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MPP 2026/27 (0077A, 9CFU)

Models for Programming Paradigms

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08b - Kleene's fixed point theorem

Comparing functions

given two functions $f, g : A \to B$, when can we say $f = g$?

$$\forall a \in A \,.\, f(a) = g(a)$$

if we see functions as relations

$$\{\ (a, f(a)) \mid a \in A\ \} \ \subseteq\ A \times B$$

we can use set equality

Example

$$f(n) = n!$$

$$f(n) = n(n+1)/2$$

$$\begin{array}{ll} & \quad n \quad f(n) \\ f = \{ & (0,1), \\ & (1,1), \\ & (2,2), \\ & (3,6), \\ & \ldots \\ & (k,k!), \\ & \ldots \qquad \} \end{array}$$

$$\begin{array}{ll} f = \{ & (0,0), \\ & (1,1), \\ & (2,3), \\ & (3,6), \\ & \ldots \\ & (k,T_k), \\ & \ldots \qquad \} \end{array}$$

Partial functions

let $f\colon A \rightharpoonup B$, or equivalently $f\colon A \to B \cup \{ \perp \}$

the function f can be undefined on some inputs

we can still see partial functions as relations

$$\{ (a, f(a)) \mid a \in A, f(a) \neq \perp \} \subseteq A \times B$$

omit pairs where $f(a)$ is undefined

Partial functions

$D = (A \rightharpoonup B) = \mathbf{Pf}(A, B) = \{f : A \rightharpoonup B\}$ partial functions

$f \sqsubseteq g$ if $f(a)$ is defined, $g(a)$ is defined and $g(a) = f(a)$

but $g(a)$ can be defined when $f(a)$ is not

if we see partial functions as relations

$$\{(x, f(x)) \mid f(x) \neq \bot\} \subseteq A \times B$$

$f \sqsubseteq g$ means essentially $f \subseteq g$

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ \perp & \text{otherwise} \end{cases}$$

$$\begin{array}{lll} & \overset{n}{} & \overset{f(n)}{} \\ f = \{ & (0, 0), & \\ & (2, 1), & \\ & (4, 2), & \\ & (6, 3), & \\ & \ldots & \\ & (2k, k), & \\ & \ldots & \} \end{array}$$

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$g(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ 2 \cdot n & \text{otherwise} \end{cases}$$

$$\begin{aligned} g = \{ \quad & (0,0), (1,2), \\ & (2,1), (3,6), \\ & (4,2), (5,10), \\ & (6,3), (7,14), \\ & \ldots \\ & (2k,k), (1+2k, 2+4k), \\ & \ldots \qquad\qquad\qquad\qquad \} \end{aligned}$$

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$g(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ 2 \cdot n & \text{otherwise} \end{cases}$$

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ \bot & \text{otherwise} \end{cases}$$

$$\begin{aligned} g = \{ \quad & (0,0), (1,2), \\ & (2,1), (3,6), \\ & (4,2), (5,10), \\ & (6,3), (7,14), \\ & \ldots \\ & (2k,k), (1+2k, 2+4k), \\ & \ldots \qquad \} \end{aligned}$$

$$\begin{aligned} f = \{ \quad & (0,0), \\ & (2,1), \\ & (4,2), \\ & (6,3), \\ & \ldots \\ & (2k,k), \\ & \ldots \qquad \} \end{aligned}$$

$f \sqsubseteq g$? ✓

$g \sqsubseteq f$? ✗

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$\emptyset \sqsubseteq \{ (0,0) \} \sqsubseteq \{ (0,0), (1,1) \} \sqsubseteq \ldots$

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$\emptyset \sqsubseteq \{ (0,0) \} \sqsubseteq \{ (0,0), (1,1) \} \sqsubseteq \{ (0,0), (1,1), (2,2) \} \sqsubseteq \ldots$

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which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$\emptyset \sqsubseteq \{(0,0)\} \sqsubseteq \{(0,0), (1,1)\} \sqsubseteq \{(0,0), (1,1), (2,2)\} \sqsubseteq \{(0,0), (1,1), (2,2), (3,3)\} \sqsubseteq \{(0,0), (1,1), (2,2), (3,3), (4,4)\} \sqsubseteq \ldots$$

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$\emptyset \sqsubseteq \{ (0,1) \} \sqsubseteq \{ (0,1), (1,1) \} \sqsubseteq \{ (0,1), (1,1), (2,2) \} \sqsubseteq \ldots$

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$\emptyset \sqsubseteq \{(0,1)\} \sqsubseteq \{(0,1), (1,1)\} \sqsubseteq \{(0,1), (1,1), (2,2)\} \sqsubseteq \{(0,1), (1,1), (2,2), (3,6)\} \sqsubseteq \ldots$$

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$\emptyset \sqsubseteq \{ (0,1) \} \sqsubseteq \{ (0,1), (1,1) \} \sqsubseteq \{ (0,1), (1,1), (2,2) \} \sqsubseteq \{ (0,1), (1,1), (2,2), (3,6) \} \sqsubseteq \{ (0,1), (1,1), (2,2), (3,6), (4,24) \} \sqsubseteq \ldots$

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$\emptyset \sqsubseteq \{ (0,1) \} \sqsubseteq \{ (0,1), (4,24) \} \sqsubseteq \{ (0,1), (1,1), (4,24) \} \sqsubseteq \{ (0,1), (1,1), (3,6), (4,24) \} \sqsubseteq \{ (0,1), (1,1), (2,2), (3,6), (4,24) \} \sqsubseteq \ldots$

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$\emptyset \sqsubseteq \{ (1,1) \} \sqsubseteq \{ (1,1), (2,4) \} \sqsubseteq \{ (1,1), (2,4), (3,81) \} \sqsubseteq \{ (1,1), (2,4), (3,81), (4,256) \} \sqsubseteq \ldots$

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$\emptyset \sqsubseteq \{ (1,6) \} \sqsubseteq \{ (1,6), (2,28) \} \sqsubseteq \{ (1,6), (2,28), (3,496) \} \sqsubseteq \{ (1,6), (2,28), (3,496), (4,8128) \} \sqsubseteq \ldots$

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$\emptyset \sqsubseteq \{ (4,2) \} \sqsubseteq \{ (4,2), (6,3) \} \sqsubseteq \{ (4,2), (6,3), (8,4) \} \sqsubseteq \{ (4,2), (6,3), (8,4), (9,3) \} \sqsubseteq \{ (4,2), (6,3), (8,4), (9,3), (10,5) \} \sqsubseteq \ldots$

which function(s) are we approximating?

Functional property

$\mathbf{Pf}(A, B) = \{f : A \rightharpoonup B\}$ partial functions

$$\mathbf{Pf}(A, B) = \{f \subseteq A \times B \mid \boxed{\forall a \in A. \forall b_1, b_2 \in B. (a, b_1) \in f \wedge (a, b_2) \in f \Rightarrow b_1 = b_2}\}$$

functional property

$f(a) \downarrow \triangleq \exists b \in B.\ (a, b) \in f$ function f is defined on a

$$f \sqsubseteq g \Leftrightarrow (\forall a \in A. \quad f(a) \downarrow \ \Rightarrow \ (g(a) \downarrow \wedge f(a) = g(a)))$$
$$\Leftrightarrow f \subseteq g$$

$(\mathbf{Pf}(A, B), \sqsubseteq)$ is a PO with bottom
what is bottom?
is it complete?

the empty relation
(the function always undefined)

Is Pf complete?

$(\mathbf{Pf}(A,B), \sqsubseteq)$

complete?

