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**PSC 2023/24** (375AA, 9CFU)

Principles for Software Composition

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# 08a - Complete Partial Orders

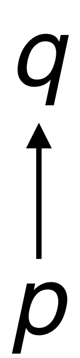
# Partial orders

# Partially ordered set (Poset or just PO)

a set

$(P, \sqsubseteq)$ — a binary relation — $\sqsubseteq \subseteq P \times P$

| | | |
|---|---|---|
| reflexive | $\forall p \in P.$ | $p \sqsubseteq p$ |
| antisymmetric | $\forall p, q \in P.$ | $p \sqsubseteq q \ \wedge \ q \sqsubseteq p \ \Rightarrow \ p = q$ |
| transitive | $\forall p, q, r \in P.$ | $p \sqsubseteq q \ \wedge \ q \sqsubseteq r \ \Rightarrow \ p \sqsubseteq r$ |




$p \sqsubseteq q$ means that p and q are **comparable** and that p is less than (or equal to) q

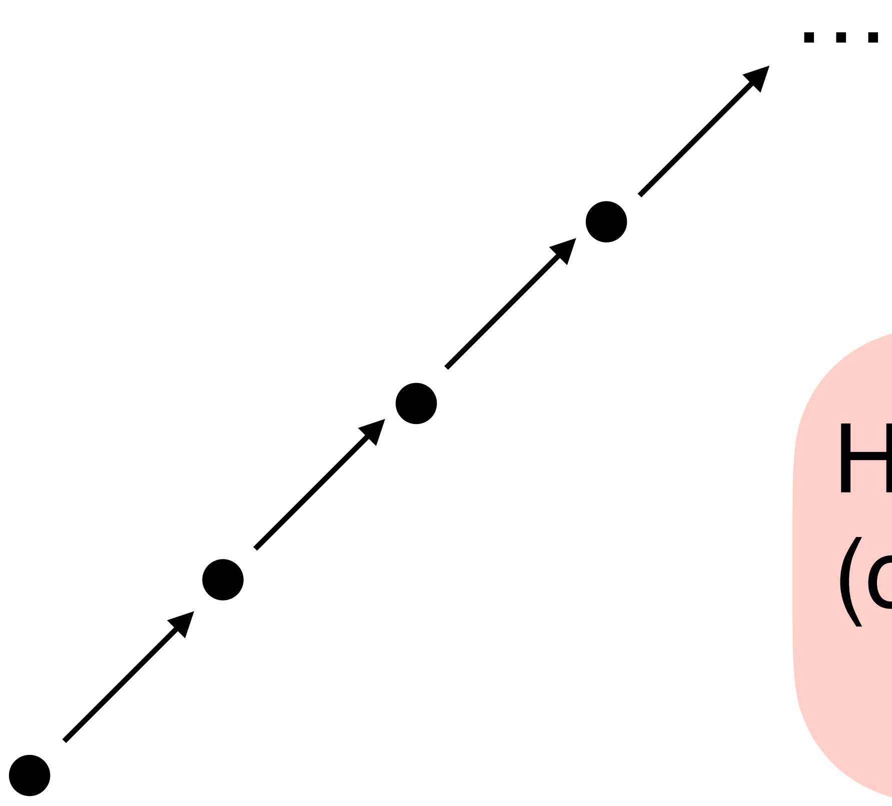
$p \sqsubset q$ means $p \sqsubseteq q \ \wedge \ p \neq q$

# Total orders

$(P, \sqsubseteq)$ PO

total $\forall p, q \in P. \quad p \sqsubseteq q \ \vee \ q \sqsubseteq p$

a PO where any two elements are **comparable**




Hasse diagram notation
(omit: reflexive arcs,
transitive arcs)

# Discrete orders

$(P, \sqsubseteq)$ PO

discrete $\forall p, q \in P. \quad p \sqsubseteq q \Leftrightarrow p = q$

each element is **comparable** only to itself

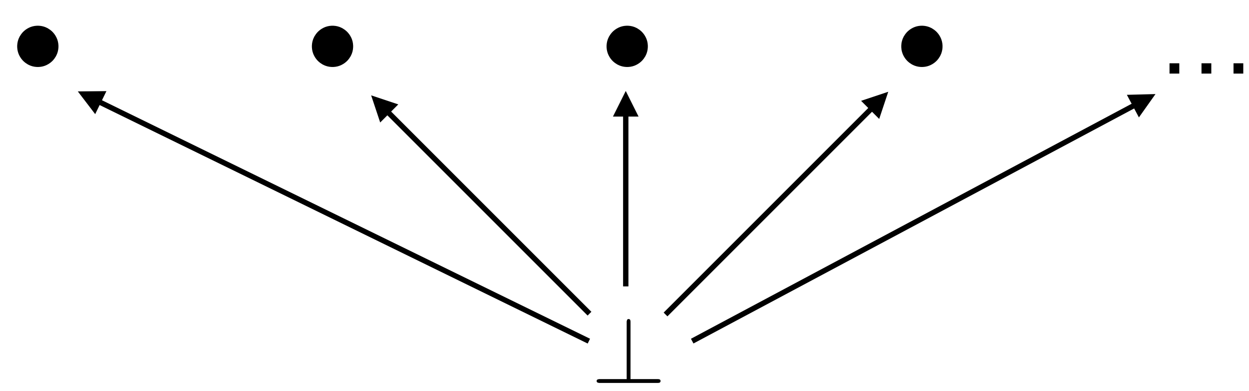
• • • • …

# Flat orders

$(P, \sqsubseteq)$ PO

flat $\forall p, q \in P. \quad p \sqsubseteq q \;\Leftrightarrow\; p = q \;\vee\; p = \bot$

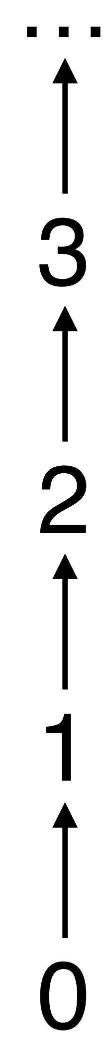
each element is **comparable** only to itself
and with a distinguished (smaller) element $\bot$



# Exercise

$(\mathbb{N}, \leq)$

| PO? | Total? | Discrete? | Flat? |
|---|---|---|---|
| ✓ | ✓ | ✗ | ✗ |

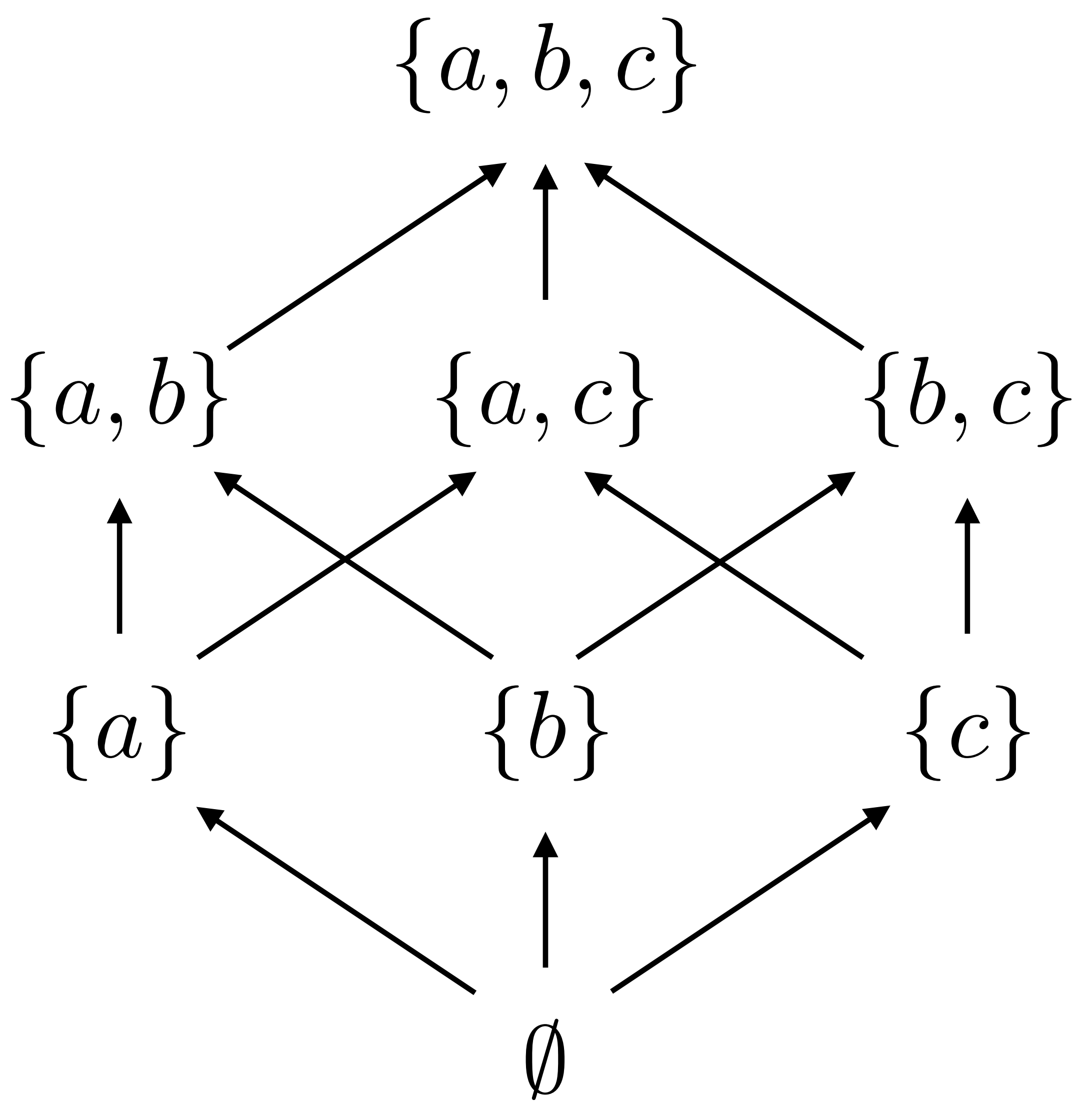


# Exercise

$(\wp(S), \subseteq)$

| | PO? | Total? | Discrete? | Flat? |
|---|---|---|---|---|
| | ✔ | $\lvert S\rvert < 2$ | $S = \emptyset$ | $\lvert S\rvert < 2$ |

