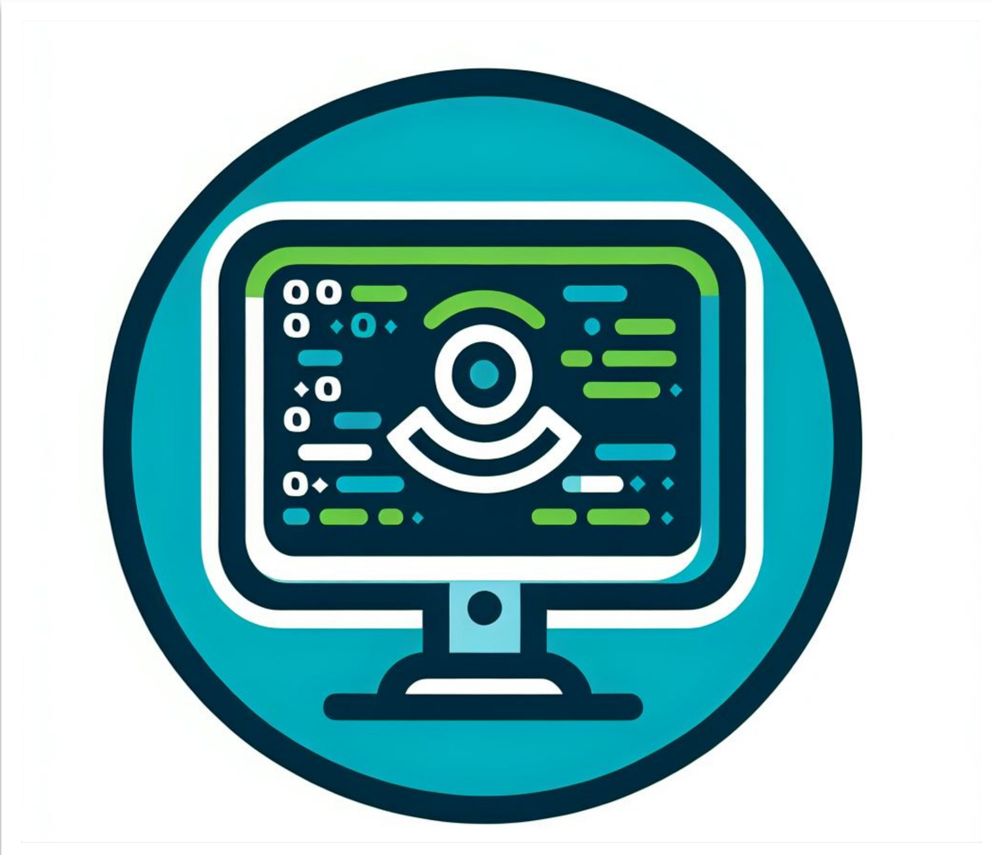

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PSC 2024/25 (375AA, 9CFU)

Principles for Software Composition

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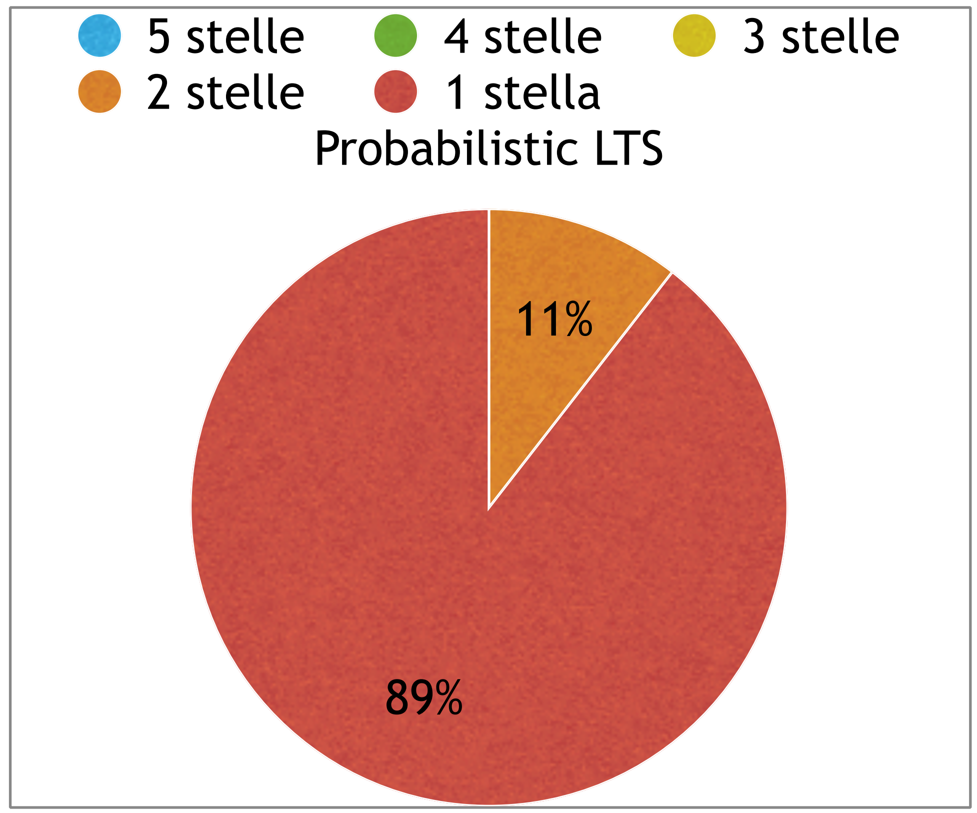
http://www.di.unipi.it/~bruni/

