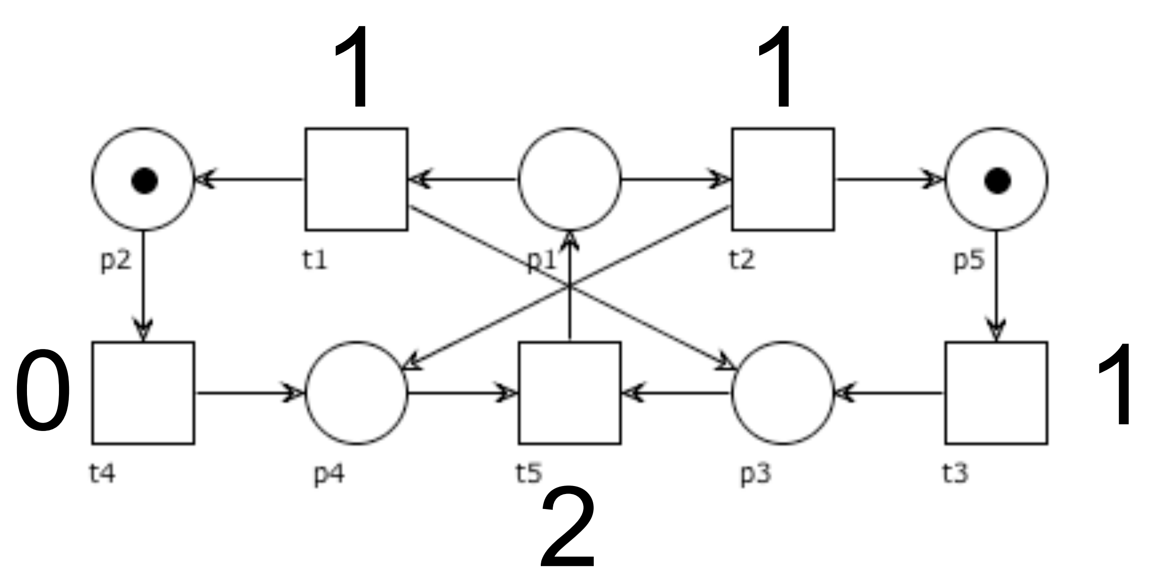

	t_1	t_2	t_3	t_4	t_5
	[1	0	0	1	1]
	[1	1	2	1	2]
→	[1	1	1	0	2]
	[1	1	1	1	2]
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Question time

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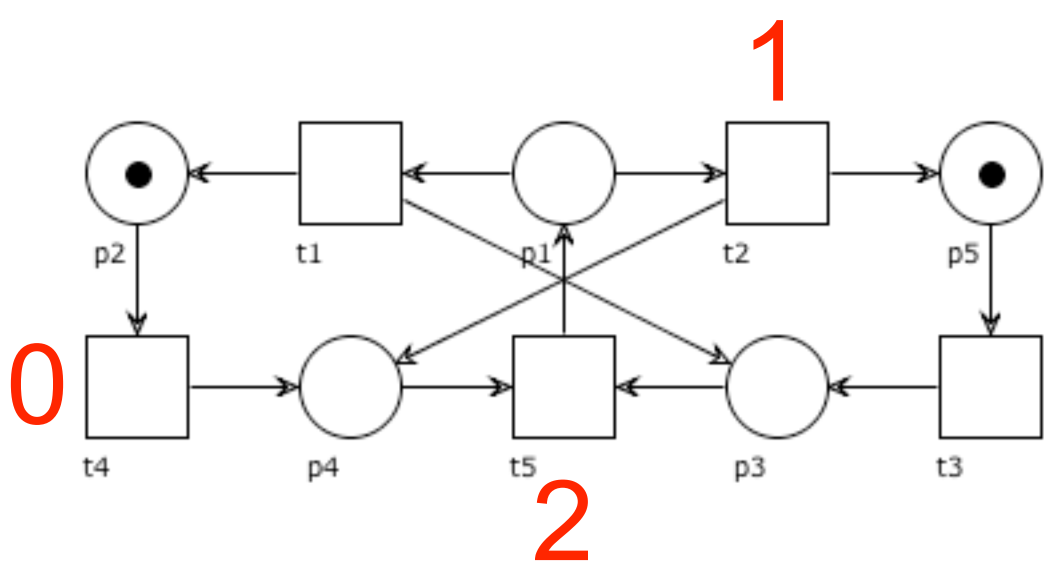