example: $S = \{a, b, c\}$




$\{a, b\} \quad \not\subseteq \quad \{b, c\}$

$\not\supseteq$

$\{a\} \quad \not\subseteq \quad \{b\}$

$\not\supseteq$

# Exercise

$(\mathbb{N}, =)$

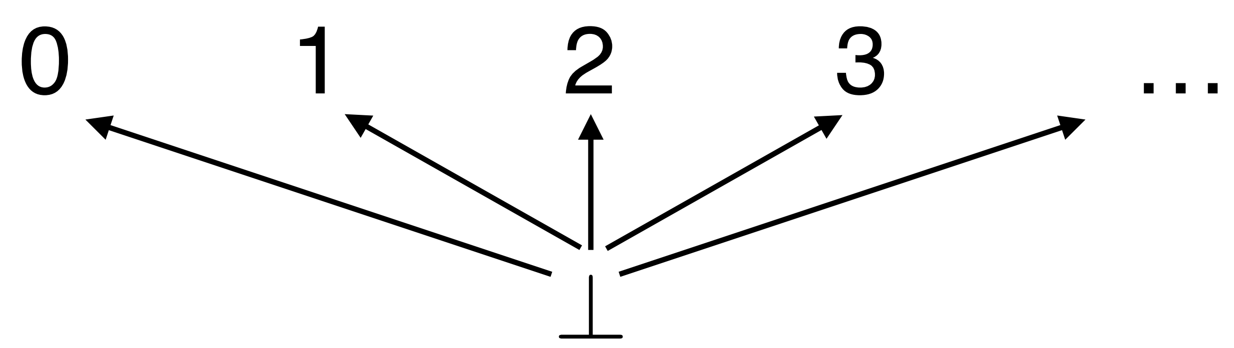
| PO? | Total? | Discrete? | Flat? |
| --- | --- | --- | --- |
| ✓ | ✗ | ✓ | ✗ |

0 1 2 3 …

# Exercise

$$(\mathbb{N} \cup \{\bot\}, \{(\bot, n) \mid n \in \mathbb{N}\}^*)$$

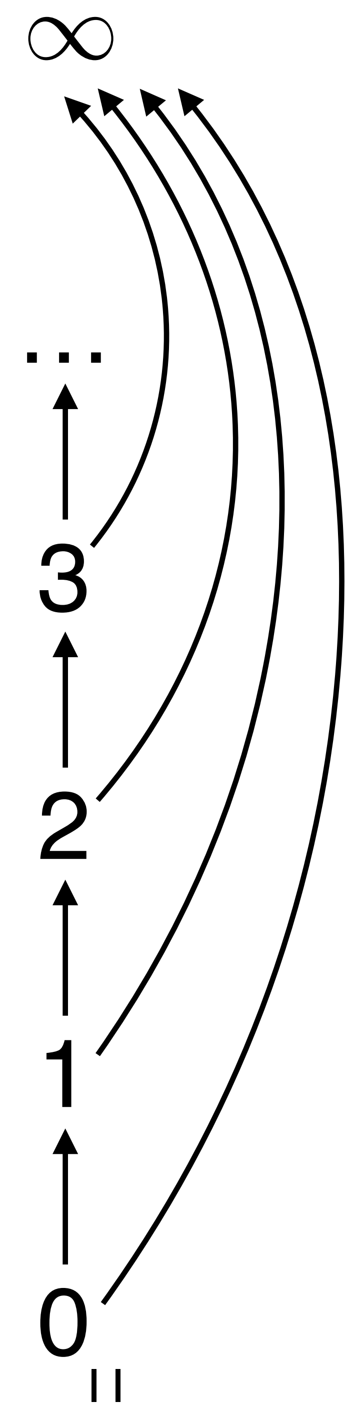
| PO? | Total? | Discrete? | Flat? |
|---|---|---|---|
| ✓ | ✗ | ✗ | ✓ |



# Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

| PO? | Total? | Discrete? | Flat? |
| --- | --- | --- | --- |
| ✓ | ✓ | ✗ | ✗ |



# Exercise

| | PO? | Total? | Discrete? | Flat? |
|---|---|---|---|---|
| $(\mathbb{N}, <)$ | ✗ | ✗ | ✗ | ✗ |
| $(\mathbb{Z}, \leq)$ | ✓ | ✓ | ✗ | ✗ |
| $(\mathbb{Z} \cup \{-\infty, \infty\}, \leq)$ | ✓ | ✓ | ✗ | ✗ |
| $(T_\Sigma, \prec)$ | ✗ | ✗ | ✗ | ✗ |
| $(\mathbb{N}, \neq)$ | ✗ | ✗ | ✗ | ✗ |

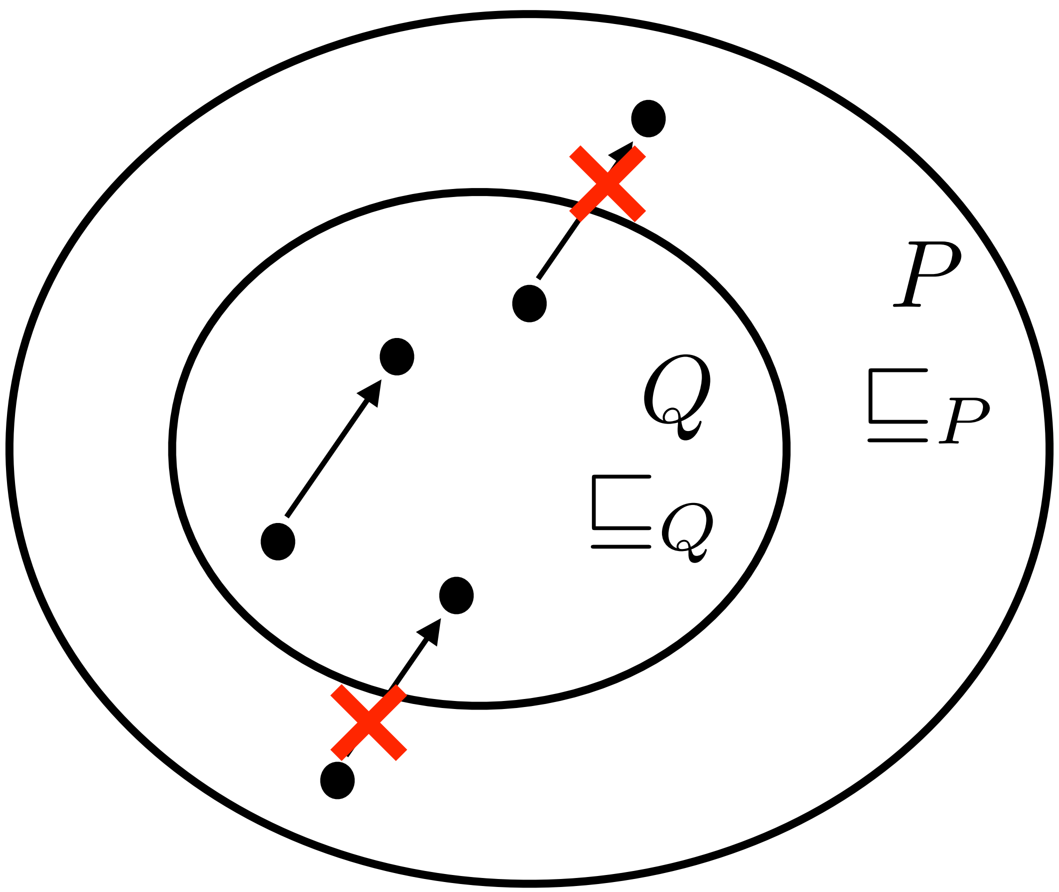
# Subset of a PO

$(P, \sqsubseteq_P)$ PO $\qquad Q \subseteq P$

let $\quad \sqsubseteq_Q \triangleq \sqsubseteq_P \cap (Q \times Q)$

TH. $(Q, \sqsubseteq_Q)$ is a PO

TH. if $(P, \sqsubseteq_P)$ is total, then $(Q, \sqsubseteq_Q)$ is total



| PO $\sqsubseteq$ | w.f. $\prec$ |
| --- | --- |
| reflexive | not reflexive (otherwise cycle!) |
| antisymmetric | antisymmetric (otherwise cycle!)<br>$p \prec q \wedge q \prec p$ is always false |
| transitive | can be transitive ( $\prec^+$ w.f.) |
| has infinite descending chains (if nonempty) | no infinite descending chain |
| | $\prec^*$ is always a PO |
| $\sqsubset$ can be w.f. | |