26 - Probabilistic bisimilarities

From your forms



(over 19 answers)

probabilistic bisimilarities

Bisimulation revisited

$(S, L, \rightarrow)$

labelled transitions

labels

states

alternative presentation of transitions

$$\alpha : S \rightarrow S \rightarrow \wp(L) \qquad \alpha\ p\ q = \{\mu \mid p \xrightarrow{\mu} q\}$$

generalization to sets of targets

$$\gamma : S \times \wp(S) \rightarrow \wp(L) \qquad \gamma(p, I) = \{\mu \mid \exists q \in I.\ p \xrightarrow{\mu} q\}$$

$$\gamma(p, I) = \bigcup_{q \in I} \alpha\ p\ q$$

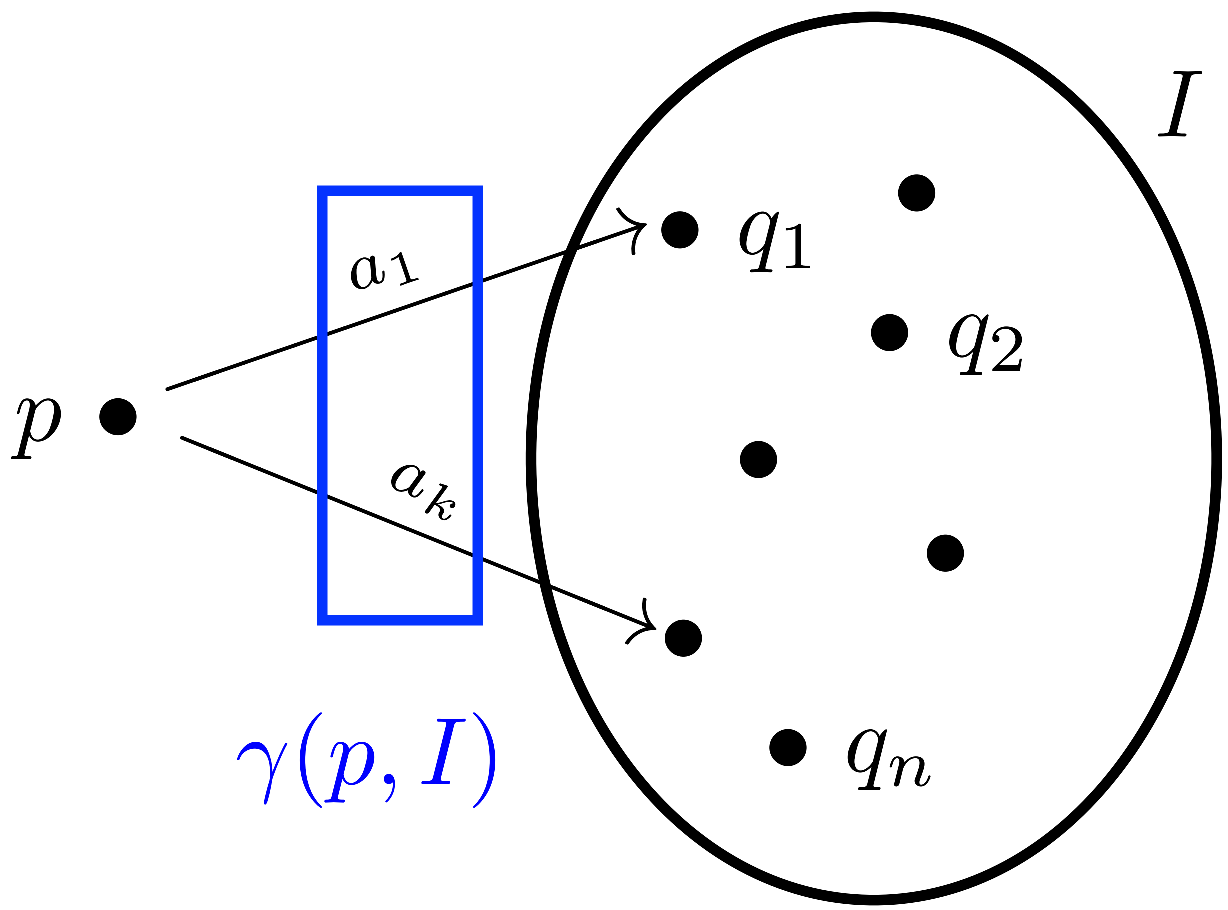
set of targets

source

Bisimulation revisited

$$\alpha : S \to S \to \wp(L) \qquad \alpha\ p\ q = \{\mu \mid p \xrightarrow{\mu} q\}$$

