

MPP 2026/27 (0077A, 9CFU)

Models for Programming Paradigms

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O8c - Immediate Consequences Operator

Kleene's fixpoint theorem

Repeated application

$$f : D \rightarrow D$$

$$f^0(d) \triangleq d$$

$$f^{n+1}(d) \triangleq f(f^n(d))$$

$$f^n(d) = \overbrace{f(\cdots (f(d)) \cdots)}^{n \text{ times}}$$

$$f^n : D \rightarrow D$$

Lemma

$(D, \sqsubseteq)$ $\text{PO}_\perp$ $f : D \rightarrow D$ monotone $\Rightarrow \{f^n(\perp)\}_{n \in \mathbb{N}}$
is a chain

proof. we need to prove $\forall n \in \mathbb{N}. f^n(\perp) \sqsubseteq f^{n+1}(\perp)$
by mathematical induction $P(n) \triangleq f^n(\perp) \sqsubseteq f^{n+1}(\perp)$

$$P(0) \triangleq f^0(\perp) \sqsubseteq f^1(\perp) \qquad f^0(\perp) = \perp \sqsubseteq f^1(\perp)$$

$\forall n \in \mathbb{N}. P(n) \Rightarrow P(n+1)$ take a generic n

assume $P(n) \triangleq f^n(\perp) \sqsubseteq f^{n+1}(\perp)$

we want to prove $P(n+1) \triangleq f^{n+1}(\perp) \sqsubseteq f^{n+2}(\perp)$

$$f^n(\perp) \sqsubseteq f^{n+1}(\perp)$$
$$\Downarrow$$

$$f^{n+1}(\perp) = f(f^n(\perp)) \sqsubseteq f(f^{n+1}(\perp)) = f^{n+2}(\perp)$$

Towards Kleene's theo.

when $(D, \sqsubseteq)$ is a $\text{CPO}_\perp$

then $\{f^n(\perp)\}_{n \in \mathbb{N}}$ is a chain

it must have a limit

$\{f^n(d)\}_{n \in \mathbb{N}}$
not necessarily
a chain!

Kleene's fix point theorem states that
if f is continuous, then the limit of the above chain
is the least fixpoint of f

Pre-fixpoints

$(D, \sqsubseteq)$ PO $f : D \rightarrow D$ monotone

fixpoint $p \in D$ $f(p) = p$

pre-fixpoint $p \in D$ $f(p) \sqsubseteq p$

Clearly any fixpoint is also a pre-fixpoint

Kleene's theorem

$(D, \sqsubseteq) \text{ CPO}_\perp \quad f : D \rightarrow D \text{ continuous}$

let $\text{fix}(f) \triangleq \bigsqcup_{n \in \mathbb{N}} f^n(\perp)$

1. $\text{fix}(f)$ is a fix point of f

$$f(\text{fix}(f)) = \text{fix}(f)$$

2. $\text{fix}(f)$ is the least pre-fixpoint of f

$$\forall d \in D. f(d) \sqsubseteq d \Rightarrow \text{fix}(f) \sqsubseteq d$$

if d is a pre-fixpoint then $\text{fix}(f)$ is smaller than d

Kleene's theorem: 1

$$1. \quad f(\text{fix}(f)) = \text{fix}(f)$$

proof.

$$f(\text{fix}(f)) = f\left(\bigsqcup_{n \in \mathbb{N}} f^n(\perp)\right) \quad \text{by def of } \text{fix}$$

$$= \bigsqcup_{n \in \mathbb{N}} f(f^n(\perp)) \quad \text{by continuity}$$

$$= \bigsqcup_{n \in \mathbb{N}} f^{n+1}(\perp) \quad \text{by def of } f^n$$

$$= \bigsqcup_{n \in \mathbb{N}} f^n(\perp) \quad \text{by prefix independence of limits}$$

$$= \text{fix}(f) \quad \text{by def of } \text{fix}$$

Kleene's theorem: 2

$$2. \quad \forall d \in D. f(d) \sqsubseteq d \Rightarrow \text{fix}(f) \sqsubseteq d$$

proof.