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Question time

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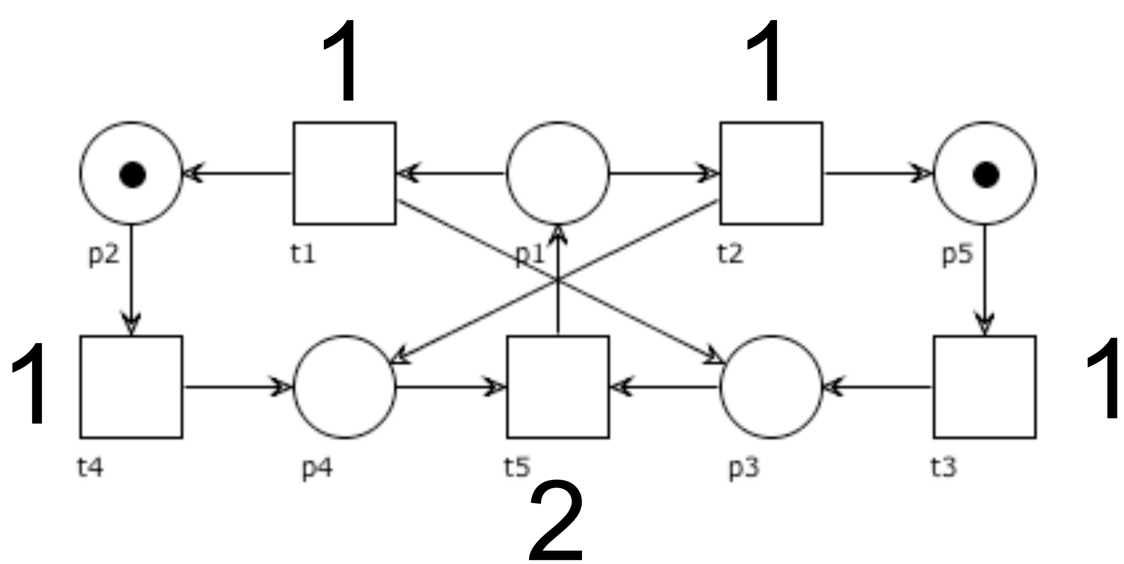

	t_1	t_2	t_3	t_4	t_5
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→	[1	1	1	1	2]
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Question time

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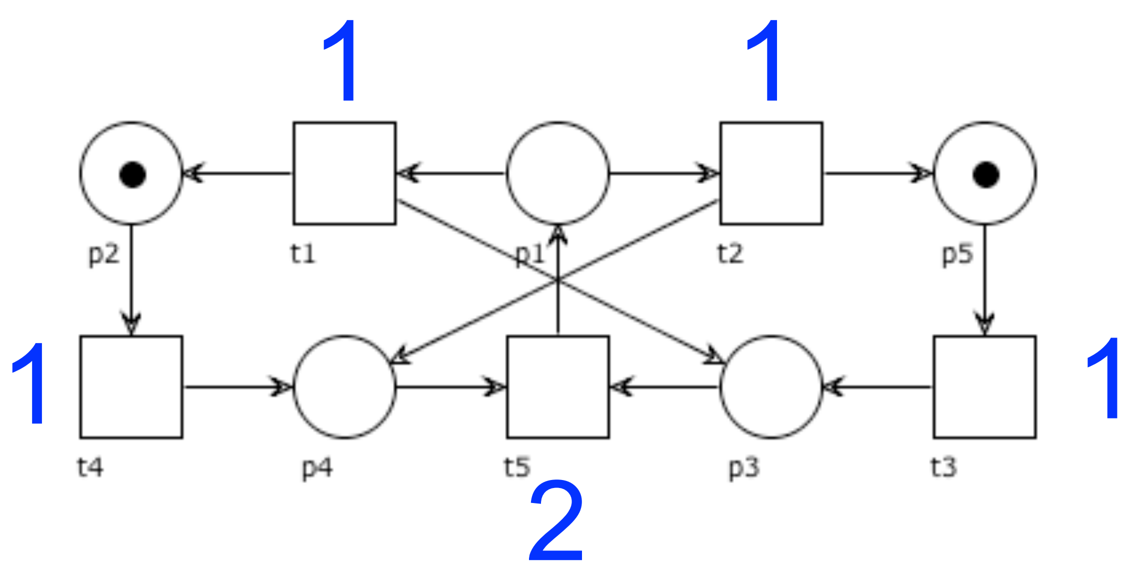