$$\gamma : S \times \wp(S) \to \wp(L) \qquad \gamma(p, I) = \bigcup_{q \in I} \alpha\ p\ q$$



$$p \xrightarrow{\gamma(p,I)} I$$

Bisimulation revisited

$\alpha\ p\ q = \{\mu \mid p \xrightarrow{\mu} q\}$ $\qquad\qquad$ $\gamma(p, I) = \bigcup_{q \in I} \alpha\ p\ q$

take the equivalence $\equiv_{\mathbf{R}}$ induced by a relation $\mathbf{R}$

(if $\mathbf{R}$ is a bisimulation, then $\equiv_{\mathbf{R}}$ is a bisimulation)

take the set of equivalence classes induced by $\equiv_{\mathbf{R}}$: $S_{|\equiv_{\mathbf{R}}}$

take $J \in S_{|\equiv_{\mathbf{R}}}$ and $p, q \in J$ (i.e., $p \equiv_{\mathbf{R}} q$)

if $p \xrightarrow{\mu} p'$ for some μ, p' then $q \xrightarrow{\mu} q'$ for some q' with $p' \equiv_{\mathbf{R}} q'$

(and vice versa) (i.e. $\exists I \in S_{|\equiv_{\mathbf{R}}}.\ p', q' \in I$)

now consider the function $\Phi : \wp(S \times S) \to \wp(S \times S)$

$$p\ \Phi(\mathbf{R})\ q \triangleq \forall I \in S_{|\equiv_{\mathbf{R}}}.\ \gamma(p, I) = \gamma(q, I)$$

by the above argument, a bisimulation is such if $\mathbf{R} \subseteq \Phi(\mathbf{R})$

$\simeq\ \triangleq \bigcup_{\mathbf{R} \subseteq \Phi(\mathbf{R})} \mathbf{R}$ is the largest bisimulation

Bisimulation revisited

$$\alpha : S \to S \to \wp(L)$$
$$\alpha\ p\ q = \{\mu \mid p \xrightarrow{\mu} q\}$$

$$\gamma : S \times \wp(S) \to \wp(L)$$
$$\gamma(p, I) = \bigcup_{q \in I} \alpha\ p\ q$$

$$\Phi : \wp(S \times S) \to \wp(S \times S)$$
$$p\ \Phi(\mathbf{R})\ q \triangleq \forall I \in S_{|\equiv_{\mathbf{R}}}.\ \gamma(p, I) = \gamma(q, I)$$

bisimulation $\quad \mathbf{R} \subseteq \Phi(\mathbf{R})$

bisimilarity $\quad \simeq\ \triangleq \bigcup_{\mathbf{R} \subseteq \Phi(\mathbf{R})} \mathbf{R}$

Bisimulation for CTMC

$$\alpha_C : S \to S \to \mathbb{R}$$

$$\alpha_C\, i\, j = \lambda_{i,j}$$

$$\gamma_C : S \times \wp(S) \to \mathbb{R}$$

$$\gamma_C(i, I) = \sum_{j \in I} \alpha_C\, i\, j = \sum_{j \in I} \lambda_{i,j}$$

$$\Phi_C : \wp(S \times S) \to \wp(S \times S)$$

$$i\ \Phi_C(\mathbf{R})\ j \triangleq \forall I \in S_{|\equiv_{\mathbf{R}}}.\ \gamma_C(i, I) = \gamma_C(j, I)$$

CTMC bisimulation $\quad \mathbf{R} \subseteq \Phi_C(\mathbf{R})$

CTMC bisimilarity $\quad \simeq_C \triangleq \bigcup_{\mathbf{R} \subseteq \Phi_C(\mathbf{R})} \mathbf{R}$

Bisimulation for CTMC

$$\alpha_C : S \to S \to \mathbb{R}$$
$$\alpha_C\ i\ j = \lambda_{i,j}$$

$$\gamma_C : S \times \wp(S) \to \mathbb{R}$$
$$\gamma_C(i, I) = \sum_{j \in I} \alpha_C\ i\ j = \sum_{j \in I} \lambda_{i,j}$$

$$\Phi_C : \wp(S \times S) \to \wp(S \times S)$$
$$i\ \Phi_C(\mathbf{R})\ j \triangleq \forall I \in S_{|\equiv_{\mathbf{R}}}.\ \gamma_C(i, I) = \gamma_C(j, I)$$