Given a chain $\{f_i\}_{i\in\mathbb{N}}$ let us consider $\bigcup_{i\in\mathbb{N}} f_i \subseteq A \times B$

we want to prove that $\bigcup_{i\in\mathbb{N}} f_i \in \mathbf{Pf}(A,B)$

i.e. that $f = \bigcup_{i\in\mathbb{N}} f_i$ satisfies the functional property

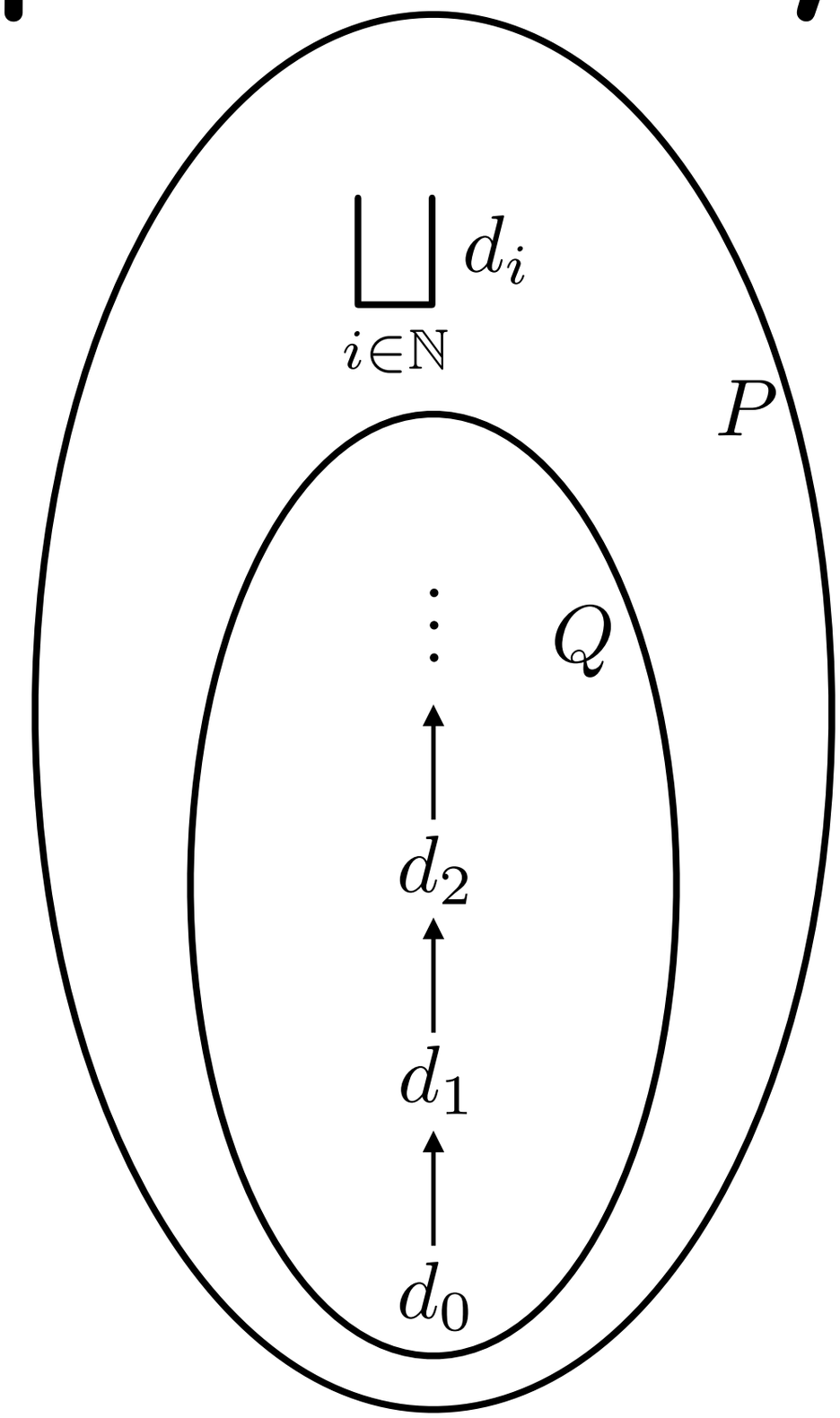
we know that each f_i is functional

$$\forall i \in \mathbb{N}.\forall a \in A.\forall b_1, b_2 \in B.(a,b_1) \in f_i \wedge (a,b_2) \in f_i \Rightarrow b_1 = b_2$$

we need to prove f is functional

$$\forall a \in A.\forall b_1, b_2 \in B.(a,b_1) \in f \wedge (a,b_2) \in f \Rightarrow b_1 = b_2$$

pictorially



is the limit in Q ?

Pf is complete

we need to prove f is functional

$$\forall a \in A. \forall b_1, b_2 \in B. (a, b_1) \in f \wedge (a, b_2) \in f \Rightarrow b_1 = b_2$$

Take $a \in A, b_1, b_2 \in B$ such that $(a, b_1) \in f \wedge (a, b_2) \in f$

we need to prove $b_1 = b_2$

$$(a, b_1) \in f = \bigcup_{i \in \mathbb{N}} f_i \Leftrightarrow \exists k \in \mathbb{N}. (a, b_1) \in f_k$$

$$m \triangleq \max\{k, h\}$$

$$(a, b_2) \in f = \bigcup_{i \in \mathbb{N}} f_i \Leftrightarrow \exists h \in \mathbb{N}. (a, b_2) \in f_h$$

Clearly $f_k \subseteq f_m$ $\quad f_h \subseteq f_m$ $\quad f_m$ is functional

$$(a, b_1) \in f_m \qquad (a, b_2) \in f_m \qquad \Rightarrow \qquad b_1 = b_2$$

Example

$$
\begin{array}{llll}
\mathbf{Pf}(\mathbb{N},\mathbb{N}) & f_0 \; \emptyset & \subseteq & \{(0,1)\} \; f_1 \\
 & & \subseteq & \{(0,1),(1,1)\} \; f_2 \\
 & & \subseteq & \{(0,1),(1,1),(2,2)\} \; f_3 \\
 & & \subseteq & \{(0,1),(1,1),(2,2),(3,6)\} \; f_4 \\
 & & \subseteq & \{(0,1),(1,1),(2,2),(3,6),(4,24)\} \; f_5 \\
 & & \subseteq & \cdots
\end{array}
$$

$\bigcup_{i \in \mathbb{N}} f_i$ is (maybe) the factorial function

note: the limit of partial functions can be a total function

Total functions

$\mathbf{Tf}(A, B) = (A \to B)$ total functions

$\mathbf{Pf}(A, B) \equiv \mathbf{Tf}(A, B_\perp)$ $B_\perp \triangleq B \uplus \{\perp\}$

$\sqsubseteq_{B_\perp} \triangleq$ flat order

$$f \sqsubseteq g \Leftrightarrow \forall a \in A.\ f(a) \sqsubseteq_{B_\perp} g(a)$$

PO? immediate to check

bottom? $f_\perp(a) = \perp$ for any $a \in A$

complete? we will prove it later

(as an instance of a more general result)

$$(\bigsqcup_{i \in \mathbb{N}} f_i)(a) \triangleq \bigsqcup_{i \in \mathbb{N}} f_i(a)$$ (flat order, limit exists)

Monotone functions

Monotone function

$(D, \sqsubseteq_D)$ PO $\quad (E, \sqsubseteq_E)$ PO $\quad f : D \to E$

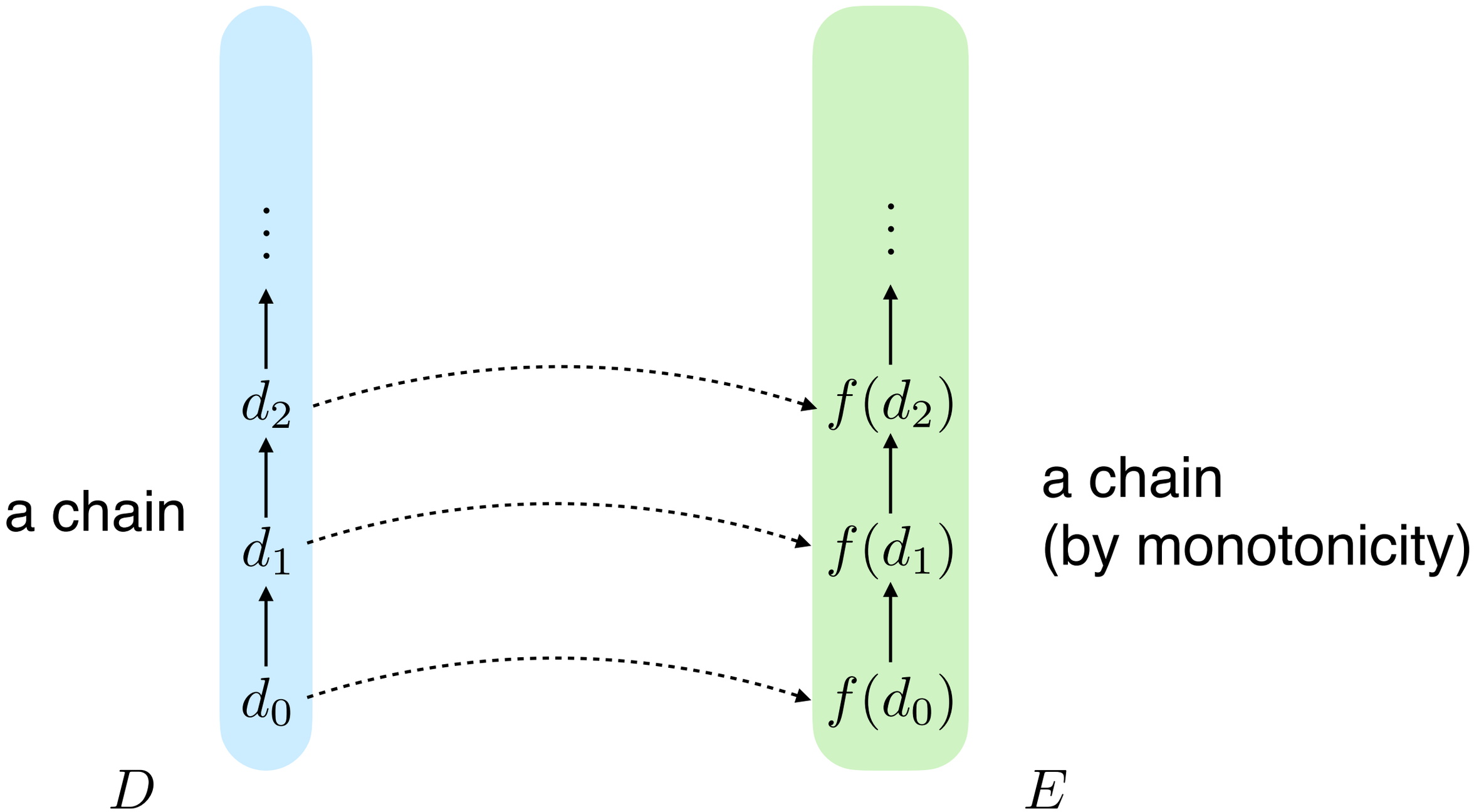
f is monotone if $\forall d_1, d_2 \in D.\ d_1 \sqsubseteq_D d_2 \Rightarrow f(d_1) \sqsubseteq_E f(d_2)$