we prove that any pre-fixpoint is an upper bound of the chain

$$\{f^n(\perp)\}_{n \in \mathbb{N}}$$

by definition $\text{fix}(f)$ is the lub of the same chain
and thus smaller than any other upper bound

Kleene's theorem: 2

$$2. \quad \forall d \in D. f(d) \sqsubseteq d \Rightarrow \text{fix}(f) \sqsubseteq d$$

take any $d \in D$ such that $f(d) \sqsubseteq d$

we prove $\forall n \in \mathbb{N}. f^n(\perp) \sqsubseteq d$ (d is an upper bound)

$P(n) \triangleq f^n(\perp) \sqsubseteq d$ by mathematical induction

$$P(0) \triangleq f^0(\perp) \sqsubseteq d \quad f^0(\perp) = \perp \sqsubseteq d$$

$\forall n \in \mathbb{N}. P(n) \Rightarrow P(n+1)$ take a generic n
assume $P(n) \triangleq f^n(\perp) \sqsubseteq d$

we want to prove $P(n+1) \triangleq f^{n+1}(\perp) \sqsubseteq d$

$$\begin{array}{ccccc} & & f^n(\perp) \sqsubseteq d & & \\ & & \downarrow \text{(monot.)} & & \\ f^{n+1}(\perp) & \stackrel{\text{(by def)}}{=} & f(f^n(\perp)) & \sqsubseteq & f(d) \sqsubseteq d \quad \text{(pre-fixpoint)} \end{array}$$

Example

$$n = 2 \cdot n$$

$$(\mathbb{N} \cup \{\infty\}, \leq)$$

$$\perp = 0$$

$\text{CPO}_{\perp}$

$$\begin{aligned} f(n) &= 2 \cdot n \\ f(\infty) &= \infty \end{aligned}$$

monotone? ok

continuous? ok

$$f^0(0) = 0$$

$$f^1(0) = f(0) = 2 \cdot 0 = 0$$

fixpoint reached!

Example

$$n = n + 1$$

$$(\mathbb{N} \cup \{\infty\}, \leq)$$

$$\perp = 0$$

$\text{CPO}_{\perp}$

$$\begin{aligned} f(n) &= n + 1 \\ f(\infty) &= \infty \end{aligned}$$

monotone? ok

continuous? ok

$$f^0(0) = 0$$

$$f^1(0) = f(0) = 0 + 1 = 1$$

$$f^2(0) = f(f^1(0)) = f(1) = 1 + 1 = 2$$

$$f^3(0) = f(f^2(0)) = f(2) = 2 + 1 = 3$$

$$\bigsqcup_{n \in \mathbb{N}} f^n(0) = \bigsqcup_{n \in \mathbb{N}} n = \infty \quad \text{fixpoint}$$

Example

$$X = X \cap \{1\}$$

$$(\wp(\mathbb{N}), \subseteq)$$

$$\perp = \emptyset \quad \text{CPO}_{\perp}$$

$$f(X) = X \cap \{1\}$$

monotone? ok
continuous? ok

$$f^0(\emptyset) = \emptyset$$

$$f^1(\emptyset) = f(\emptyset) = \emptyset \cap \{1\} = \emptyset$$

fixpoint reached!

Example

$$X = \mathbb{N} \setminus X$$

$$(\wp(\mathbb{N}), \subseteq)$$

$$\perp = \emptyset \quad \text{CPO}_{\perp}$$

$$f(X) = \mathbb{N} \setminus X$$

monotone? NO

the larger X the smaller $f(X)$

$$f^0(\emptyset) = \emptyset$$

$$f^1(\emptyset) = f(\emptyset) = \mathbb{N} \setminus \emptyset = \mathbb{N}$$

$$f^2(\emptyset) = f(f^1(\emptyset)) = f(\mathbb{N}) = \mathbb{N} \setminus \mathbb{N} = \emptyset$$

not a chain!

Example

$$X = X \cup \{1\}$$

$$(\wp(\mathbb{N}), \subseteq)$$

$$\perp = \emptyset \quad \text{CPO}_{\perp}$$

$$f(X) = X \cup \{1\}$$

monotone? ok
continuous? ok

$$f^0(\emptyset) = \emptyset$$

$$f^1(\emptyset) = f(\emptyset) = \emptyset \cup \{1\} = \{1\}$$

$$f^2(\emptyset) = f(f^1(\emptyset)) = f(\{1\}) = \{1\} \cup \{1\} = \{1\}$$

fixpoint reached!