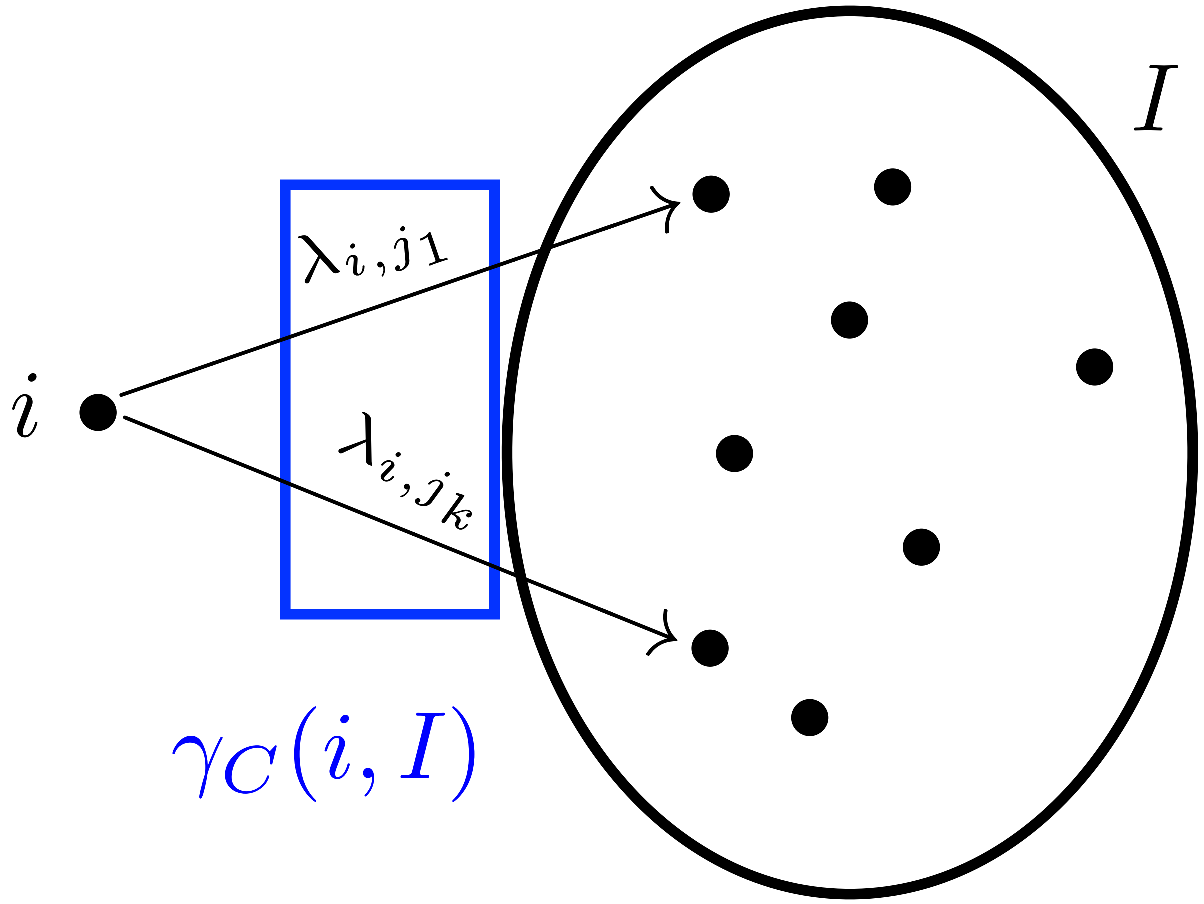
CTMC bisimulation $\quad \mathbf{R} \subseteq \Phi_C(\mathbf{R})$

CTMC bisimilarity $\quad \simeq_C \triangleq \bigcup_{\mathbf{R} \subseteq \Phi_C(\mathbf{R})} \mathbf{R}$