Monotone = Order preserving

$$\left.\begin{array}{l} \{d_i\}_{i\in\mathbb{N}} \text{ a chain in } D \\ f \text{ monotone} \end{array}\right\} \Rightarrow \{f(d_i)\}_{i\in\mathbb{N}} \text{ a chain in } E$$

When $D = E$ we say $f : D \to D$ is a function on D

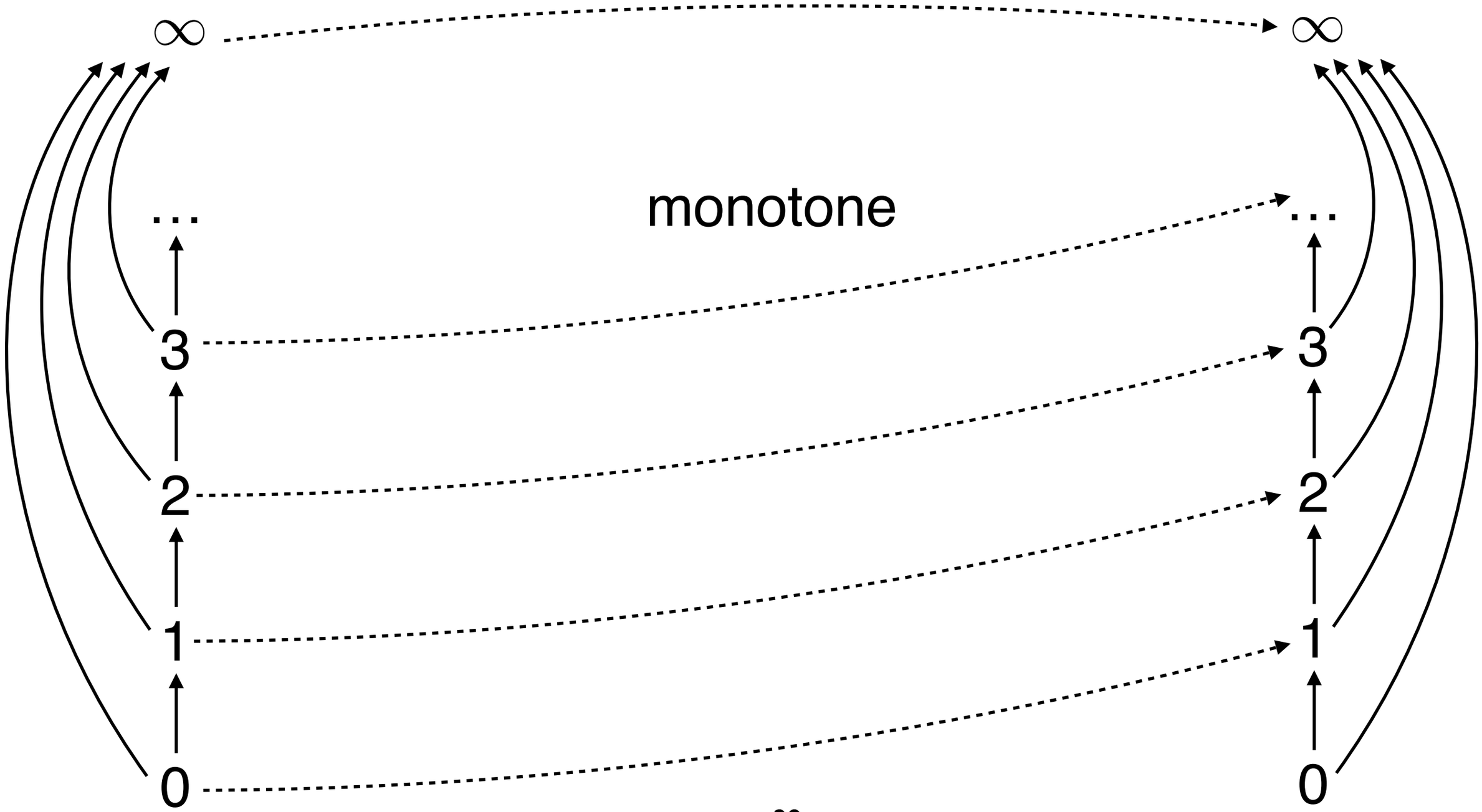
Monotonicity illustrated



Example

$$f(n) = n + 1$$
$$f(\infty) = \infty$$

$(\mathbb{N} \cup \{\infty\}, \leq)$ $\qquad$ $(\mathbb{N} \cup \{\infty\}, \leq)$