Intermezzo

McCarthy 91 function

$$f = M f \quad (\mathbf{Pf}(\mathbb{N}, \mathbb{N}), \sqsubseteq) \quad \perp = \emptyset \quad \text{CPO}_\perp$$

$$M f n = \begin{cases} n - 10 & \text{if } n > 100 \\ f(f(n + 11)) & \text{if } n \leq 100 \text{ and } f(n + 11) \downarrow \\ \perp & \text{otherwise} \end{cases}$$

$$M \text{ monotone?} \quad f \sqsubseteq g \stackrel{?}{\Rightarrow} M(f) \sqsubseteq M(g)$$

$$M \text{ continuous?} \quad M \left(\bigsqcup_i f_i \right) \stackrel{?}{\sqsubseteq} \bigsqcup_i M(f_i)$$

McCarthy 91 function

$$M\ f\ n = \begin{cases} n - 10 & \text{if } n > 100 \\ f(f(n + 11)) & \text{if } n \leq 100 \text{ and } f(n + 11) \downarrow \\ \perp & \text{otherwise} \end{cases}$$

M monotone? 

we assume $f \sqsubseteq g$ and prove $M(f) \sqsubseteq M(g)$

take $(n, m) \in M(f)$ we prove $(n, m) \in M(g)$

if $n > 100$

$m = n - 10$ and $(n, n - 10) \in M(g)$

if $n \leq 100$

$m = f(\overbrace{f(n + 11)}^k)$

thus $\exists k. \{(n + 11, k), (k, m)\} \subseteq f \subseteq g$
and $(n, m) \in M(g)$

McCarthy 91 function

$$M f n = \begin{cases} n - 10 & \text{if } n > 100 \\ f(f(n + 11)) & \text{if } n \leq 100 \text{ and } f(n + 11) \downarrow \\ \perp & \text{otherwise} \end{cases}$$

M continuous? 

let $f \triangleq \bigsqcup_i f_i$

take $(n, m) \in M(f)$

$$M \left(\bigsqcup_i f_i \right) \stackrel{?}{\subseteq} \bigsqcup_i M(f_i)$$

we prove $(n, m) \in \bigsqcup_i M(f_i)$

if $n > 100$ $m = n - 10$ and $(n, n - 10) \in M(f_1) \subseteq \bigsqcup_i M(f_i)$
 if $n \leq 100$ $m = f(f(n + 11))$

thus $\exists k. \{(n + 11, k), (k, m)\} \subseteq f$

thus $\exists j_1, j_2. (n + 11, k) \in f_{j_1}, (k, m) \in f_{j_2}$

let $j \triangleq \max\{j_1, j_2\}$ thus $\{(n + 11, k), (k, m)\} \subseteq f_j$

and $(n, m) \in M(f_j) \subseteq \bigsqcup_i M(f_i)$

McCarthy 91 function

$$f = M f \quad (\mathbf{Pf}(\mathbb{N}, \mathbb{N}), \sqsubseteq) \quad \perp = \emptyset \quad \text{CPO}_{\perp}$$

$$M f n = \begin{cases} n - 10 & \text{if } n > 100 \\ f(f(n + 11)) & \text{if } n \leq 100 \text{ and } f(n + 11) \downarrow \\ \perp & \text{otherwise} \end{cases}$$

$$M^0(\emptyset) = \emptyset$$

$$M^1(\emptyset) = \{(n, n - 10) \mid n > 100\} = \{(101, 91), (102, 92), \dots\}$$

$$M^2(\emptyset) = \{(100, 91), (101, 91), (102, 92), \dots\}$$

$$M^3(\emptyset) = \{(99, 91), (100, 91), (101, 91), (102, 92), \dots\}$$

...

$$M^{11}(\emptyset) = \{(91, 91), (92, 91), \dots, (101, 91), (102, 92), \dots\}$$

$$M^{12}(\emptyset) = \{(80, 91), (81, 91), \dots, (101, 91), (102, 92), \dots\}$$