Bisimulation for CTMC

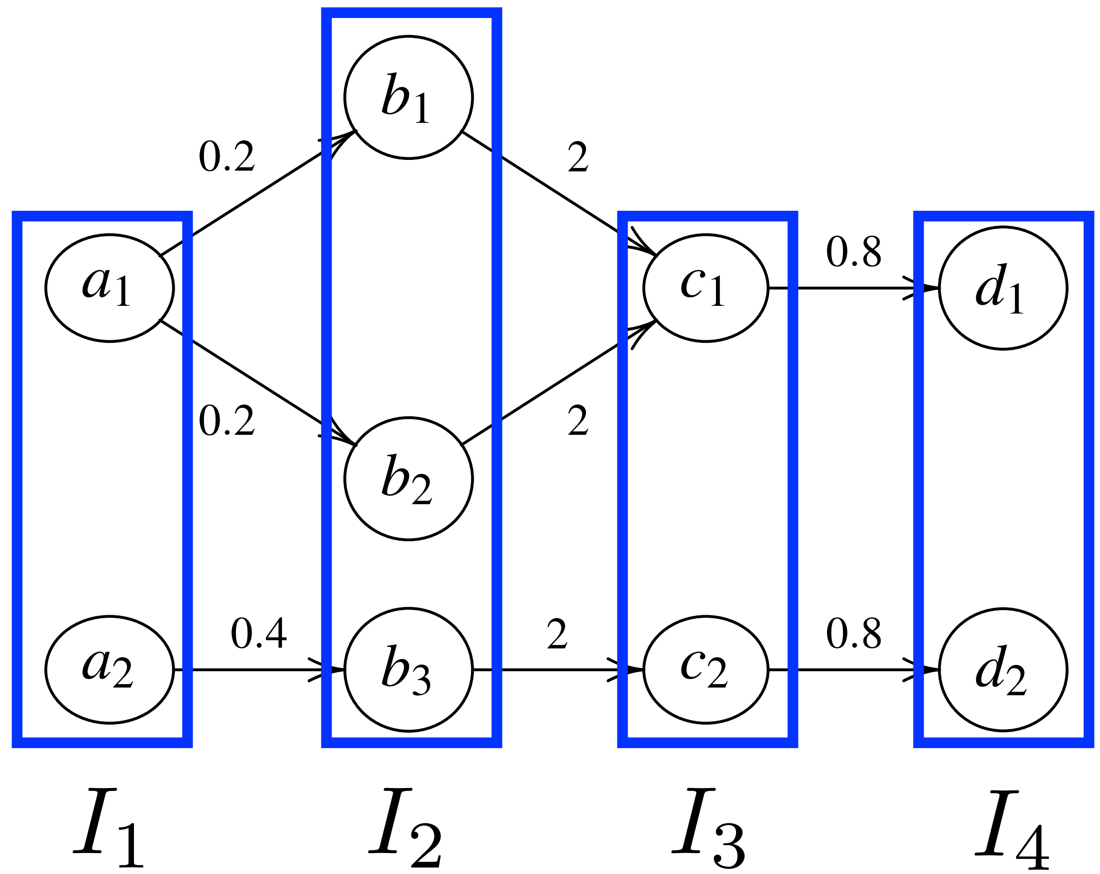
$$\alpha_C : S \to S \to \mathbb{R} \qquad \alpha_C \; i \; j = \lambda_{i,j}$$

$$\gamma_C : S \times \wp(S) \to \mathbb{R} \qquad \gamma_C(i, I) = \sum_{j \in I} \alpha_C \; i \; j = \sum_{j \in I} \lambda_{i,j}$$



$$i \xrightarrow{\sum_{j \in I} \lambda_{i,j}} I$$

Bisimulation for CTMC



$$\equiv_{\mathbf{R}} = \{\ \{a_1, a_2\}\ ,\ \{b_1, b_2, b_3\}\ ,\ \{c_1, c_2\}\ ,\ \{d_1, d_2\}\ \}$$

$$\forall i.\quad \gamma_C(a_1, I_i) \stackrel{?}{=} \gamma_C(a_2, I_i)$$

$$\gamma_C(b_1, I_i) \stackrel{?}{=} \gamma_C(b_2, I_i) \stackrel{?}{=} \gamma_C(b_3, I_i)$$

$$\gamma_C(c_1, I_i) \stackrel{?}{=} \gamma_C(c_2, I_i)$$

$$\gamma_C(d_1, I_i) \stackrel{?}{=} \gamma_C(d_2, I_i)$$

Bisimulation for DTMC

$$\alpha_D : S \to \mathbb{D}(S) \cup \{\star\}$$

$$\alpha_C : S \to S \to \mathbb{R}$$

$$\alpha_D\ i = d \quad \alpha_C\ i\ j = \lambda_{i,j} \quad \alpha_D\ i = \star$$

$$\gamma_D : S \times \wp(S) \to [0,1] \cup \{\star\}$$

$$\gamma_C : S \to \wp(S) \to \mathbb{R}$$

$$\gamma_D(i, I) = \sum_{j \in I} \alpha_D\ i\ j$$

$$\gamma_C\ i\ I = \sum_{j \in I} \alpha_C\ i\ j = \sum_{j \in I} \lambda_{i,j}$$

$$\gamma_D(i, I) = \star$$

$$\Phi_D : \wp(S \times S) \to \wp(S \times S)$$

$$i\ \Phi_D(\mathbf{R})\ j \triangleq \forall I \in S_{|\equiv_{\mathbf{R}}}.\ \gamma_D(i, I) = \gamma_D(j, I)$$