# Element properties
# (least, minimal, …)

# Least element

$(P, \sqsubseteq)$ PO $\quad Q \subseteq P \quad \ell \in Q$

$\ell$ is a least element of $Q$ if $\forall q \in Q. \quad \ell \sqsubseteq q$

TH. (uniqueness of least element)

$(P, \sqsubseteq)$ PO $\quad Q \subseteq P \quad \ell_1, \ell_2$ least elements of $Q$ implies $\ell_1 = \ell_2$

$\ell_1$ least element of $Q \quad \Rightarrow \quad \ell_1 \sqsubseteq \ell_2$

$\ell_2$ least element of $Q \quad \Rightarrow \quad \ell_2 \sqsubseteq \ell_1$

$$\Rightarrow \quad \ell_1 = \ell_2$$

by antisymmetry

# Bottom

$(P, \sqsubseteq)$ PO

the least element of $P$
(if it exists) is called bottom and denoted $\bot$

sometimes written $\bot_P$

Examples

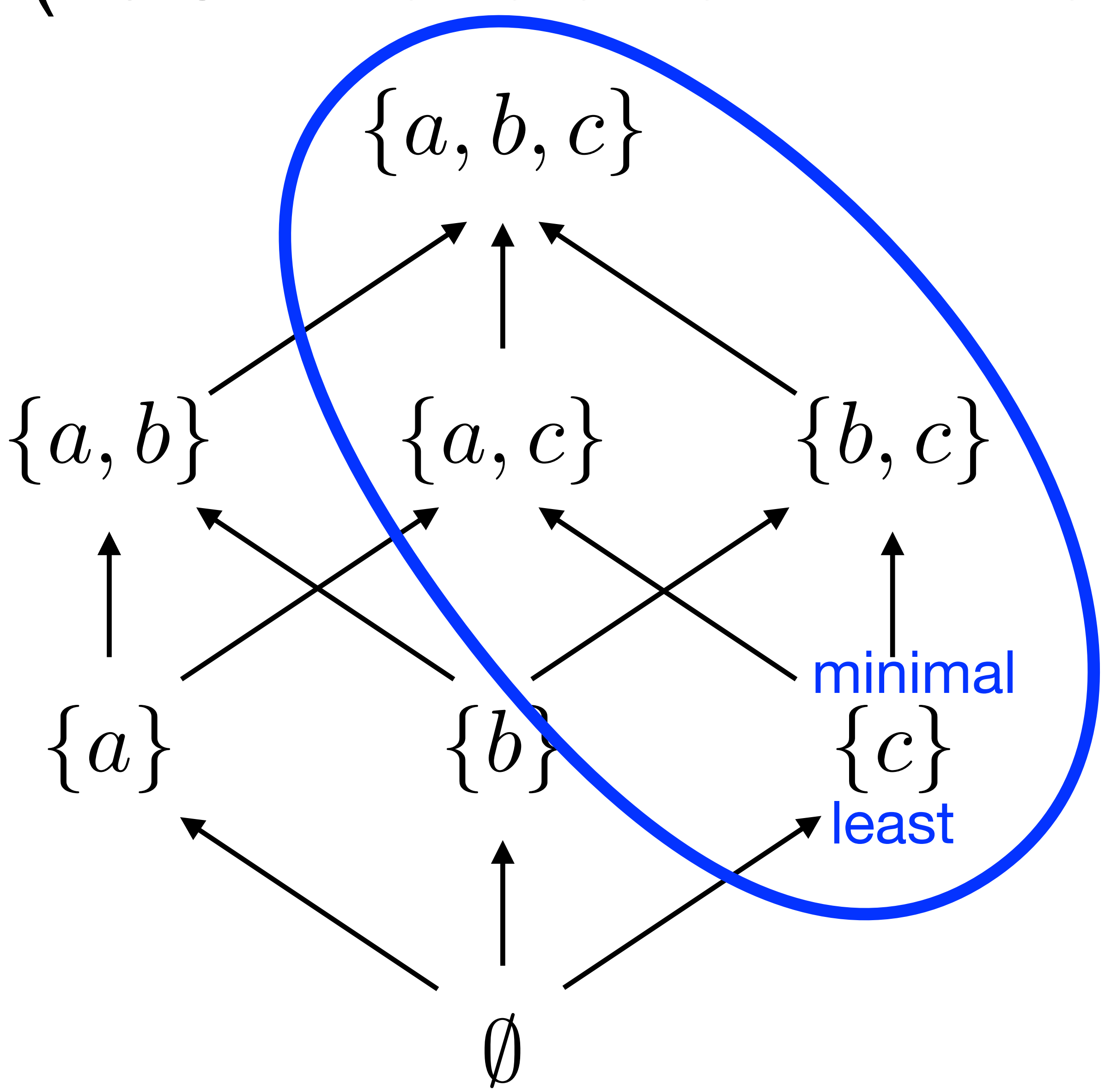
| PO | bottom? |
|---|---|
| $(\mathbb{N} \cup \{\infty\}, \leq)$ | $0$ |
| $(\wp(S), \subseteq)$ | $\varnothing$ |
| $(\mathbb{Z}, \leq)$ | ✖ |
| $(\mathbb{Z} \cup \{-\infty, \infty\}, \leq)$ | $-\infty$ |

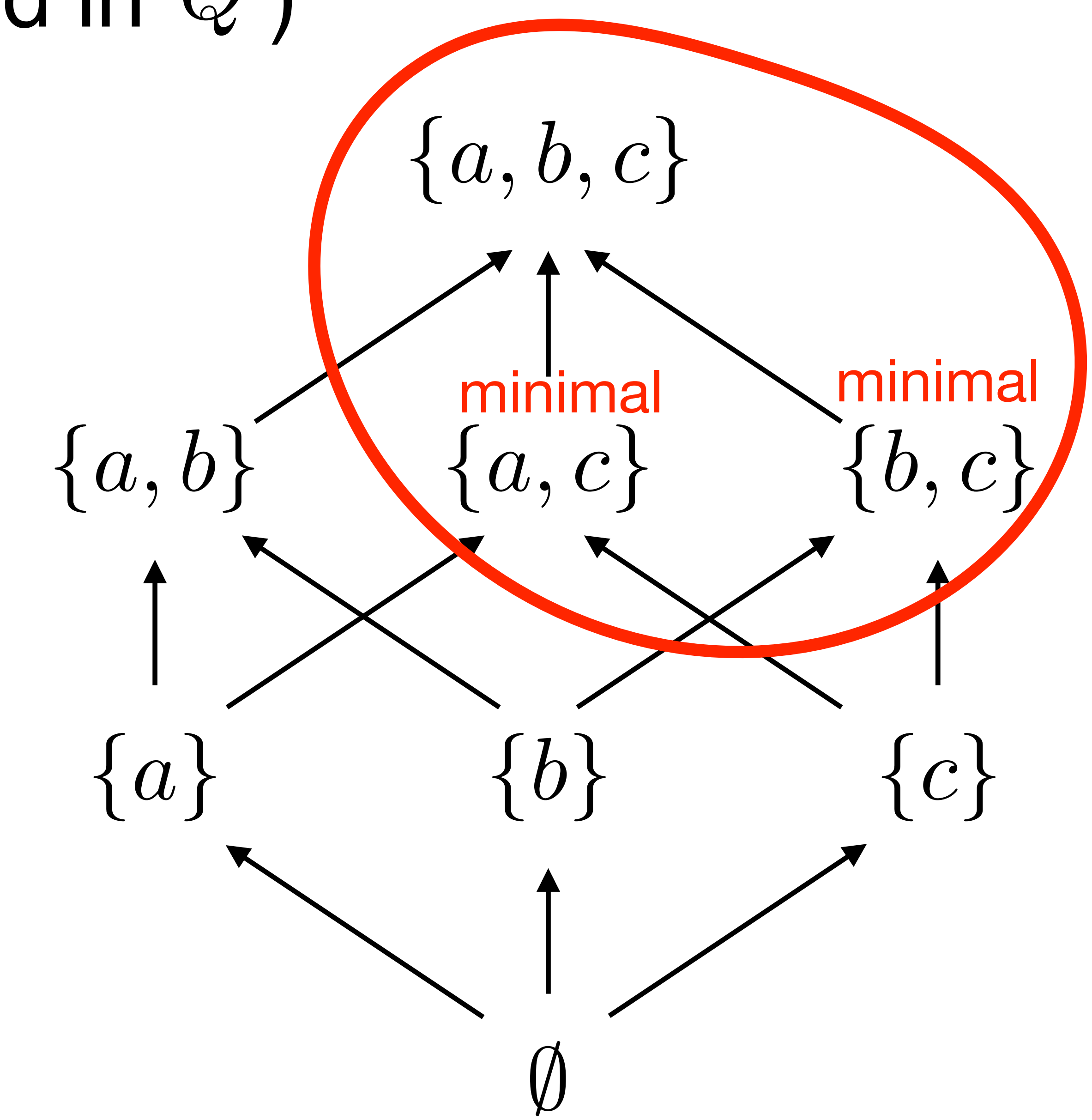
# Minimal element

$(P, \sqsubseteq)$ PO $\quad Q \subseteq P \quad m \in Q$

$m$ is a minimal element of $Q$ if $\forall q \in Q. \quad q \sqsubseteq m \Rightarrow q = m$