...

$$M^{20}(\emptyset) = \{(0, 91), (1, 91), \dots, (101, 91), (102, 92), \dots\}$$

fixpoint reached! (total function)

Alternative perspective

$$f = M f \quad (\mathbf{Pf}(\mathbb{N}, \mathbb{N}), \sqsubseteq) \quad \perp = \emptyset \quad \text{CPO}_\perp$$

$$M f n = \begin{cases} n - 10 & \text{if } n > 100 \\ f(f(n + 11)) & \text{if } n \leq 100 \text{ and } f(n + 11) \downarrow \\ \perp & \text{otherwise} \end{cases}$$

formulas: $(n, m) \in f$

or just (n, m) for brevity

$$R_M \triangleq \left\{ \frac{}{(n, n - 10)} n > 100 \quad , \quad \frac{(n + 11, k) \quad (k, m)}{(n, m)} n \leq 100 \right\}$$

$$f(n) = m \quad \Leftrightarrow \quad (n, m) \in I_{R_M}$$

(set of theorems of R_M)

Immediate Consequences Operator (ICO)

Immediate Consequences Operator

F a set of formulas

$(\wp(F), \subseteq)$ $\text{CPO}_\perp$

R a logical system
(with some restrictions)

$\hat{R} : \wp(F) \rightarrow \wp(F)$ ICO

$$\hat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \dots x_n}{y} \in R. \{x_1, \dots, x_n\} \subseteq S \right\}$$

$\hat{R}$ applies the rules to the facts in S in all possible ways

$\hat{R}(S)$: all conclusions we can draw in one step
from hypotheses on S

We prove the least fixpoint of $\hat{R}$ is I_R
(set of theorems of R)

Example

Strings of balanced parentheses

$$F = \{s \in \mathcal{L} \mid s \in \{ (,) \}^* \}$$

$$R = \left\{ \frac{}{\epsilon \in \mathcal{L}} , \frac{s \in \mathcal{L}}{(s) \in \mathcal{L}} , \frac{s_1 \in \mathcal{L} \quad s_2 \in \mathcal{L}}{s_1 s_2 \in \mathcal{L}} \right\}$$

$$S = \emptyset$$

$$\hat{R}(S) = \{ \epsilon \in \mathcal{L} \}$$

$$\hat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \dots x_n}{y} \in R. \{x_1, \dots, x_n\} \subseteq S \right\}$$



Exercise

Strings of balanced parentheses

$$F = \{s \in \mathcal{L} \mid s \in \{(\,,\,)\}^*\}$$

$$R = \left\{ \frac{}{\epsilon \in \mathcal{L}} , \frac{s \in \mathcal{L}}{(s) \in \mathcal{L}} , \frac{s_1 \in \mathcal{L} \quad s_2 \in \mathcal{L}}{s_1 s_2 \in \mathcal{L}} \right\}$$

$$S = \{ \epsilon \in \mathcal{L} \}$$

$$\hat{R}(S) = ? \{ \epsilon \in \mathcal{L} , () \in \mathcal{L} \}$$

$$\hat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \dots x_n}{y} \in R. \{x_1, \dots, x_n\} \subseteq S \right\}$$



Exercise

Strings of balanced parentheses

$$F = \{s \in \mathcal{L} \mid s \in \{(\,,\,)\}^*\}$$

$$R = \left\{ \frac{}{\epsilon \in \mathcal{L}} , \frac{s \in \mathcal{L}}{(s) \in \mathcal{L}} , \frac{s_1 \in \mathcal{L} \quad s_2 \in \mathcal{L}}{s_1 s_2 \in \mathcal{L}} \right\}$$

$$S = \{ () \in \mathcal{L} \}$$

$$\hat{R}(S) = ? \{ \epsilon \in \mathcal{L} , (()) \in \mathcal{L} , ()() \in \mathcal{L} \}$$

$$\hat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \dots x_n}{y} \in R. \{x_1, \dots, x_n\} \subseteq S \right\}$$