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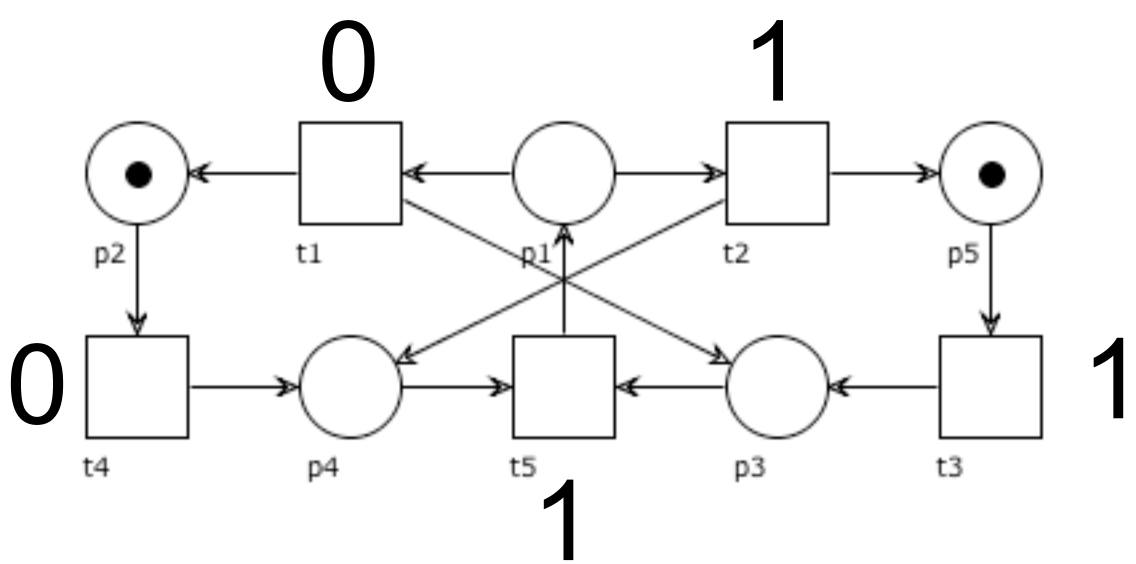
$$\forall p \in P,\ \sum_{t \in \bullet p} \mathbf{J}(t) \stackrel{?}{=} \sum_{t \in p \bullet} \mathbf{J}(t)$$

Question time

Which of the following are T-invariants?



t_1	t_2	t_3	t_4	t_5
[1	0	0	1	1]
[1	1	2	1	2]
[1	1	1	0	2]
[1	1	1	1	2]
→[0	1	1	0	1]

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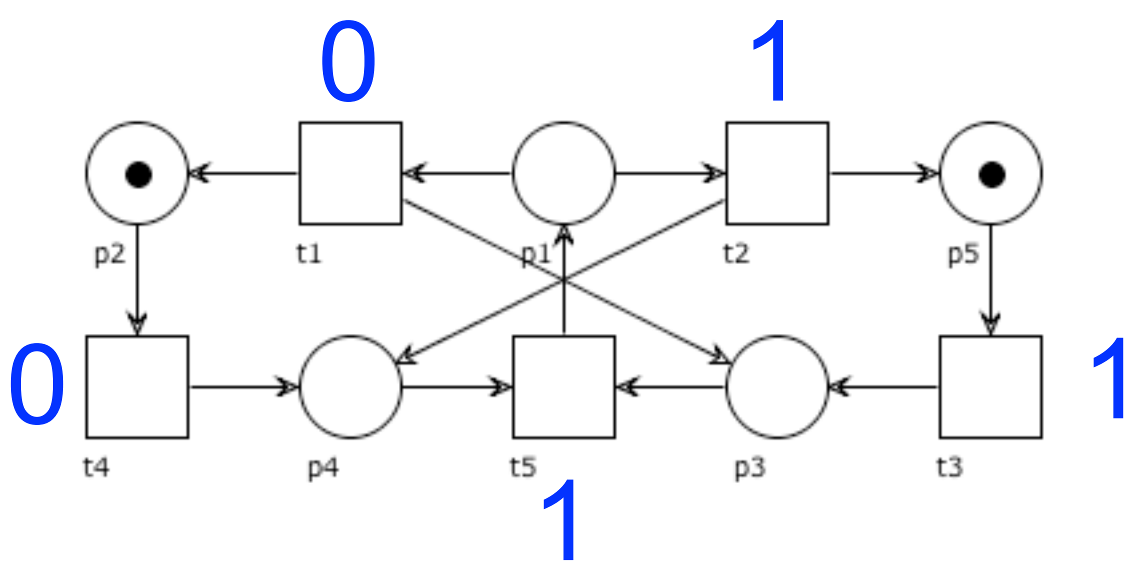