Skill levels

advanced
multitasking

intermediate
does many things

novice
does one thing

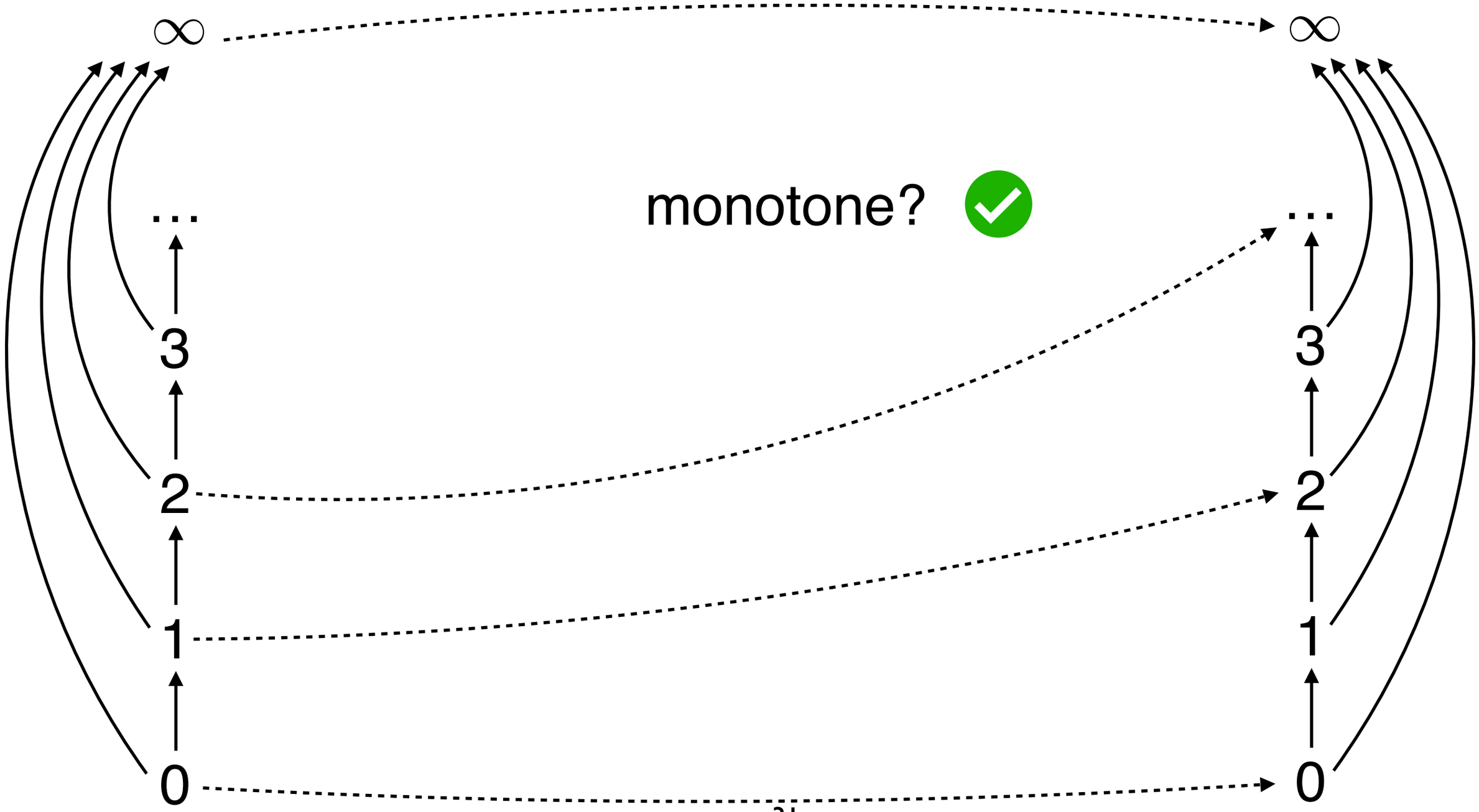
beginner
does nothing

Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

$$f(n) = 2 \cdot n$$
$$f(\infty) = \infty$$

$(\mathbb{N} \cup \{\infty\}, \leq)$



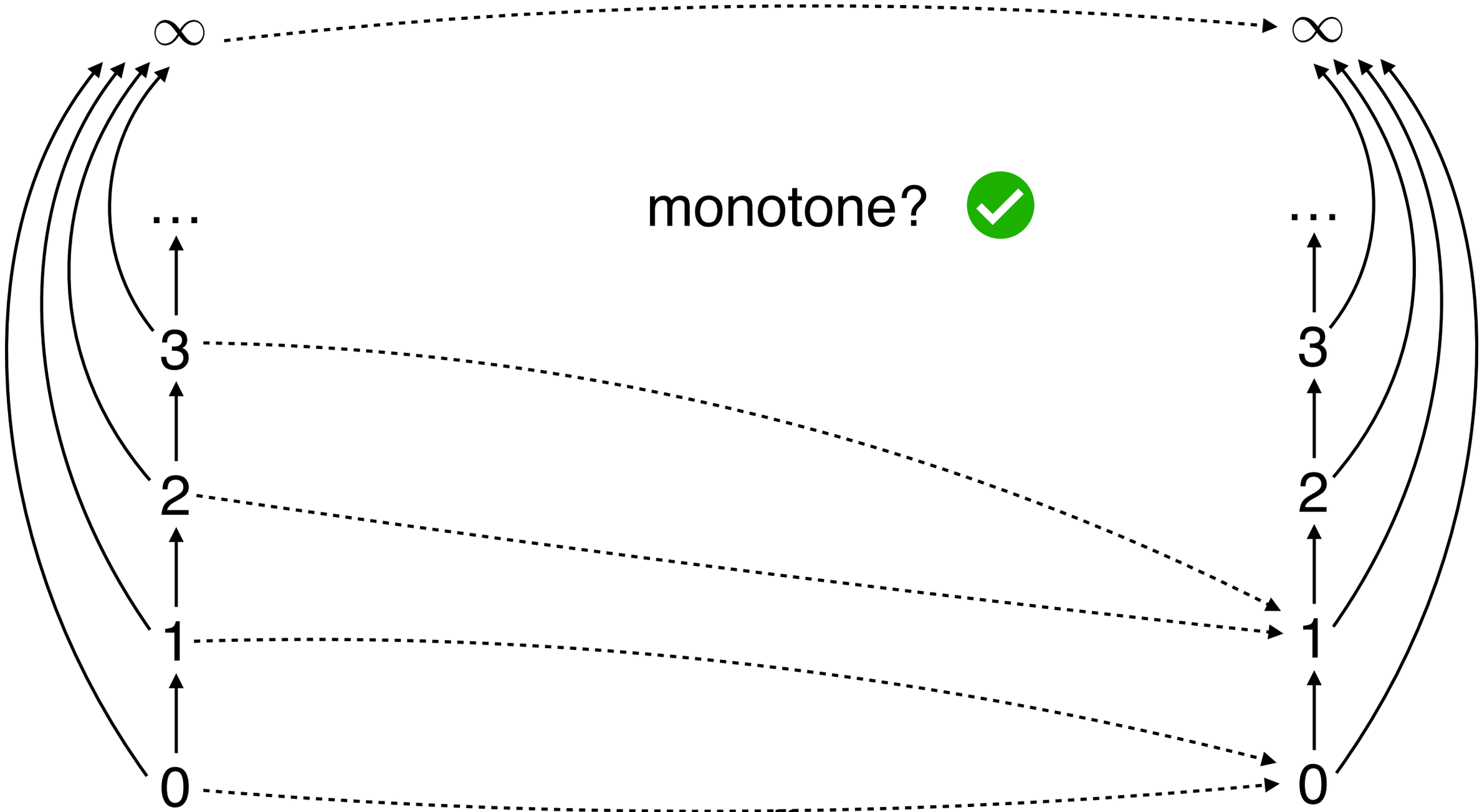
monotone? ✔

Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

$$f(n) = n/2$$
$$f(\infty) = \infty$$

$(\mathbb{N} \cup \{\infty\}, \leq)$

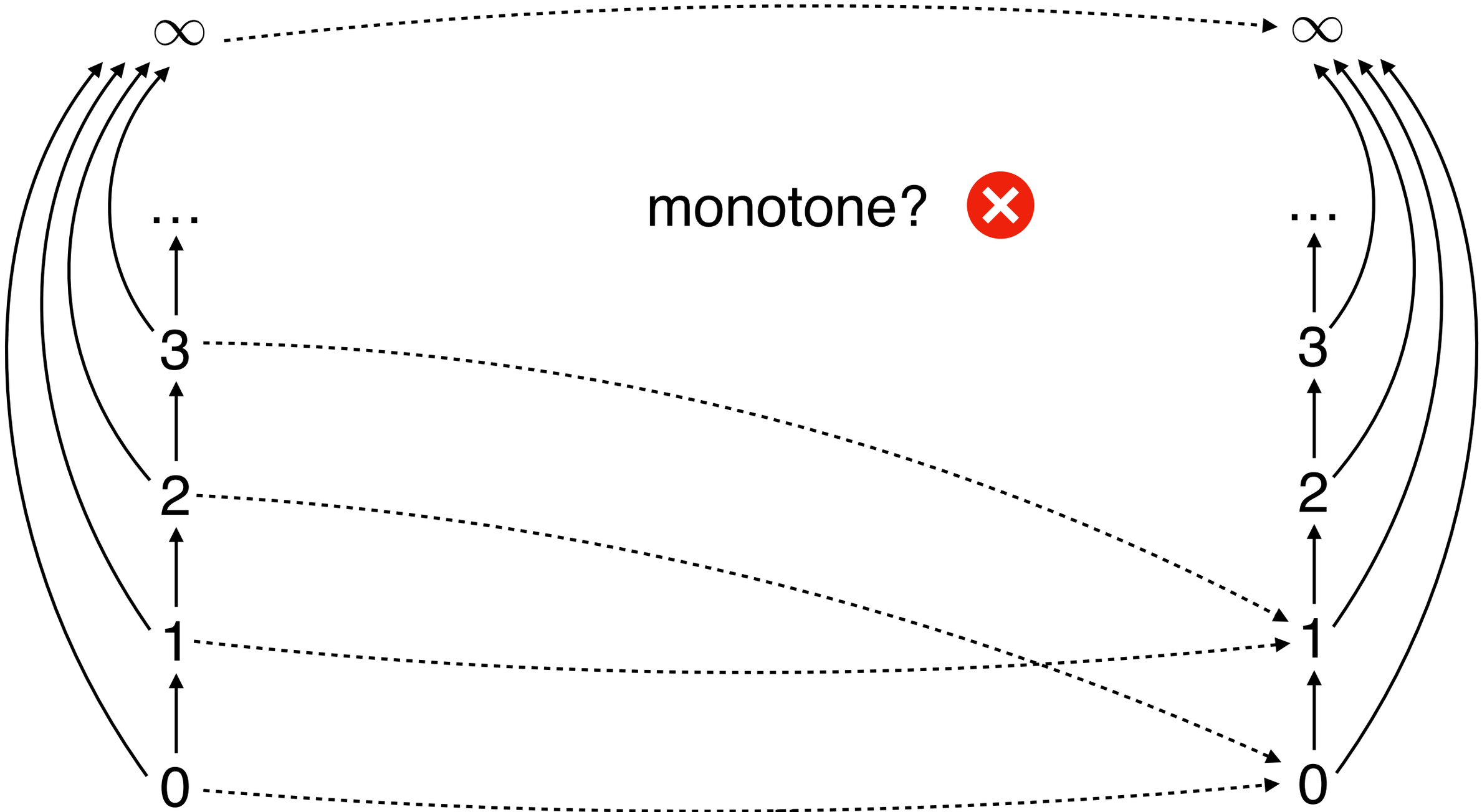


monotone? ✔

Exercise

$$f(n) = n \% 2$$
$$f(\infty) = \infty$$

$(\mathbb{N} \cup \{\infty\}, \leq)$ $\qquad$ $(\mathbb{N} \cup \{\infty\}, \leq)$