(no smaller element can be found in $Q$ )



# Least vs minimal

| least $\forall q \in Q. \quad \ell \sqsubseteq q$ | minimal $\forall q \in Q. \quad q \sqsubseteq m \Rightarrow q = m$ |
| --- | --- |
| unique | not necessarily unique |
| minimal | |
| | not necessarily least<br>can be least |

# Reverse order

TH. $(P, \sqsubseteq)$ PO implies $(P, \sqsupseteq)$ PO

note:
$\sqsupseteq \; = \; \sqsubseteq^{-1}$

proof. it is immediate to check that $\sqsupseteq$

is reflexive
is antisymmetric
is transitive

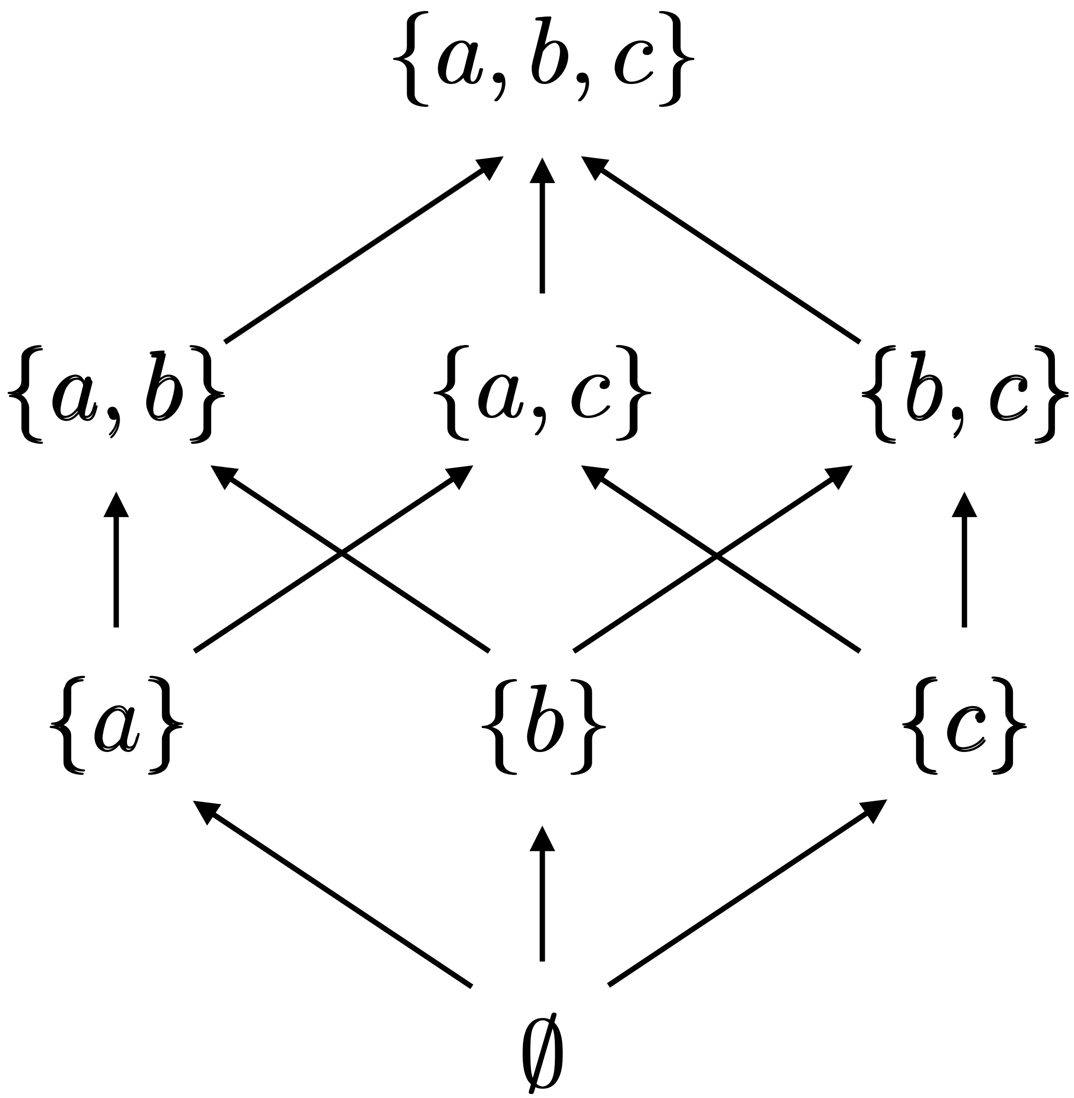
$(P, \sqsubseteq)$ PO $\quad Q \subseteq P$

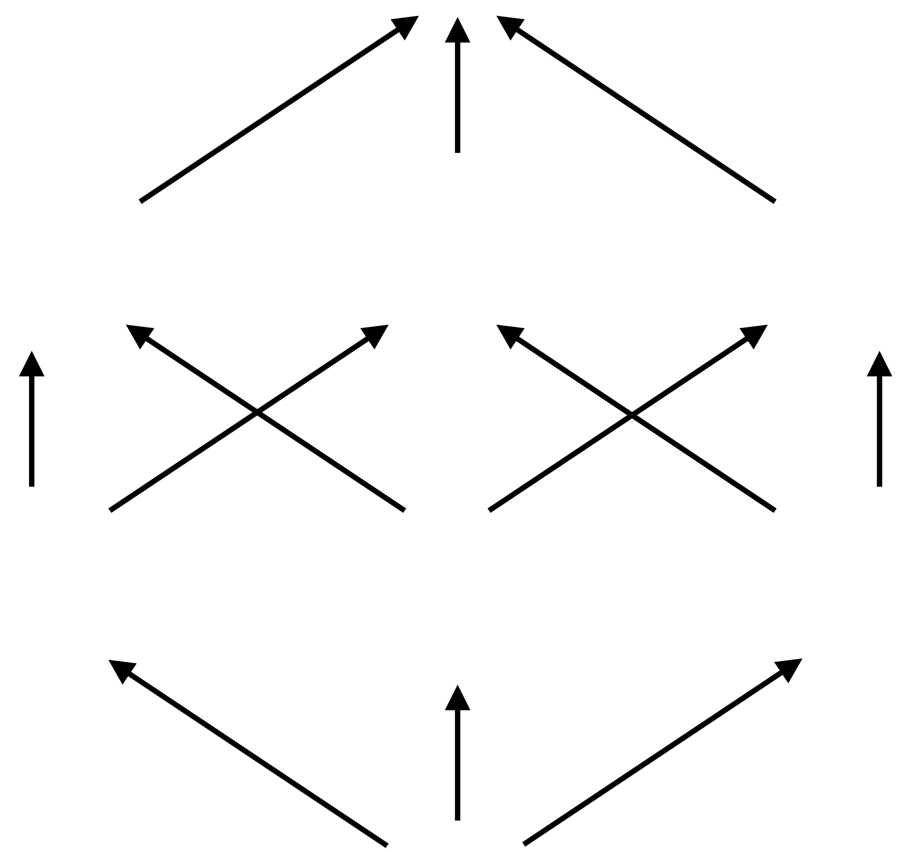
greatest element: least element of $Q$ w.r.t. $(P, \sqsupseteq)$

top element: $\top$ greatest element of $P$ (if it exists)