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T-invariants and system properties

Pigeonhole principle

If n items are put into m slots, with $n > m$,
then at least one slot must contain more than one item

Pigeonhole principle

If a path traverses n states,
but there exist only m different states, with $n > m$,
then at least one state is traversed twice

$$n = k + 1 > k = m$$

$$M_0 \to M_1 \to \cdots \to M_i \to \cdots \to M_j \to \cdots \to M_k$$

$$M_i = M_j$$

Reproduction lemma

Lemma: Let (P, T, F, M_0) be a bounded system.
If $M_0 \xrightarrow{\sigma}$ for some infinite sequence σ, then
there is a semi-positive T-invariant $\mathbf{J}$ such that $\langle\, \mathbf{J} \,\rangle \subseteq \{\, t \mid t \in \sigma \,\}$.

Assume $\sigma = t_1\, t_2\, t_3 \ldots$ and $M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} M_2 \xrightarrow{t_3} \ldots$

By boundedness: $[\, M_0 \,\rangle$ is finite.

By the pigeonhole principle, there are $0 \leq i < j$ s.t. $M_i = M_j$

Let $\sigma' = t_{i+1} \ldots t_j$. Then $M_i \xrightarrow{\sigma'} M_j = M_i$

By the marking equation lemma: $\vec{\sigma'}$ is a T-invariant. (fund. prop. of T-inv.)
It is semi-positive, because σ' is not empty $(i < j)$.
Clearly, $\langle\, \mathbf{J} \,\rangle$ only includes transitions in σ.

Boundedness, liveness and positive T-invariant

Theorem: If a bounded system is live, then it has a positive T-invariant

By boundedness: $[\,M_0\,\rangle$ is finite and we let $k = |[\,M_0\,\rangle|$.

By liveness: $M_0 \xrightarrow{\sigma_1} M_1$ with $\vec{\sigma_1}(t) > 0$ for any $t \in T$
Similarly: $M_1 \xrightarrow{\sigma_2} M_2$ with $\vec{\sigma_2}(t) > 0$ for any $t \in T$
Similarly: $M_0 \xrightarrow{\sigma_1} M_1 \xrightarrow{\sigma_2} M_2 ... \xrightarrow{\sigma_k} M_k$

By the pigeonhole principle, there are $0 \leq i < j \leq k$ s.t. $M_i = M_j$
Let $\sigma = \sigma_{i+1} ... \sigma_j$. Then $M_i \xrightarrow{\sigma} M_j = M_i$

By the marking equation lemma: $\vec{\sigma}$ is a T-invariant. (fund. prop. of T-inv.)
It is positive, because $\vec{\sigma}(t) \geq \vec{\sigma_j}(t) > 0$ for any $t \in T$.

T-invariants: recap

Boundedness + liveness => positive T-invariant

No positive T-invariant => non (live + bounded)