DTMC bisimulation $\quad \mathbf{R} \subseteq \Phi_D(\mathbf{R})$

DTMC bisimilarity $\quad \simeq_D \triangleq \bigcup_{\mathbf{R} \subseteq \Phi_D(\mathbf{R})} \mathbf{R}$

Bisimulation for DTMC

$$\alpha_D : S \to \mathbb{D}(S) \cup \{\star\}$$

$$\alpha_D\ i = d \qquad\qquad \alpha_D\ i = \star$$

$$\gamma_D : S \times \wp(S) \to [0,1] \cup \{\star\}$$

$$\gamma_D(i, I) = \sum_{j \in I} \alpha_D\ i\ j = \sum_{j \in I} a_{i,j} \qquad\qquad \gamma_D(i, I) = \star$$

$$\Phi_D : \wp(S \times S) \to \wp(S \times S)$$

$$i\ \Phi_D(\mathbf{R})\ j \triangleq \forall I \in S_{|\equiv_{\mathbf{R}}}.\ \gamma_D(i, I) = \gamma_D(j, I)$$

DTMC bisimulation $\mathbf{R} \subseteq \Phi_D(\mathbf{R})$

DTMC bisimilarity $\simeq_D \triangleq \bigcup_{\mathbf{R} \subseteq \Phi_D(\mathbf{R})} \mathbf{R}$

Bisimulation for DTMC

any two deadlock states i, j are DTMC bisimilar

$$\forall I.\ \gamma_D(i, I) = \star = \gamma_D(j, I)$$

any deadlock state i is separated from any non-deadlock state k

$$\exists I.\ \gamma_D(i, I) = \star \neq \gamma_D(k, I) \in [0, 1]$$

if there are no deadlock states, then $\simeq_D = S \times S$

reactive PTS

DTMC with actions

also called Markov decision processes

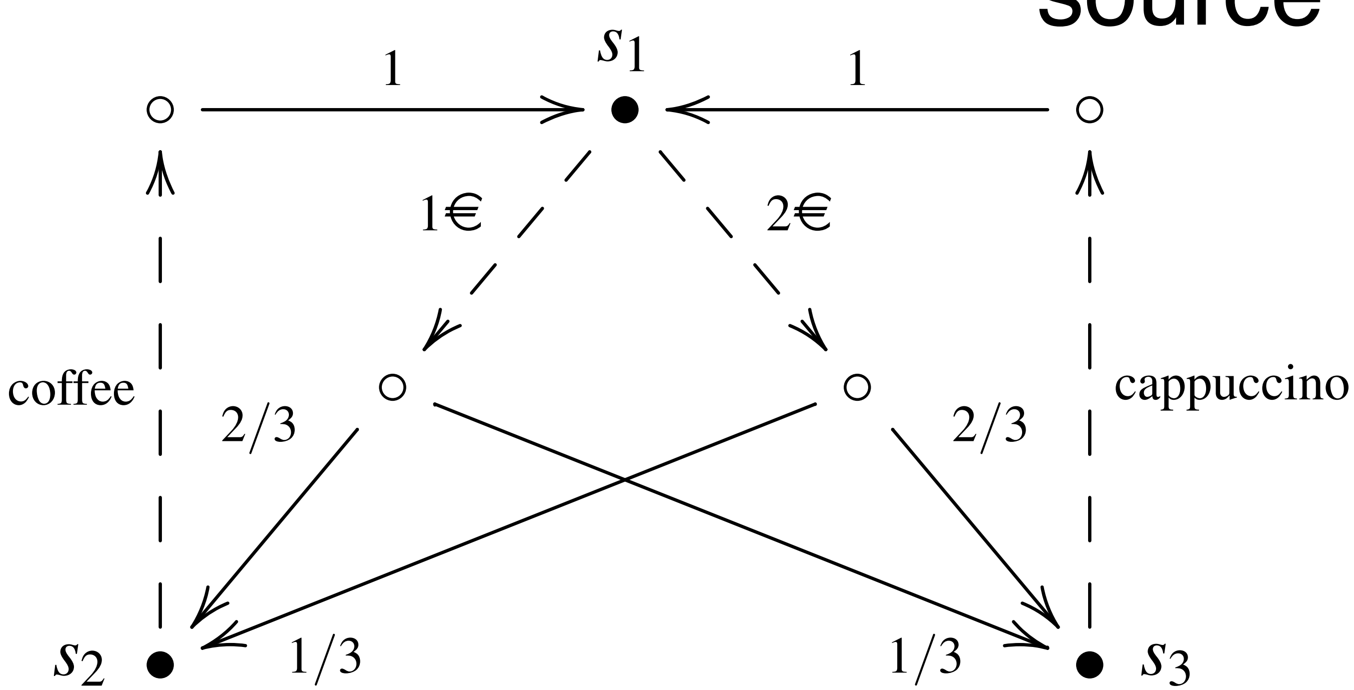
Reactive probabilistic transition systems