Exercise

Strings of balanced parentheses

$$F = \{s \in \mathcal{L} \mid s \in \{(\,,\,)\}^*\}$$

$$R = \left\{ \frac{}{\epsilon \in \mathcal{L}} , \frac{s \in \mathcal{L}}{(s) \in \mathcal{L}} , \frac{s_1 \in \mathcal{L} \quad s_2 \in \mathcal{L}}{s_1 s_2 \in \mathcal{L}} \right\}$$

$$S = \{ \text{)}(\in \mathcal{L} \}$$

$$\hat{R}(S) = ? \{ \epsilon \in \mathcal{L} , ()() \in \mathcal{L} ,)()(\in \mathcal{L} \}$$

$$\hat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \dots x_n}{y} \in R. \{x_1, \dots, x_n\} \subseteq S \right\}$$

Example

Strings of balanced parentheses

$$F = \{s \in \mathcal{L} \mid s \in \{ (,) \}^*\}$$

$$R = \left\{ \frac{}{\epsilon \in \mathcal{L}} , \frac{s \in \mathcal{L}}{(s) \in \mathcal{L}} , \frac{s_1 \in \mathcal{L} \quad s_2 \in \mathcal{L}}{s_1 s_2 \in \mathcal{L}} \right\}$$

$$S = \{)(\in \mathcal{L} , () \in \mathcal{L} \}$$

$$\hat{R}(S) = \{ \epsilon \in \mathcal{L} , ()() \in \mathcal{L} , (()) \in \mathcal{L} ,)() (\in \mathcal{L} ,)(() \in \mathcal{L} , (()) \in \mathcal{L} \}$$

$$\hat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \dots x_n}{y} \in R. \{x_1, \dots, x_n\} \subseteq S \right\}$$

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$F = \{(n, m) \mid n, m \in \mathbb{N}\}$$

$$R = \left\{ \overline{(0, 0)}, \overline{(1, 1)}, \frac{(n, h) \quad (n+1, k)}{(n+2, h+k)} \right\}$$

$$S = \emptyset$$

$$\hat{R}(S) = \{ (0, 0), (1, 1) \}$$

$$\hat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \dots x_n}{y} \in R. \{x_1, \dots, x_n\} \subseteq S \right\}$$



Exercise

Pf($\mathbb{N}, \mathbb{N}$)

$$F = \{(n, m) \mid n, m \in \mathbb{N}\}$$

$$R = \left\{ \overline{(0, 0)}, \overline{(1, 1)}, \frac{(n, h) \quad (n+1, k)}{(n+2, h+k)} \right\}$$

$$S = \{ (2, 1) \}$$

$$\hat{R}(S) = ? \{ (0, 0), (1, 1) \}$$

$$\hat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \dots x_n}{y} \in R. \{x_1, \dots, x_n\} \subseteq S \right\}$$



Exercise

Pf($\mathbb{N}, \mathbb{N}$)

$$F = \{(n, m) \mid n, m \in \mathbb{N}\}$$

$$R = \left\{ \overline{(0, 0)}, \overline{(1, 1)}, \frac{(n, h) \quad (n+1, k)}{(n+2, h+k)} \right\}$$

$$S = \{ (5, 5), (6, 8) \}$$

$$\hat{R}(S) = ? \{ (0, 0), (1, 1), (7, 13) \}$$

$$\hat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \dots x_n}{y} \in R. \{x_1, \dots, x_n\} \subseteq S \right\}$$

ICO is monotone

TH. $\hat{R}$ is monotone

proof.

Take $S_1 \subseteq S_2$

we want to prove $\hat{R}(S_1) \subseteq \hat{R}(S_2)$

take $y \in \hat{R}(S_1)$ we want to prove $y \in \hat{R}(S_2)$

$\Downarrow$

$\exists \frac{x_1 \dots x_n}{y} \in R$ with $\{x_1, \dots, x_n\} \subseteq S_1$

$\Downarrow S_1 \subseteq S_2$

$\{x_1, \dots, x_n\} \subseteq S_2$

$\Downarrow$

$y \in \hat{R}(S_2)$

ICO is continuous

TH. $\hat{R}$ is continuous (under some hypotheses)

proof. Take $\{S_i\}_{i \in \mathbb{N}}$ a chain in $\wp(F)$

We want to prove $\bigcup_{i \in \mathbb{N}} \hat{R}(S_i) = \hat{R} \left(\bigcup_{i \in \mathbb{N}} S_i \right)$

$\bigcup_{i \in \mathbb{N}} \hat{R}(S_i) \subseteq \hat{R} \left(\bigcup_{i \in \mathbb{N}} S_i \right)$ because $\hat{R}$ is monotone

$\bigcup_{i \in \mathbb{N}} \hat{R}(S_i) \supseteq \hat{R} \left(\bigcup_{i \in \mathbb{N}} S_i \right)$?