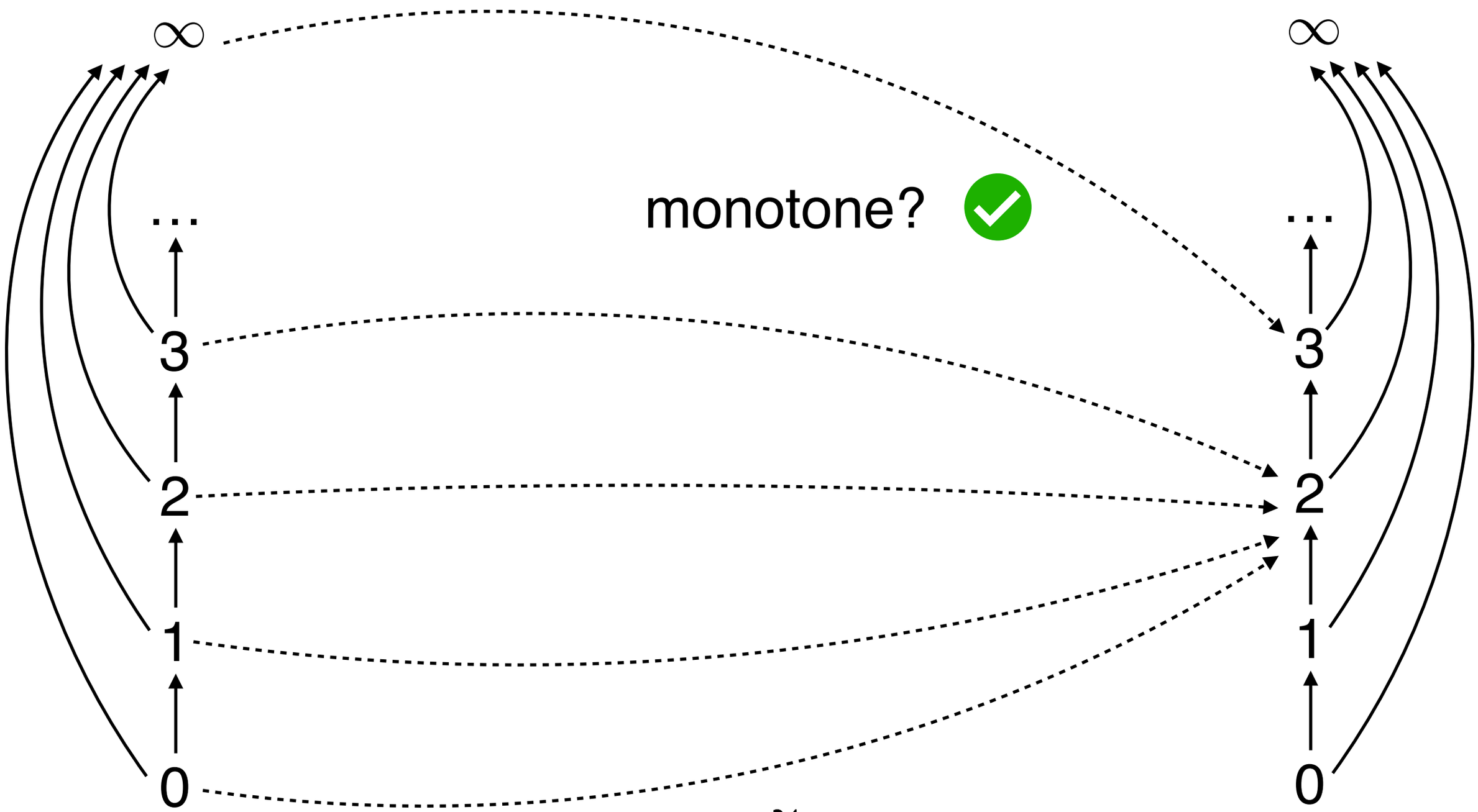


monotone? ✗

Exercise

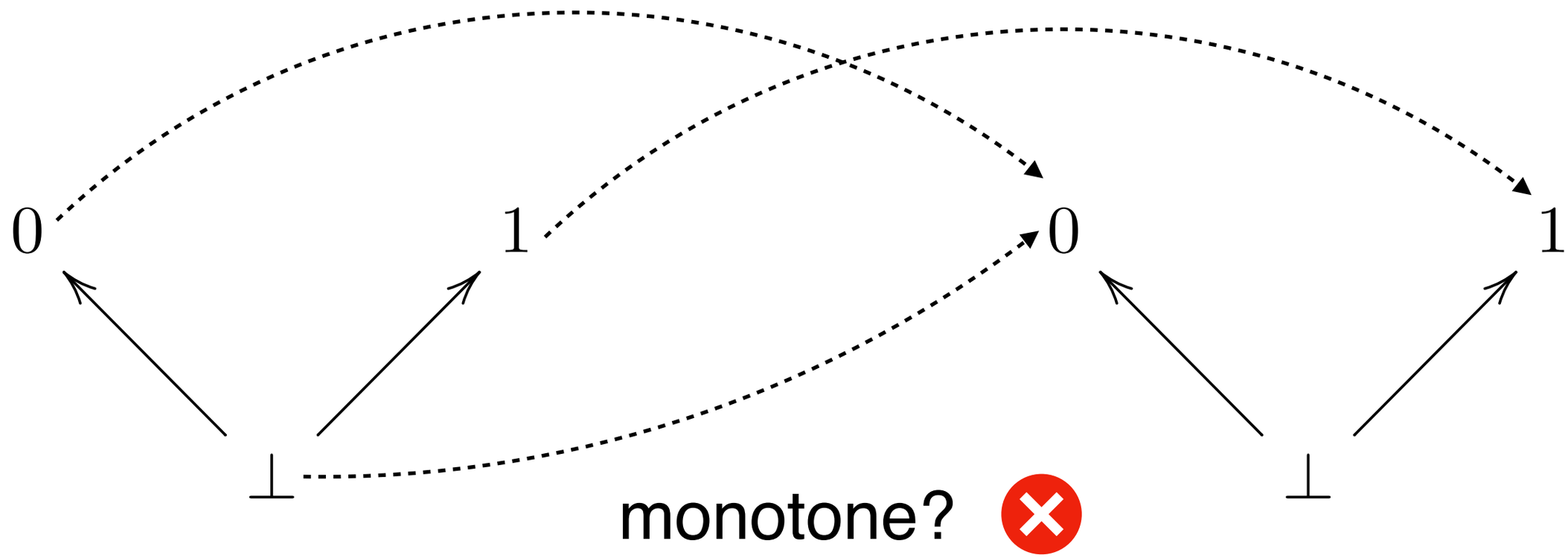
$$f(n) = 2$$
$$f(\infty) = 3$$

$(\mathbb{N} \cup \{\infty\}, \leq)$ $\qquad$ $(\mathbb{N} \cup \{\infty\}, \leq)$



monotone? ✓

Exercise



D $\qquad$ $f : D \to D$ $\qquad$ D

$$f(\bot) = f(0) = 0$$
$$f(1) = 1$$

$\bot \sqsubseteq 1$ $\qquad$ $f(\bot) = 0 \not\sqsubseteq 1 = f(1)$

Exercise

$(\wp(\mathbb{N}), \subseteq)$ $\quad f(S) = \{\, m \in \mathbb{N} \mid \exists n \in S, m \leq n \,\}$ $\quad (\wp(\mathbb{N}), \subseteq)$

monotone? ✅

Exercise

$(\wp(\mathbb{N}), \subseteq)$ $\quad f(S) = \{\, m \in \mathbb{N} \mid \forall n \in S, n < m \,\}$ $\quad (\wp(\mathbb{N}), \subseteq)$

monotone? ✗

Composition

TH. Any composition of monotone functions is monotone

$(D, \sqsubseteq_D)$ PO
$(E, \sqsubseteq_E)$ PO
$(F, \sqsubseteq_F)$ PO

$f : D \to E$ monotone
$g : E \to F$ monotone

$\Rightarrow$ $h = g \circ f : D \to F$ monotone

proof. we need to prove $\forall x, y \in D.\ x \sqsubseteq_D y \Rightarrow h(x) \sqsubseteq_F h(y)$

take $x \sqsubseteq_D y$

we want to prove $h(x) \sqsubseteq_F h(y)$

then $f(x) \sqsubseteq_E f(y)$ because f is monotone

then $g(f(x)) \sqsubseteq_F g(f(y))$ because g is monotone

$$\underset{h(x)}{\underset{=}{g(f(x))}} \sqsubseteq_F \underset{h(y)}{\underset{=}{g(f(y))}}$$