maximal element: minimal element of $Q$ w.r.t. $(P, \sqsupseteq)$

# Reversed powerset



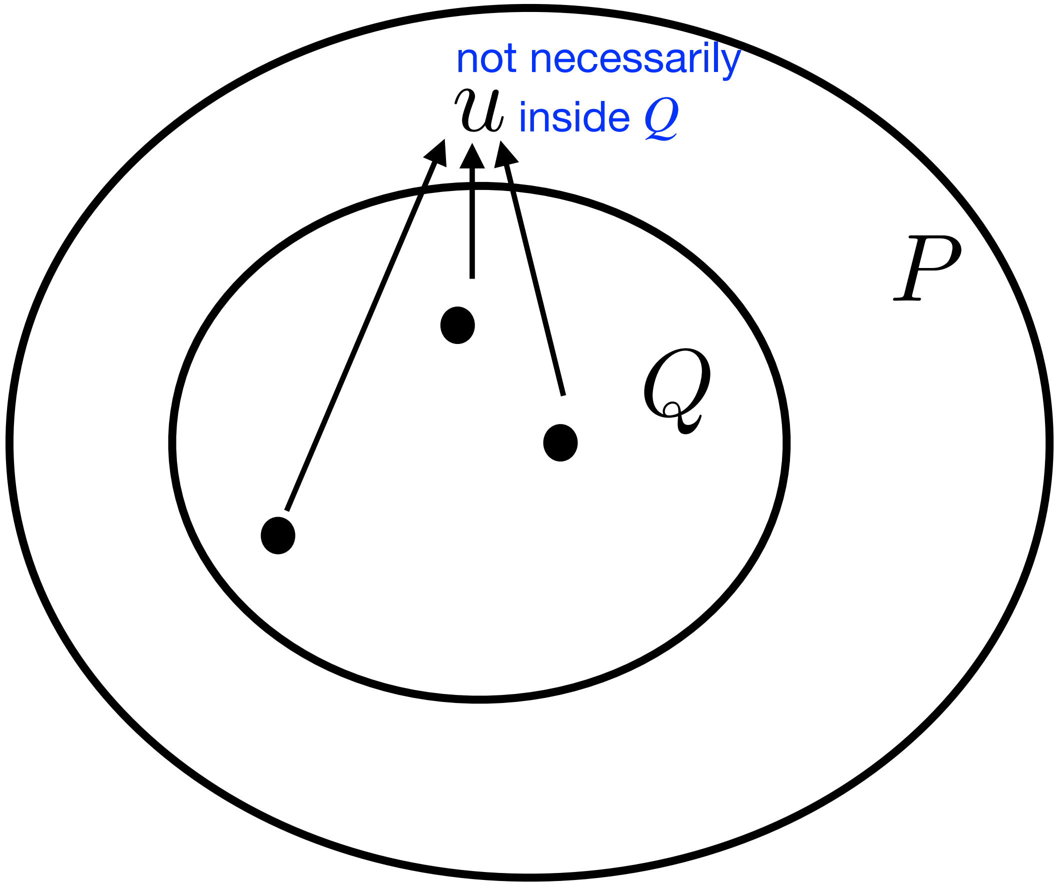



# Upper bound

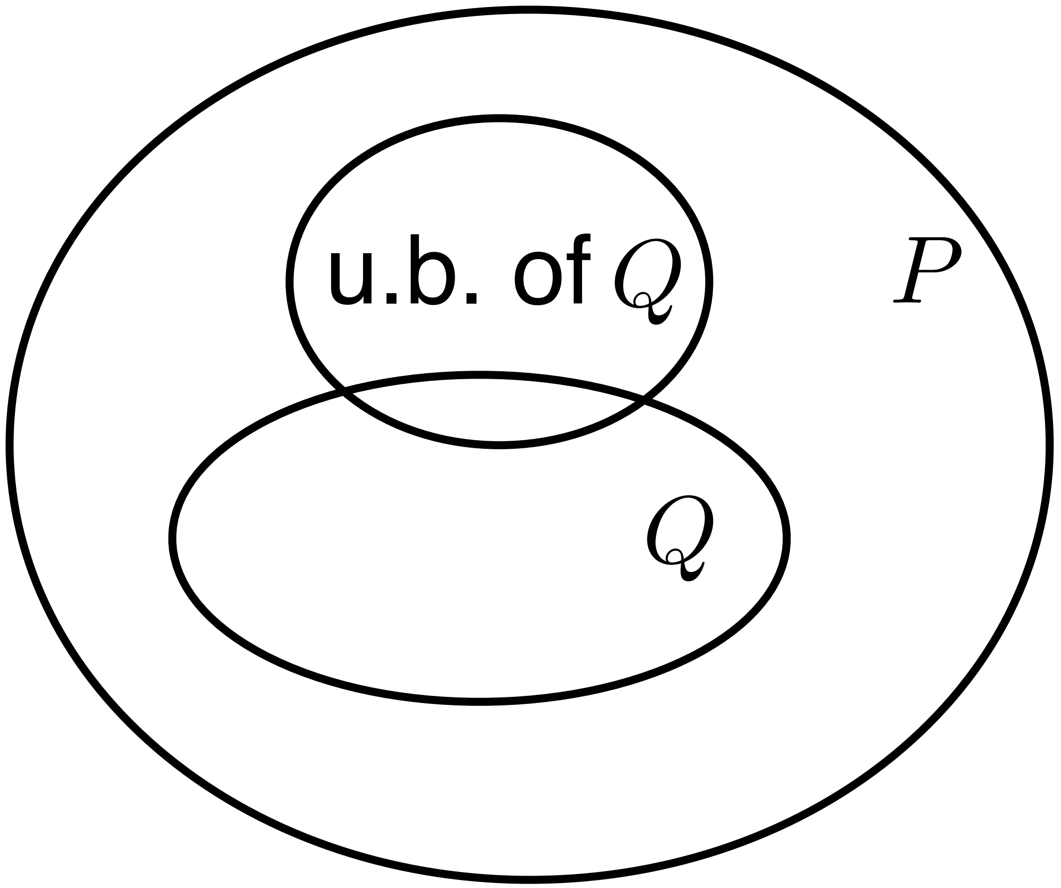
$(P, \sqsubseteq)$ PO $\quad Q \subseteq P \quad u \in P$

$u$ is an upper bound of $Q$ if $\forall q \in Q.\ \ q \sqsubseteq u$

(all the elements of $Q$ are smaller than $u$ )




$Q$ may have many upper bounds



# Least upper bound

$(P, \sqsubseteq)$ PO $\qquad Q \subseteq P \qquad p \in P$

$p$ is the least upper bound (lub) of $Q$ if

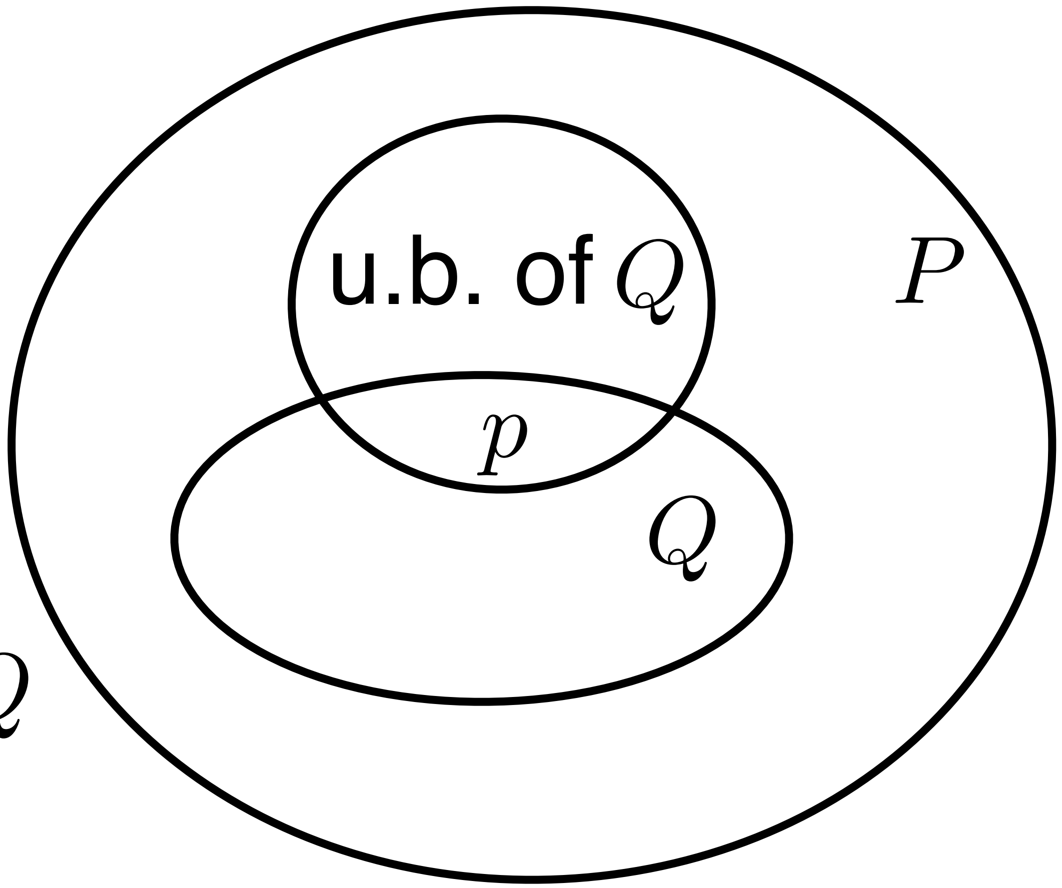
1. it is an upper bound of $Q$ $\qquad \forall q \in Q. \quad q \sqsubseteq p$
2. it is smaller than any other upper bound of $Q$

$$\forall u \in P. \quad (\forall q \in Q. q \sqsubseteq u) \Rightarrow p \sqsubseteq u$$