ICO is continuous (ctd)

$$\bigcup_{i \in \mathbb{N}} \hat{R}(S_i) \supseteq \hat{R} \left(\bigcup_{i \in \mathbb{N}} S_i \right)$$

Take $y \in \hat{R} \left(\bigcup_{i \in \mathbb{N}} S_i \right)$ we want to prove $y \in \bigcup_{i \in \mathbb{N}} \hat{R}(S_i)$

$\Downarrow$

$$\exists \frac{x_1 \dots x_n}{y} \in R \text{ with } \{x_1, \dots, x_n\} \subseteq \bigcup_{i \in \mathbb{N}} S_i$$

thus $\forall j \in [1, n]. \exists k_j \in \mathbb{N}. x_j \in S_{k_j}$ take $k = \max\{k_1, \dots, k_n\}$

clearly $\{x_1, \dots, x_n\} \subseteq S_k$

thus $y \in \hat{R}(S_k) \subseteq \bigcup_{i \in \mathbb{N}} \hat{R}(S_i)$

**possible iff each rule has
finitely many premises**

Provable theorems

I_R set of theorems in R

I_R^n theorems provable with a derivation of height at most n

$$I_R^0 = \emptyset$$

$$I_R^{n+1} = I_R^n \cup \hat{R}(I_R^n)$$

↑
provable using one more inference step

↑
theorems provable with a derivation of height at most n

clearly $I_R = \bigcup_{n \in \mathbb{N}} I_R^n$

n-depth theorems

TH. Let $P(n) \triangleq I_R^n = \hat{R}^n(\emptyset) \quad \forall n \in \mathbb{N}. P(n)$

proof.

by mathematical induction

$$P(0) \triangleq I_R^0 = \hat{R}^0(\emptyset)$$

$$I_R^0 = \emptyset = \hat{R}^0(\emptyset)$$

$$\forall n \in \mathbb{N}. P(n) \Rightarrow P(n+1)$$

take a generic n
we assume $P(n) \triangleq I_R^n = \hat{R}^n(\emptyset)$

we want to prove $P(n+1) \triangleq I_R^{n+1} = \hat{R}^{n+1}(\emptyset)$

$$I_R^{n+1} = I_R^n \cup \hat{R}(I_R^n)$$

by def

$$= \hat{R}^n(\emptyset) \cup \hat{R}(\hat{R}^n(\emptyset))$$

by inductive hypothesis

$$= \hat{R}^n(\emptyset) \cup \hat{R}^{n+1}(\emptyset)$$

by def

$$= \hat{R}^{n+1}(\emptyset)$$

because $\hat{R}^n(\emptyset) \subseteq \hat{R}^{n+1}(\emptyset)$

ICO's fixpoint

TH. $fix(\hat{R}) = I_R$ (under some hypotheses)

**each rule must have
finitely many premises**

proof.

by Kleene's fixpoint theorem we know that $fix(\hat{R})$ exists

$$fix(\hat{R}) = \bigcup_{n \in \mathbb{N}} \hat{R}^n(\emptyset) \text{ by def}$$

$$= \bigcup_{n \in \mathbb{N}} I_R^n \text{ by previous result}$$

$$= I_R \text{ by def}$$

Example

Strings of balanced parentheses

$$F = \{s \in \mathcal{L} \mid s \in \{ (,) \}^* \}$$

$$R = \left\{ \frac{}{\epsilon \in \mathcal{L}} , \frac{s \in \mathcal{L}}{(s) \in \mathcal{L}} , \frac{s_1 \in \mathcal{L} \quad s_2 \in \mathcal{L}}{s_1 s_2 \in \mathcal{L}} \right\}$$