No positive T-invariant => non-live OR unbounded

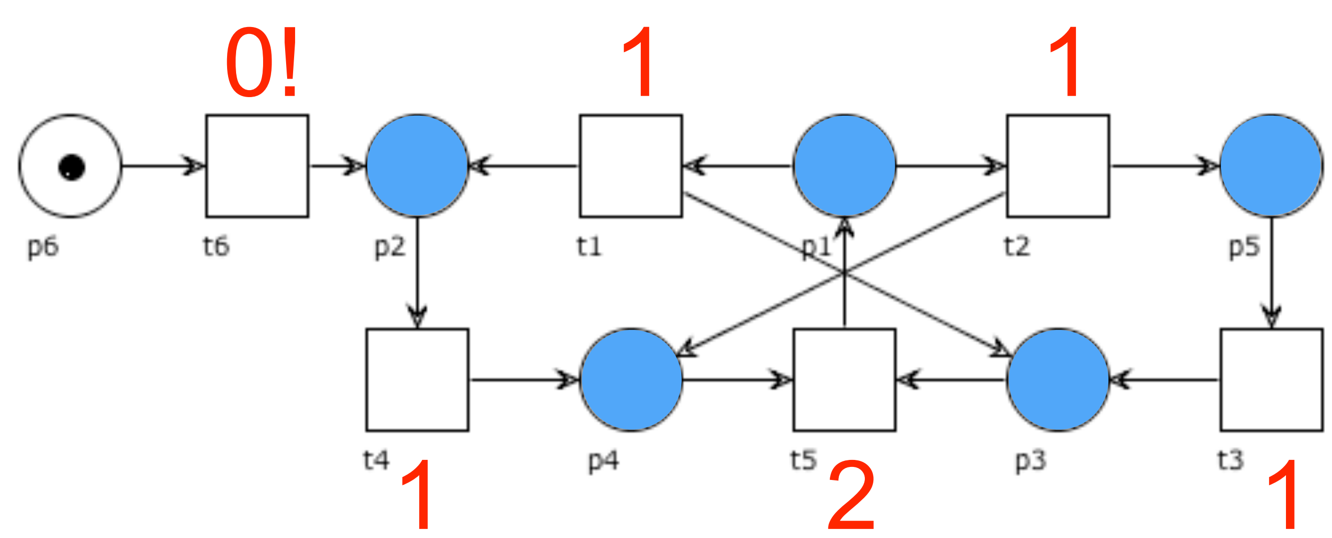
No positive T-invariant + liveness => unbounded

No positive T-invariant + boundedness => non-live

No positive T-inv. + positive S-inv. => non-live

Example

The system below has a positive S-invariant
but no positive T-invariant:
thus it is bounded but cannot be live



I = [2 1 1 1 1 1]

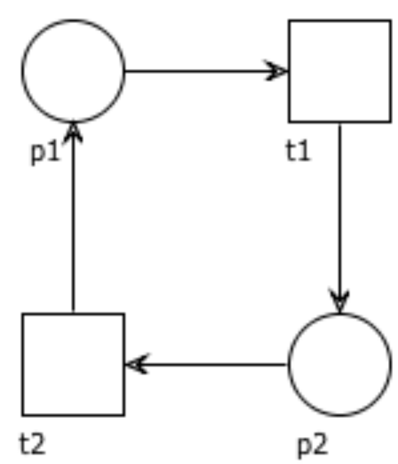
J = [??]

T-invariants: pay attention to implication

No positive T-invariant => **maybe** non live

Exercises

This system has a positive T-invariant.
Is it live and bounded?