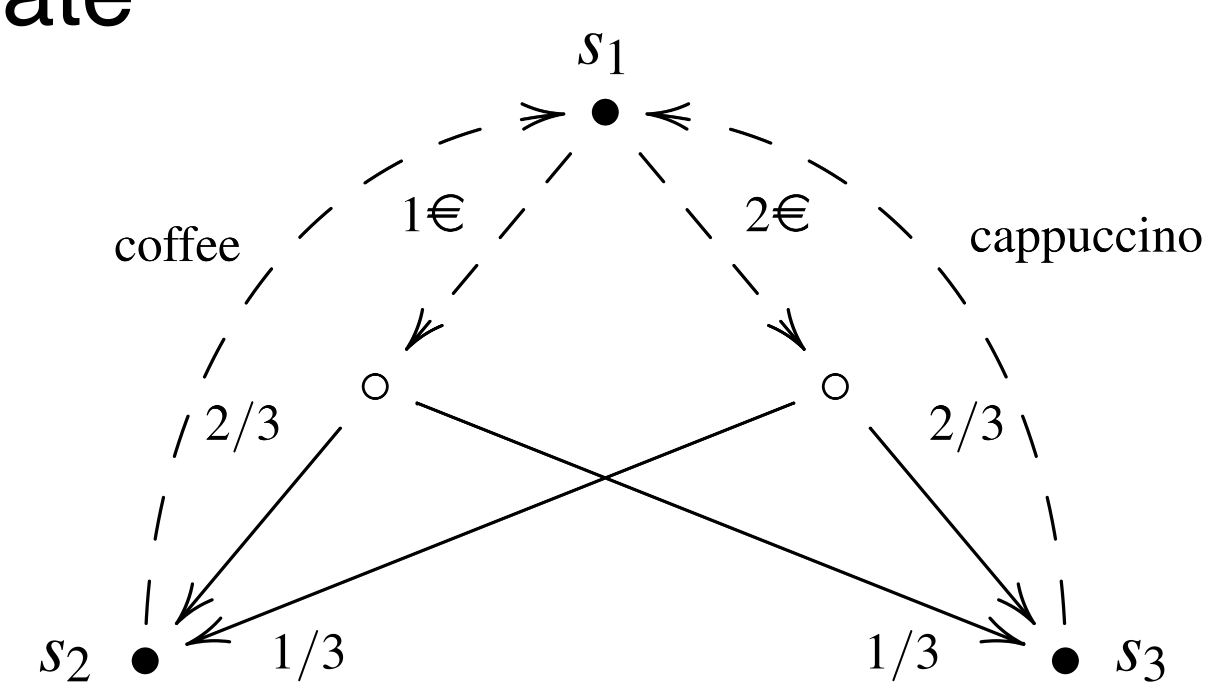
$$\alpha_R : S \to L \to \mathbb{D}(S) \cup \{\star\}$$

reaction

label (stimulus)

source state





Reactive bisimulation

$$\alpha_R : S \to L \to \mathbb{D}(S) \cup \{\star\}$$

$$\alpha_R \; s \; \ell = d \qquad\qquad \alpha_R \; s \; \ell = \star$$

$$\gamma_R : S \times L \times \wp(S) \to [0,1]$$

$$\gamma_R(s, \ell, I) = \sum_{u \in I} \alpha_R \; s \; \ell \; u \qquad\qquad \gamma_R(s, \ell, I) = 0$$

$$\Phi_R : \wp(S \times S) \to \wp(S \times S)$$

$$s \;\, \Phi_R(\mathbf{R}) \;\, u \triangleq \forall \ell \in L.\ \forall I \in S_{|\equiv_\mathbf{R}}.\ \gamma_R(s, \ell, I) = \gamma_R(u, \ell, I)$$

reactive bisimulation $\quad \mathbf{R} \subseteq \Phi_R(\mathbf{R})$

reactive bisimilarity $\quad \simeq_R \triangleq \bigcup_{\mathbf{R} \subseteq \Phi_R(\mathbf{R})} \mathbf{R}$

Larsen-Skou logic: syntax

$$
\begin{array}{rcll}
\varphi & ::= & \mathbf{tt} & \text{true} \\
& | & \varphi_1 \wedge \varphi_2 & \text{conjunction} \\
& | & \neg\varphi & \text{negation} \\
& | & \langle \underbrace{\ell}_{\text{label}} \rangle_{\underbrace{q}_{\text{probability}}} \varphi & \text{diamond operator}
\end{array}
$$

Larsen-Skou: semantics

$$s \models \varphi$$

defined inductively on the structure of the formula

$s \models \mathbf{tt}$ — any state satisfies true

$s \models \varphi_1 \land \varphi_2 \quad \text{iff} \quad s \models \varphi_1 \text{ and } s \models \varphi_2$ — s satisfies both φ_1 and φ_2