$$\hat{R}^0(\emptyset) = \emptyset$$

$$\hat{R}^1(\emptyset) = \{ \epsilon \in \mathcal{L} \}$$

$$\hat{R}^2(\emptyset) = \{ \epsilon \in \mathcal{L} , () \in \mathcal{L} \}$$

$$\hat{R}^3(\emptyset) = \{ \epsilon \in \mathcal{L} , () \in \mathcal{L} , (()) \in \mathcal{L} , ()() \in \mathcal{L} \}$$

$$\begin{aligned} \hat{R}^4(\emptyset) = \{ & \epsilon \in \mathcal{L} , () \in \mathcal{L} , (()) \in \mathcal{L} , ()() \in \mathcal{L} , ((())) \in \mathcal{L} , \\ & (()()) \in \mathcal{L} , ()(()) \in \mathcal{L} , (())() \in \mathcal{L} , ()()() \in \mathcal{L} , \\ & (())()() \in \mathcal{L} , ()()()() \in \mathcal{L} , (())()() \in \mathcal{L} , ()()()() \in \mathcal{L} \} \end{aligned}$$

Example

$$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$$

$$F = \{(n, m) \mid n, m \in \mathbb{N}\}$$

$$R = \left\{ \overline{(0, 0)} \ , \ \overline{(1, 1)} \ , \ \frac{(n, h) \quad (n+1, k)}{(n+2, h+k)} \right\}$$

$$\hat{R}^0(\emptyset) = \emptyset$$

$$\hat{R}^1(\emptyset) = \{ (0, 0) \ , \ (1, 1) \}$$

$$\hat{R}^2(\emptyset) = \{ (0, 0) \ , \ (1, 1) \ , \ (2, 1) \}$$

$$\hat{R}^3(\emptyset) = \{ (0, 0) \ , \ (1, 1) \ , \ (2, 1) \ , \ (3, 2) \}$$

$$\hat{R}^4(\emptyset) = \{ (0, 0) \ , \ (1, 1) \ , \ (2, 1) \ , \ (3, 2) \ , \ (4, 3) \}$$

$$\hat{R}^5(\emptyset) = \{ (0, 0) \ , \ (1, 1) \ , \ (2, 1) \ , \ (3, 2) \ , \ (4, 3) \ , \ (5, 5) \}$$

$$\hat{R}^6(\emptyset) = \{ (0, 0) \ , \ (1, 1) \ , \ (2, 1) \ , \ (3, 2) \ , \ (4, 3) \ , \ (5, 5) \ , \ (6, 8) \}$$

Example

$$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$$

$$F = \{(n, m) \mid n, m \in \mathbb{N}\}$$

$$R_M \triangleq \left\{ \begin{array}{l} \overline{(n, n-10)} \quad n > 100 \quad , \quad \frac{\overbrace{(n+11, k)}^{(111,101)(110,100)} \quad \overbrace{(k, m)}^{(101,91)(100,?)}}{(n, m)} \quad n \leq 100 \end{array} \right\}$$

$\underbrace{(100,91)(99,?)}$

$$\hat{R}_M^0(\emptyset) = \emptyset$$

$$\hat{R}_M^1(\emptyset) = \{(n, n-10) \mid n > 100\} = \{(101,91), (102,92), \dots\}$$

$$\hat{R}_M^2(\emptyset) = \{(100,91), (101,91), (102,92), \dots\}$$

$$\hat{R}_M^3(\emptyset) = \{(99,91), (100,91), (101,91), (102,92), \dots\}$$

$$\hat{R}_M^{11}(\emptyset) = \overset{\dots}{\{ (91,91), (92,91), \dots, (99,91), (100,91), (101,91), (102,92), \dots \}}$$

$$\hat{R}_M^{12}(\emptyset) = \{ (80,91), (81,91), \dots, (99,91), (100,91), (101,91), (102,92), \dots \}$$

$$\hat{R}_M^{20}(\emptyset) = \overset{\dots}{\{ (0,91), (1,91), \dots, (99,91), (100,91), (101,91), (102,92), \dots \}}$$

fixpoint reached! (total function)