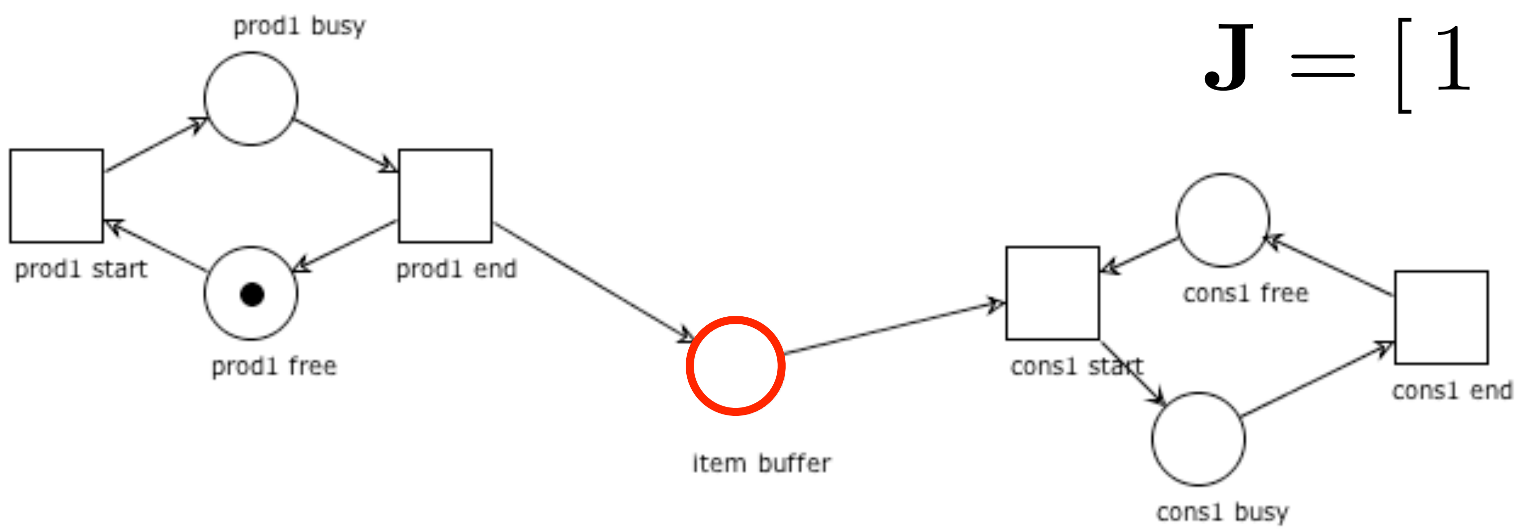


$$\mathbf{J} = [\,1 \quad 1\,]$$

t_1 and t_2 are dead

Exercises

This system has a positive T-invariant.
Is it live and bounded?



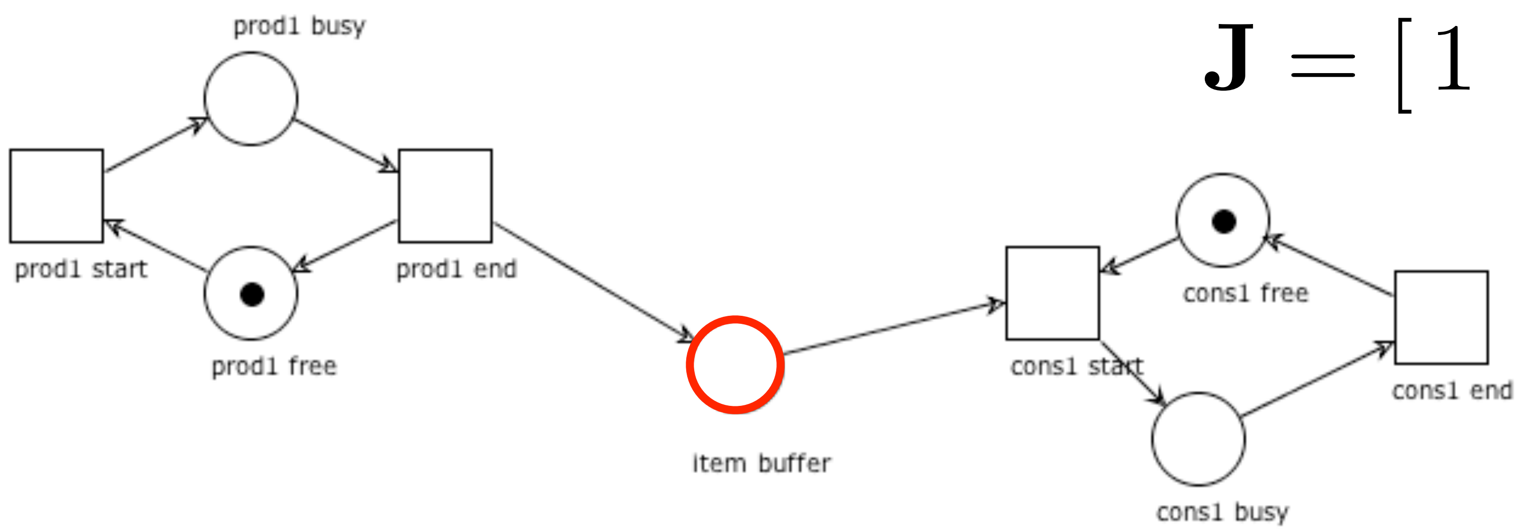
$$\mathbf{J} = [1 \quad 1 \quad 1 \quad 1]$$

"cons1 start" and "cons1 end" are dead
"item buffer" is unbounded

Exercises

This system has a positive T-invariant.
Is it live and bounded?