$s \models \neg\varphi \quad \text{iff} \quad s \not\models \varphi$ — s does not satisfy φ

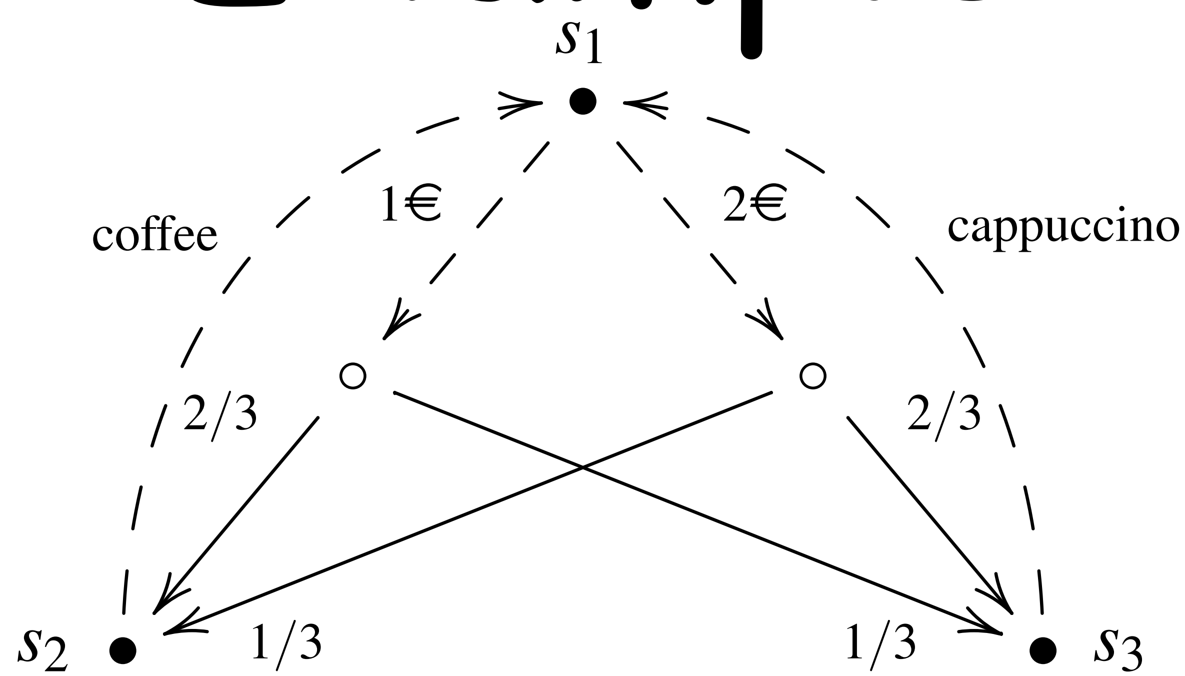
$s \models \langle \ell \rangle_q \varphi \quad \text{iff} \quad \gamma_R(s, \ell, [\![\varphi]\!]) \geq q$

$\text{where} \quad [\![\varphi]\!] \triangleq \{u \in S \mid u \models \varphi\}$

the sum of all probabilities to reach a state u that satisfies φ is greater than or equal to q

$\Diamond_\ell \varphi$ is just $\langle \ell \rangle_1 \varphi$

Example



$$s_1 \stackrel{?}{\models} \langle 1€ \rangle_{\frac{1}{2}} \langle \text{coffee} \rangle_1 \mathbf{tt} \qquad \gamma_R(s_1, 1€, [\![\langle \text{coffee} \rangle_1 \mathbf{tt}]\!]) \stackrel{?}{\geq} \frac{1}{2}$$

$$\begin{aligned}
[\![\langle \text{coffee} \rangle_1 \mathbf{tt}]\!] &= \{u \mid u \models \langle \text{coffee} \rangle_1 \mathbf{tt}\} \\
&= \{u \mid \gamma_R(u, \text{coffee}, [\![\mathbf{tt}]\!]) \geq 1\} \\
&= \{u \mid \gamma_R(u, \text{coffee}, S) \geq 1\} \\
&= \{s_2\}
\end{aligned}$$

$$\gamma_R(s_1, 1€, \{s_2\}) = \frac{2}{3} \geq \frac{1}{2}$$

Reactive bis as logic

TH. $s_1 \simeq_R s_2$ iff $\forall \varphi.\ s_1 \models \varphi \Leftrightarrow s_2 \models \varphi$

it is even sufficient to consider formulas without negation!

Logical characterisation of reactive bisimilarity

consequences:

to show that two reactive PTS are reactive bisimilar: exhibit a reactive bisimulation that relates them

to show that two reactive PTS are not reactive bisimilar: exhibit a LS formula that distinguishes between them