Continuous functions

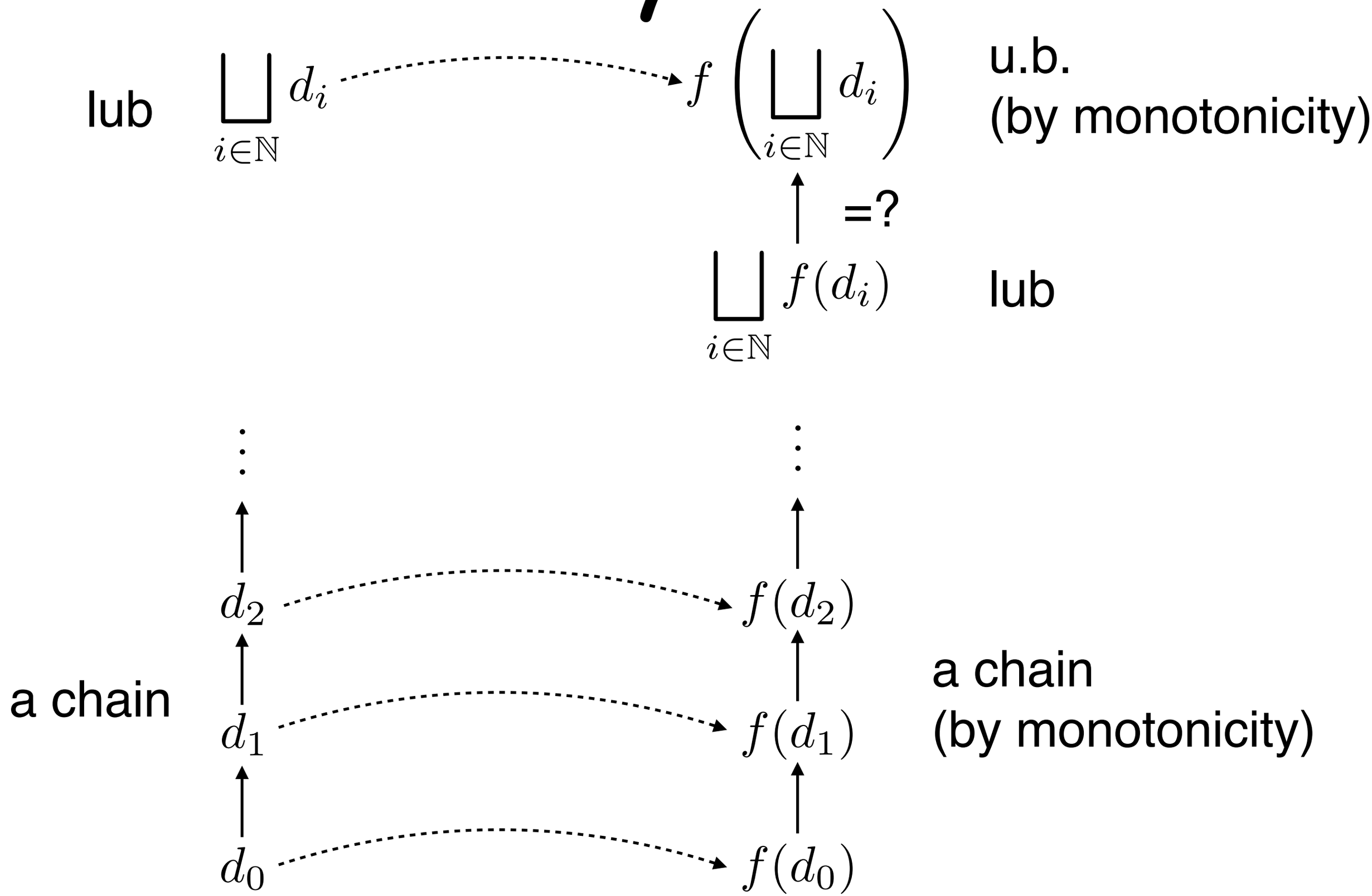
Continuous function

$(D, \sqsubseteq_D)$ CPO $(E, \sqsubseteq_E)$ CPO $f : D \to E$ monotone

f is continuous if $\underbrace{\forall \{d_i\}_{i \in \mathbb{N}}}_{\text{chain}}.\ f\left(\underbrace{\bigsqcup_{i \in \mathbb{N}} d_i}_{\text{limit in } D}\right) = \underbrace{\bigsqcup_{i \in \mathbb{N}} f(d_i)}_{\text{limit in } E}$

Continuous = limit preserving

Continuity illustrated



Continuity illustrated

lub $\bigsqcup_{i\in\mathbb{N}} d_i$ $\longrightarrow$ $f\left(\bigsqcup_{i\in\mathbb{N}} d_i\right)$

$$\uparrow \; =?$$

$$\bigsqcup_{i\in\mathbb{N}} f(d_i)$$

$$\bigsqcup_{i\in\mathbb{N}} f(d_i) \sqsubseteq_E f\left(\bigsqcup_{i\in\mathbb{N}} d_i\right)$$

follows from monotonicity (and CPO)

a chain

$$\begin{array}{ccc}
\vdots & & \vdots \\
\uparrow & & \uparrow \\
d_2 & \dashrightarrow & f(d_2) \\
\uparrow & & \uparrow \\
d_1 & \dashrightarrow & f(d_1) \\
\uparrow & & \uparrow \\
d_0 & \dashrightarrow & f(d_0)
\end{array}$$

Continuity illustrated

lub $\bigsqcup_{i\in\mathbb{N}} d_i$ $\longrightarrow$ $f\left(\bigsqcup_{i\in\mathbb{N}} d_i\right)$

$$\uparrow \; =?$$

$$\bigsqcup_{i\in\mathbb{N}} f(d_i)$$

$$\bigsqcup_{i\in\mathbb{N}} f(d_i) \overset{?}{\sqsupseteq}_E f\left(\bigsqcup_{i\in\mathbb{N}} d_i\right)$$