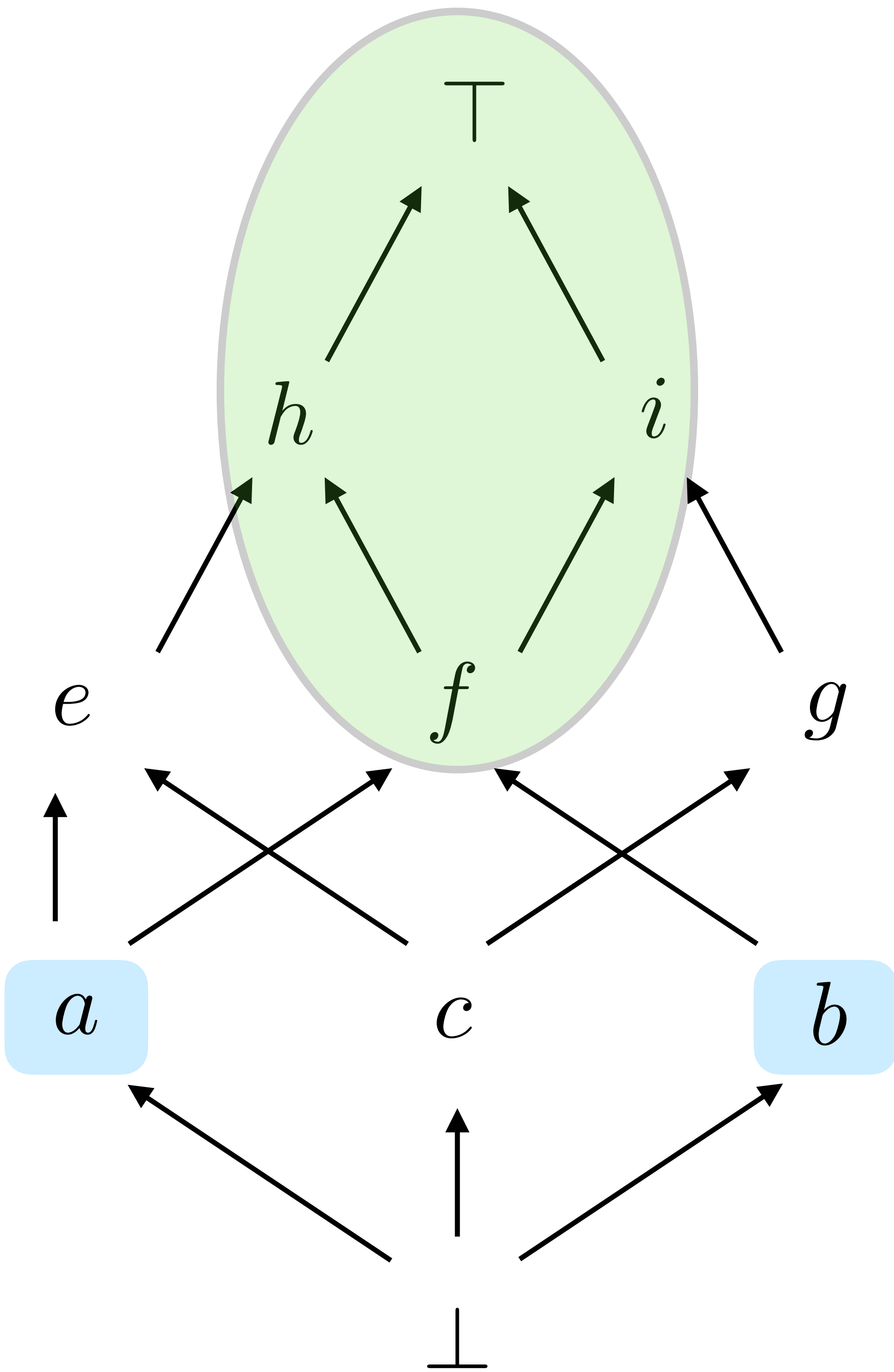
we write $p = lub\ Q$

intuitively, it is the least element that represents all of $Q$

$p$ not necessarily an element of $Q$



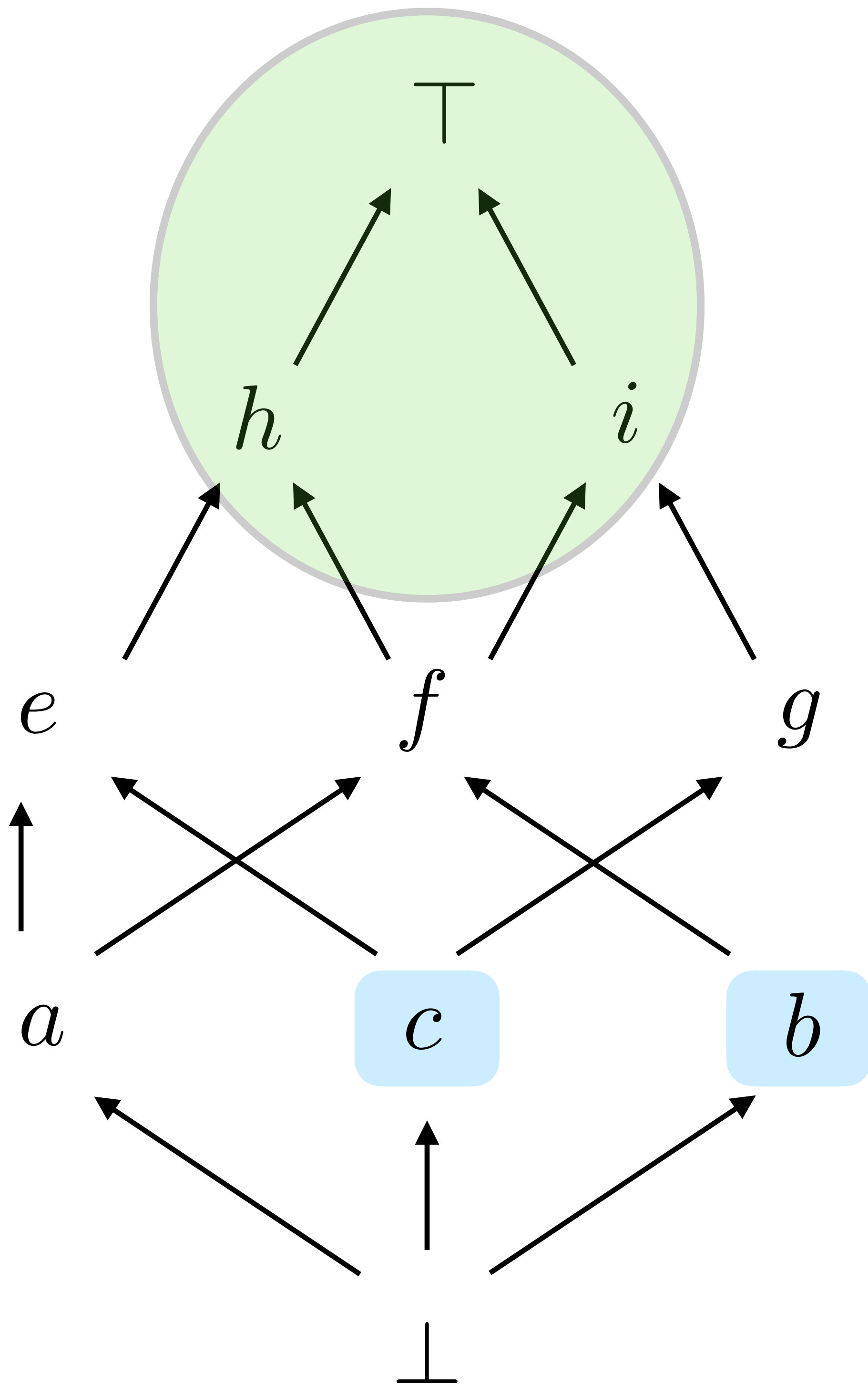
# Exercise




Upper bounds of $\{a, b\}$ ? $\{f, h, i, \top\}$

lub? $f$

# Exercise



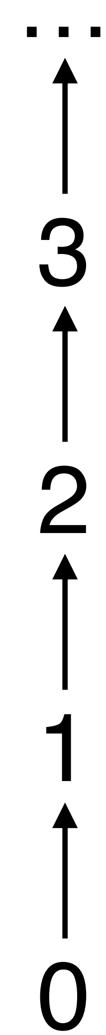

Upper bounds of $\{b, c\}$ ? $\{h, i, \top\}$

lub? no lub!