$$\mathbf{J} = [1 \quad 1 \quad 1 \quad 1]$$



"item buffer" is unbounded

Exercises

This system has a positive T-invariant.
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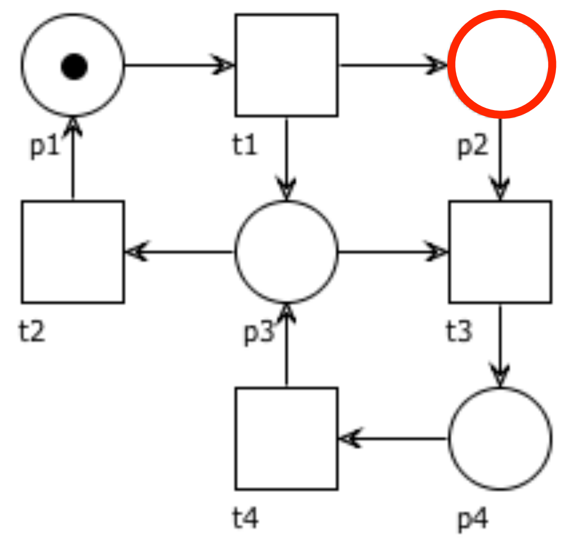


$$\mathbf{J} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

p_2 is unbounded

Recap: T-invariants can be used to

Disprove boundedness
(live system with no positive T-invariant)

Disprove liveness
(boundedness with no positive T-invariant)
(positive S-invariant but no positive T-invariant)

Another theorem on strong connectedness (whose proof we omit)

Strong connectedness via invariants

Theorem: If a weakly connected net has a positive S-invariant **I** and a positive T-invariant **J** then it is strongly connected

Consequences

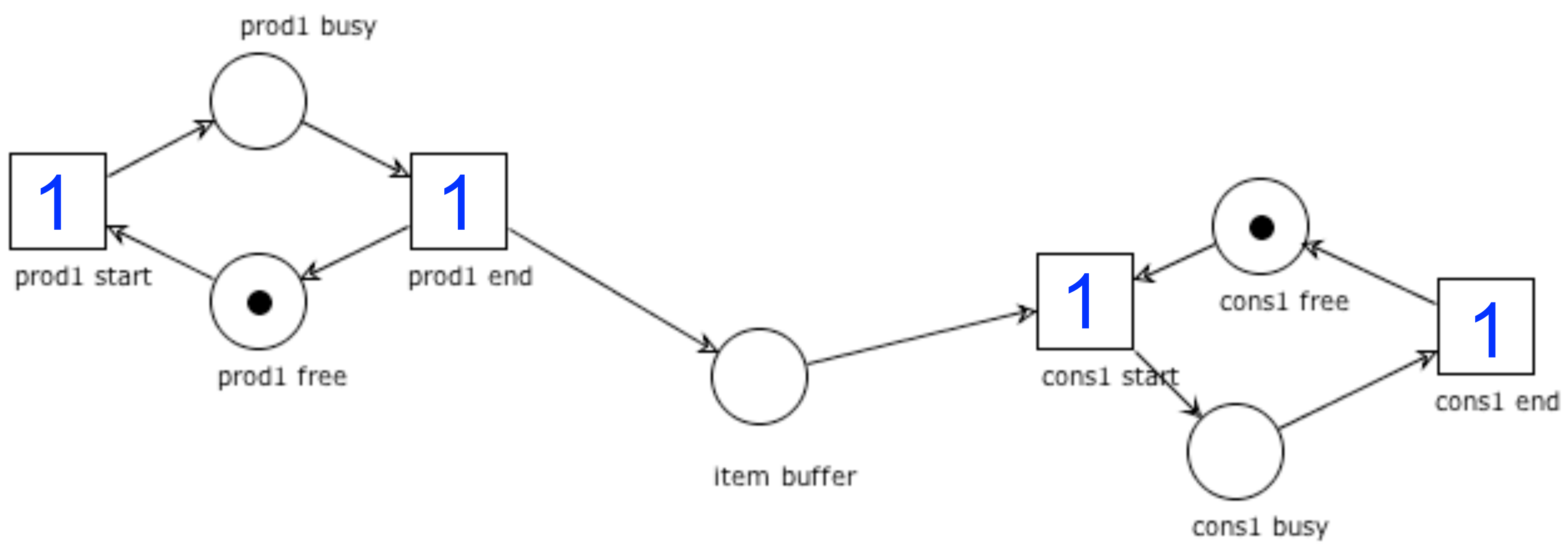
If a (weakly-connected) net is not strongly connected

then

we cannot find (two) positive S- and T-invariants

Example

It is now immediate to check that this system
(weakly connected, not strongly connected)
has a positive T-invariant, but not a positive S-Invariant



Example

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