continuity

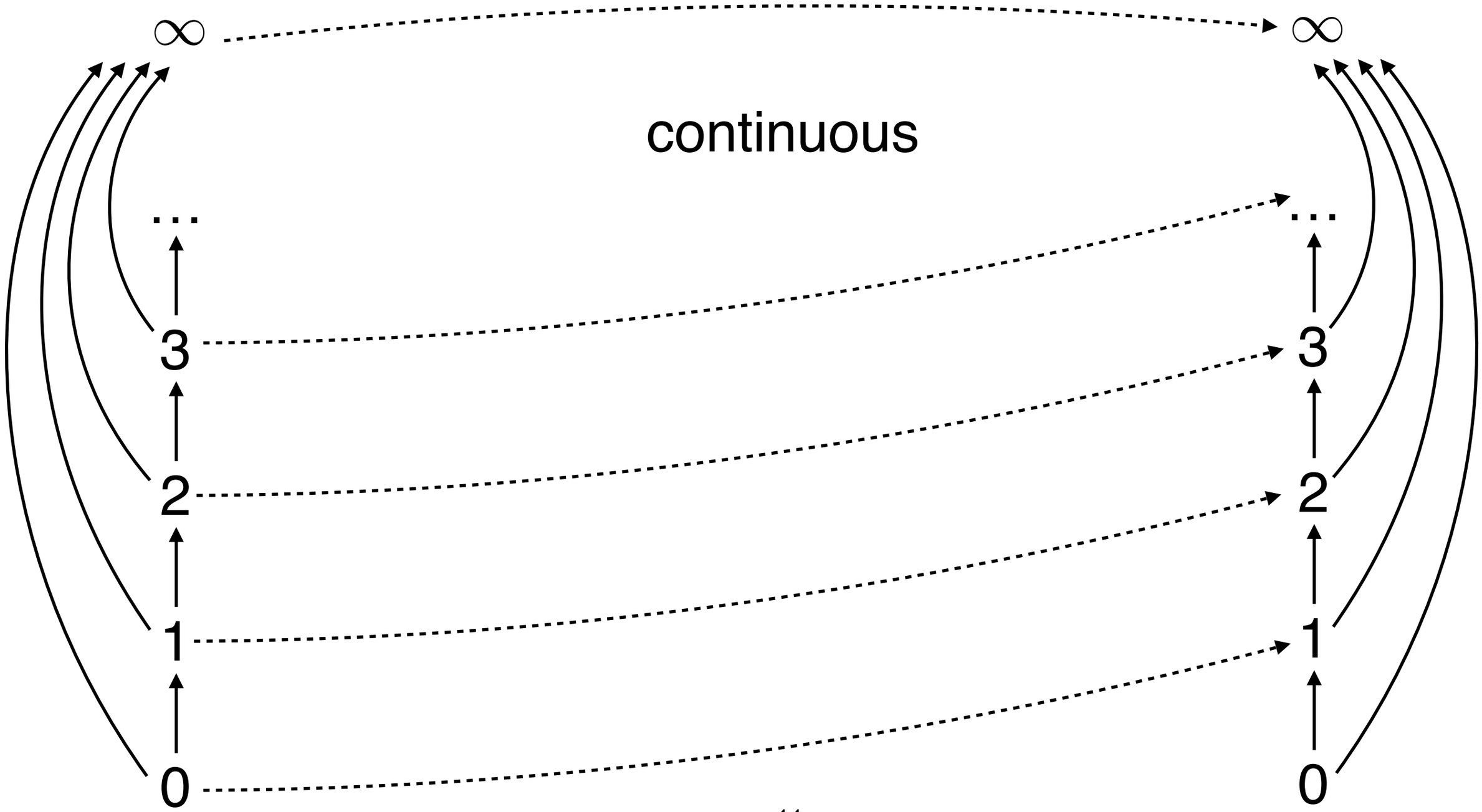
a chain

$$\begin{array}{ccc}
\vdots & & \vdots \\
\uparrow & & \uparrow \\
d_2 & \dashrightarrow & f(d_2) \\
\uparrow & & \uparrow \\
d_1 & \dashrightarrow & f(d_1) \\
\uparrow & & \uparrow \\
d_0 & \dashrightarrow & f(d_0)
\end{array}$$

Example

$$f(n) = n + 1$$
$$f(\infty) = \infty$$

$(\mathbb{N} \cup \{\infty\}, \leq)$ $\qquad$ $(\mathbb{N} \cup \{\infty\}, \leq)$

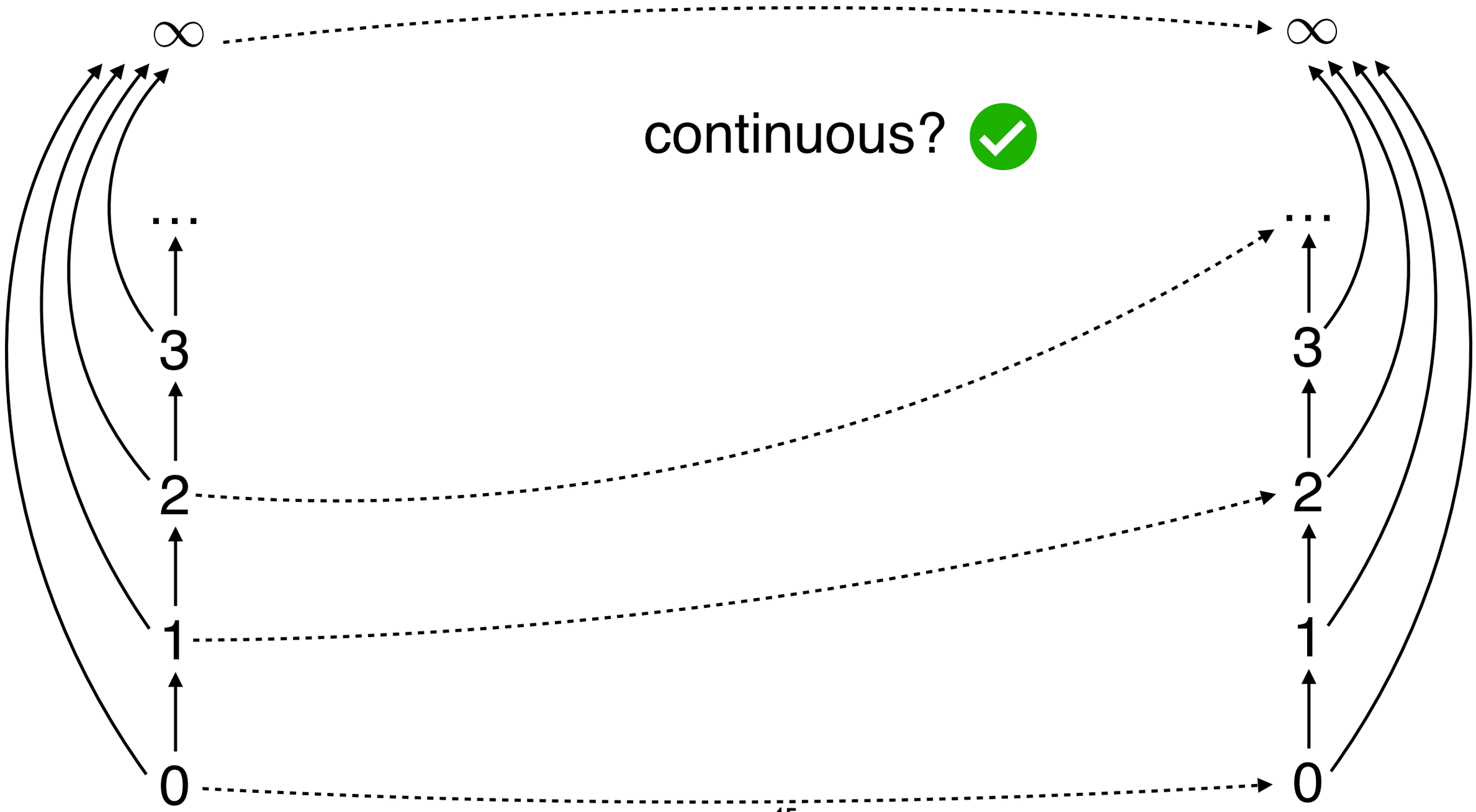


Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

$$f(n) = 2 \cdot n$$
$$f(\infty) = \infty$$

$(\mathbb{N} \cup \{\infty\}, \leq)$

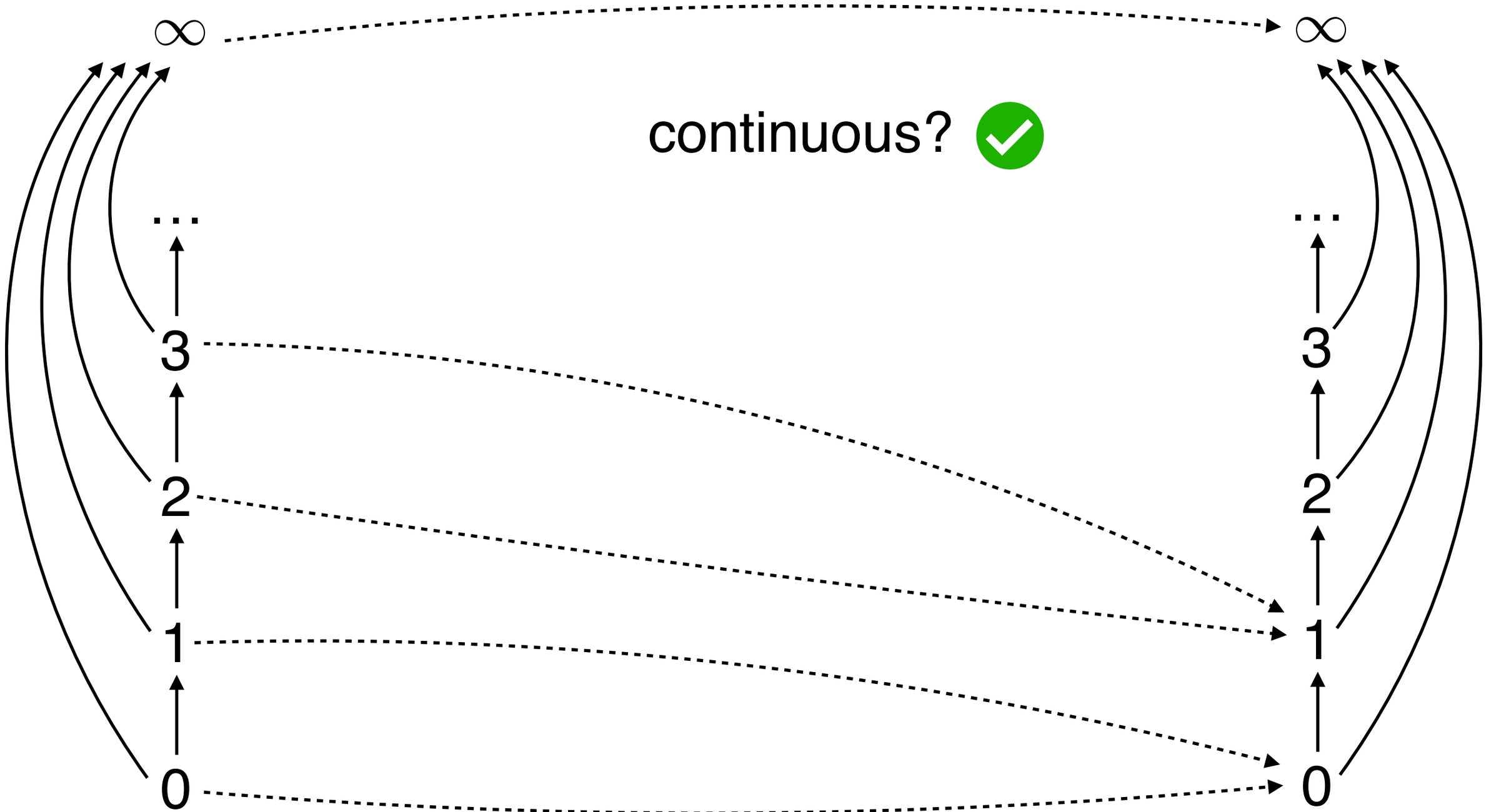


continuous?