# Exercise

$(\mathbb{N}, \leq)$ $Q \subseteq \mathbb{N}$ lub? if $Q$ finite $lub\ Q = \max Q$

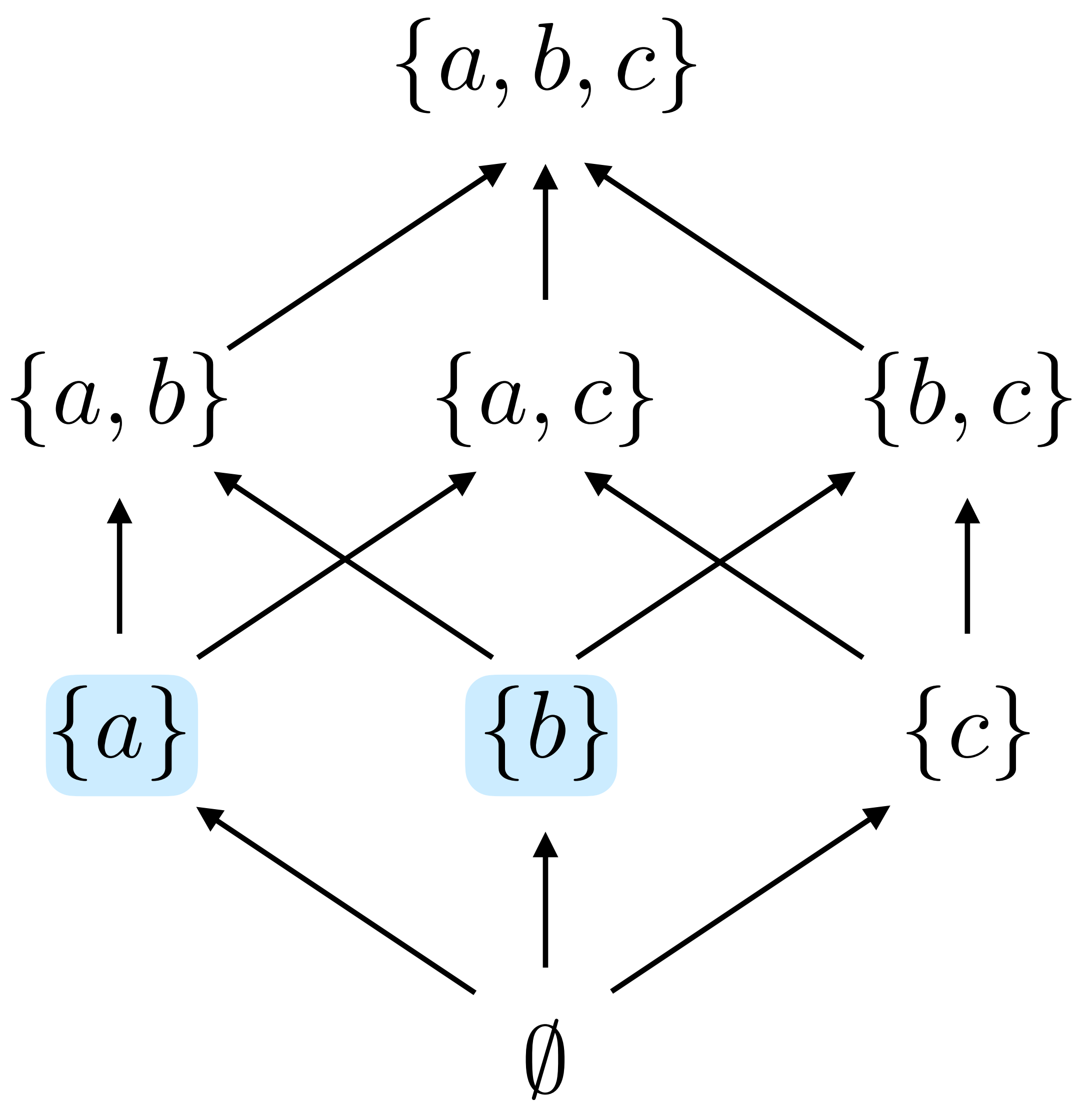
otherwise no lub



# Exercise

$(\wp(S), \subseteq)$ $\quad Q \subseteq \wp(S)$ $\quad$ lub?

$$lub\ Q = \bigcup_{T \in Q} T$$




$$lub\ \{\{a\}, \{b\}\} = \{a, b\}$$

# Complete partial orders (CPO)

# Completeness: the idea

$D$ a domain

$\sqsubseteq$ a way to compare element

$x \sqsubseteq y$ — $x$ is a (less precise) approximation of $y$; $x$ and $y$ are consistent, but $y$ is more accurate than $x$

} PO

$$x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq \cdots$$

- $x_0$: first approximation
- $x_1$: second approximation
- $x_2$: third approximation
- $x_n$: (n+1)th approximation

does any sequence of approximations tend to some limit?

# Chain

$(P, \sqsubseteq)$ PO $\{d_i\}_{i \in \mathbb{N}}$ is a chain if $\forall i \in \mathbb{N}.\ d_i \sqsubseteq d_{i+1}$

$$d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots$$

any chain is an infinite list

finite chain: there are only finitely many distinct elements

$$\exists k \in \mathbb{N}.\ \forall i \geq k.\ d_i = d_{i+1}$$

or equivalently

$$\exists k \in \mathbb{N}.\ \forall i \geq k.\ d_i = d_k$$

# Example

$(\mathbb{N}, \leq)$

$0 \leq 2 \leq 4 \leq \cdots \leq 2n \leq \cdots$ is an infinite chain

$0 \leq 1 \leq 3 \leq 3 \leq 5 \leq \cdots \leq 5 \leq \cdots$ is a finite chain

any chain has infinite length

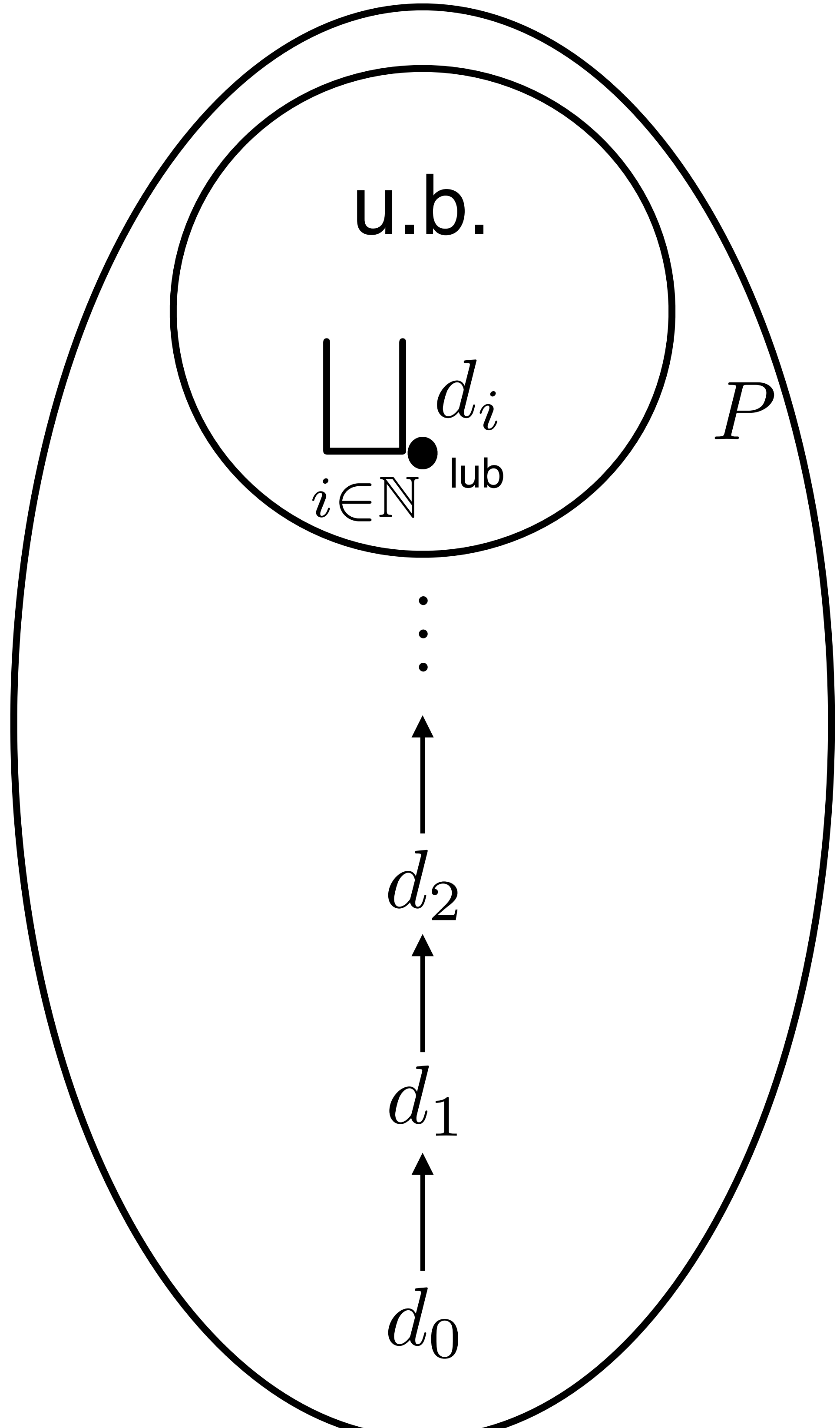
# Limit of a chain

$(P, \sqsubseteq)$ PO $\{d_i\}_{i \in \mathbb{N}}$ a chain

we denote by $\bigsqcup_{i \in \mathbb{N}} d_i$ the lub of $\{d_i\}_{i \in \mathbb{N}}$ if it exists

and call it the limit of the chain

# Limit illustrated



# Example

$(\mathbb{N}, \leq)$

$0 \leq 2 \leq 4 \leq \cdots \leq 2n \leq \cdots$

has no lub
(empty set of upper bounds)

$0 \leq 1 \leq 3 \leq 3 \leq 5 \leq \cdots \leq 5 \leq \cdots$

has lub 5
(which upper bounds?)

# Lemma on finite chains

Lemma (any finite chain has a limit)

$(P, \sqsubseteq)$ PO $\{d_i\}_{i \in \mathbb{N}}$ a finite chain $\Rightarrow$ $\bigsqcup_{i \in \mathbb{N}} d_i$ exists

proof.

$\{d_i\}_{i \in \mathbb{N}}$ finite $\Rightarrow$ $\exists k.\ \forall i.\ d_{i+k} = d_k$

the elements of the chain are totally ordered

$d_k$ is the greatest element of the chain

$d_k$ is an upper bound $\forall i.\ d_i \sqsubseteq d_k$

$d_k$ is the least upper bound

take $u$ such that $\forall i.\ d_i \sqsubseteq u$ then $d_k \sqsubseteq u$

# Prefix independence

Lemma (prefix independence) $(P, \sqsubseteq)$ PO $\{d_i\}_{i\in\mathbb{N}}$ a chain

if $\bigsqcup_{i\in\mathbb{N}} d_i$ exists $\Rightarrow \forall k.\ \bigsqcup_{i\in\mathbb{N}} d_{i+k} = \bigsqcup_{i\in\mathbb{N}} d_i$

$$d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots \sqsubseteq d_k \sqsubseteq d_{k+1} \sqsubseteq \cdots \qquad \bigsqcup_{i\in\mathbb{N}} d_i$$

=

$$d_k \sqsubseteq d_{k+1} \sqsubseteq \cdots \qquad \bigsqcup_{i\in\mathbb{N}} d_{i+k}$$

# Prefix independence

Lemma (prefix independence) $(P, \sqsubseteq)$ PO $\{d_i\}_{i\in\mathbb{N}}$ a chain

$$\text{if } \bigsqcup_{i\in\mathbb{N}} d_i \text{ exists } \Rightarrow \forall k.\ \bigsqcup_{i\in\mathbb{N}} d_{i+k} = \bigsqcup_{i\in\mathbb{N}} d_i$$

proof.

take a generic $k$

we prove that $\{d_i\}_{i\in\mathbb{N}}$ and $\{d_{i+k}\}_{i\in\mathbb{N}}$ have the same u.b.

(and thus the same lub)

1. if $u$ is an u.b. of $\{d_i\}_{i\in\mathbb{N}}$ then is an u.b. of $\{d_{i+k}\}_{i\in\mathbb{N}}$
   because $\{d_{i+k}\}_{i\in\mathbb{N}} \subseteq \{d_i\}_{i\in\mathbb{N}}$
2. if $u$ is an u.b. of $\{d_{i+k}\}_{i\in\mathbb{N}}$ we need to show $\forall j.\ d_j \sqsubseteq u$
   for $j \geq k$ it is obvious
   if $j < k$ then $d_j \sqsubseteq d_k \sqsubseteq u$ because $d_k \in \{d_{i+k}\}_{i\in\mathbb{N}}$

# Complete partial order

$(P, \sqsubseteq)$ PO $\quad$ $P$ is complete if each chain has a limit (lub)

TH. Any finite chain has a limit
(the last element in the sequence)

If $P$ has only finite chains it is complete

If $P$ is finite it is complete

Any discrete order is complete

Any flat order is complete

# Example

$(\mathbb{N}, \leq)$ is not complete
(it is enough to exhibit a chain with no limit)

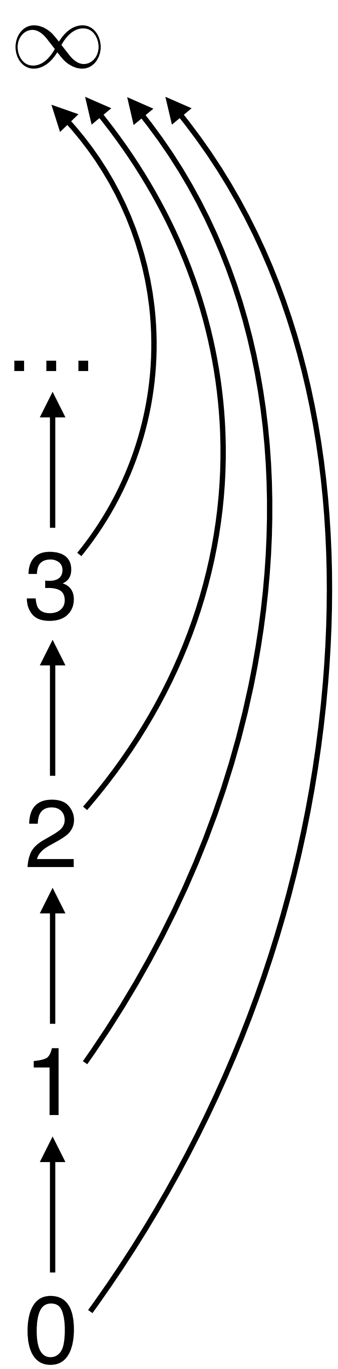
$$0 \leq 2 \leq 4 \leq \cdots \leq 2n \leq \cdots$$

has no lub
(empty set of u.b.)

# Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

complete?




any infinite chain has limit $\infty$

(set of u.b. $\{\infty\}$ )

# Exercise

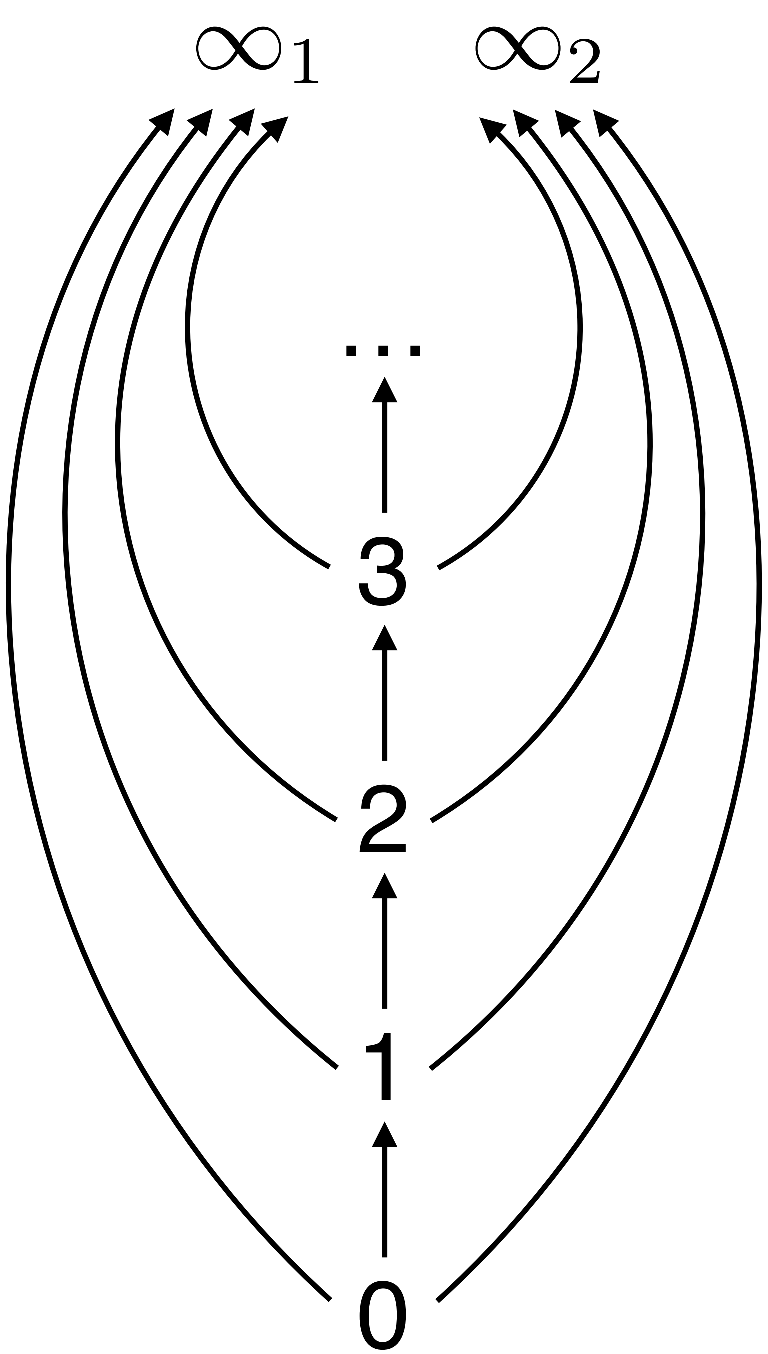
$(\wp(S), \subseteq)$ complete? ✓

$\{S_i\}_{i\in\mathbb{N}}$

$$\bigsqcup_{i\in\mathbb{N}} S_i = \bigcup_{i\in\mathbb{N}} S_i = \{x \mid \exists k \in \mathbb{N}. x \in S_k\}$$

# Exercise

$(\mathbb{N} \cup \{\infty_1, \infty_2\}, \leq)$ complete? ❌




any infinite chain has no limit
(set of u.b. $\{\infty_1, \infty_2\}$ )