Example

$$f(n) = n/2$$
$$f(\infty) = \infty$$

$(\mathbb{N} \cup \{\infty\}, \leq)$ $\qquad$ $(\mathbb{N} \cup \{\infty\}, \leq)$

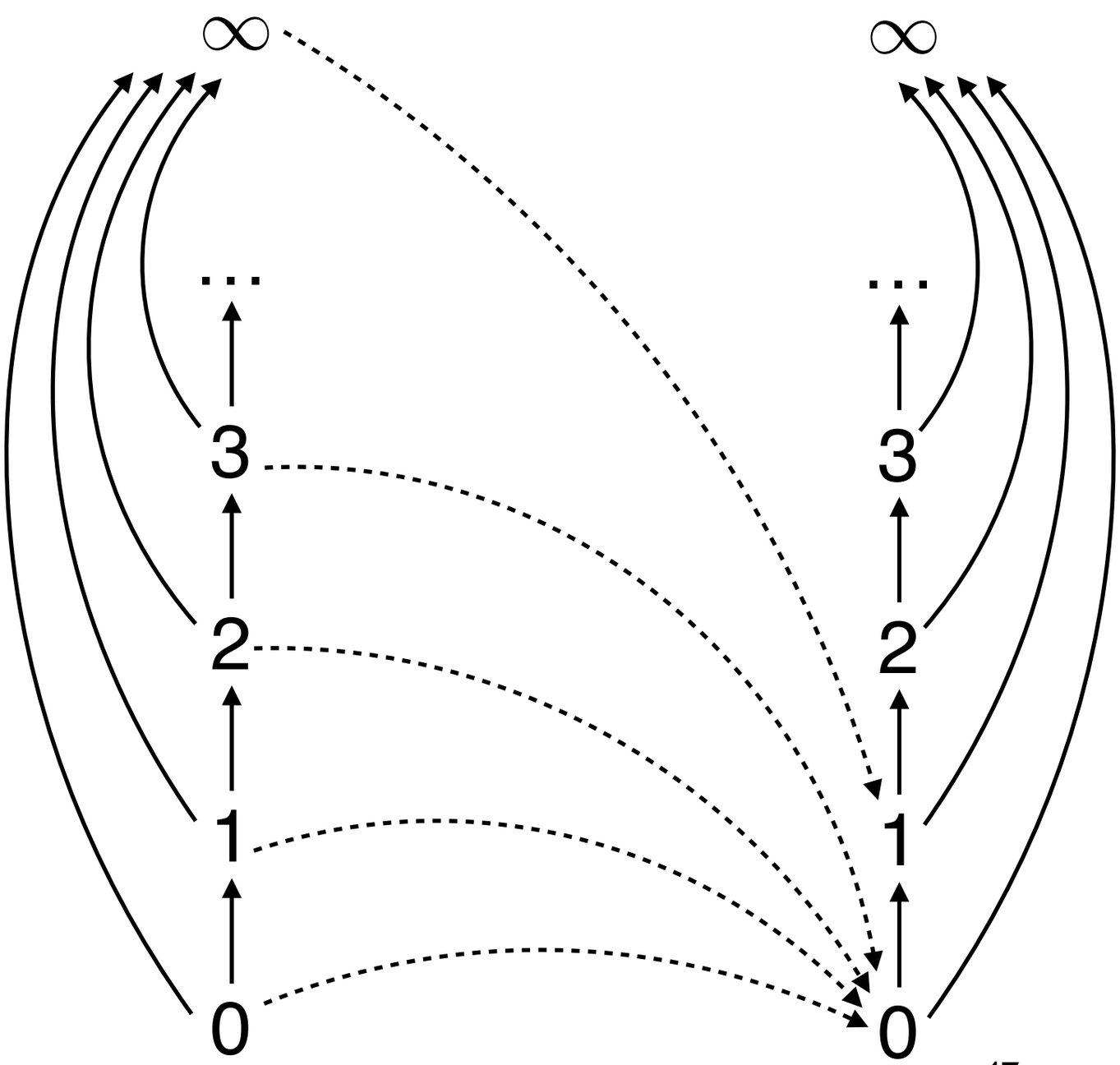


continuous? ✅

Example

$(\mathbb{N} \cup \{\infty\}, \leq)$

monotone function, not continuous



$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{N} \\ 1 & \text{if } x = \infty \end{cases}$$

$$d_i = 2 \cdot i \qquad \bigsqcup_{i \in \mathbb{N}} d_i = \infty$$

$$f\left(\bigsqcup_{i \in \mathbb{N}} d_i\right) = f(\infty) = 1$$

$$f(d_i) = 0$$

$$\bigsqcup_{i \in \mathbb{N}} f(d_i) = \bigsqcup_{i \in \mathbb{N}} 0 = 0$$

Exercise

$(\wp(\mathbb{N}), \subseteq)$ $\quad f(S) = \{\, m \in \mathbb{N} \mid \exists n \in S, m \leq n \,\}$ $\quad (\wp(\mathbb{N}), \subseteq)$

continuous? ✅

Exercise

$(\wp(\mathbb{N}), \subseteq)$

$$f(S) = \begin{cases} S & \text{if } S \text{ finite} \\ \mathbb{N} & \text{otherwise} \end{cases}$$

$(\wp(\mathbb{N}), \subseteq)$

continuous? ✗

Lemma

$(D, \sqsubseteq_D)$ CPO
no infinite chains

$(E, \sqsubseteq_E)$ PO

$f : D \to E$
monotone $\Rightarrow$ f
continuous

proof. Take a chain $\{d_i\}_{i \in \mathbb{N}}$

$$\{d_i\}_{i \in \mathbb{N}} \text{ is finite} \quad \Rightarrow \quad \exists k \in \mathbb{N}. \bigsqcup_{i \in \mathbb{N}} d_i = d_k$$

$$\Downarrow$$

$$\{f(d_i)\}_{i \in \mathbb{N}} \text{ is finite} \quad \Rightarrow \quad \bigsqcup_{i \in \mathbb{N}} f(d_i) = f(d_k) = f\left(\bigsqcup_{i \in \mathbb{N}} d_i\right)$$

Composition

TH. Any composition of continuous functions is continuous

$(D, \sqsubseteq_D)$ CPO
$(E, \sqsubseteq_E)$ CPO
$(F, \sqsubseteq_F)$ CPO

$f : D \to E$ continuous
$g : E \to F$ continuous

$\Rightarrow$ $h = g \circ f : D \to F$ continuous

proof. take a chain $\{d_i\}_{i \in \mathbb{N}}$

we need to prove $h\left(\bigsqcup_{i \in \mathbb{N}} d_i\right) = \bigsqcup_{i \in \mathbb{N}} h(d_i)$

$$h\left(\bigsqcup_{i \in \mathbb{N}} d_i\right) = g\left(f\left(\bigsqcup_{i \in \mathbb{N}} d_i\right)\right) = g\left(\bigsqcup_{i \in \mathbb{N}} f(d_i)\right) = \bigsqcup_{i \in \mathbb{N}} g(f(d_i))$$
$$= \bigsqcup_{i \in \mathbb{N}} h(d_i)$$

Kleene’s fixpoint theorem

Repeated application

$$f : D \to D$$

$$f^0(d) \triangleq d$$

$$f^{n+1}(d) \triangleq f(f^n(d))$$

$$f^n(d) = \overbrace{f(\cdots(f}^{n \text{ times}}(d))\cdots)$$

$$f^n : D \to D$$

Lemma

$(D, \sqsubseteq)$ $\mathrm{PO}_\perp$ $f : D \to D$ monotone $\Rightarrow$ $\{f^n(\perp)\}_{n \in \mathbb{N}}$ is a chain

proof. we need to prove $\forall n \in \mathbb{N}.\ f^n(\perp) \sqsubseteq f^{n+1}(\perp)$

by mathematical induction $P(n) \triangleq f^n(\perp) \sqsubseteq f^{n+1}(\perp)$

$P(0) \triangleq f^0(\perp) \sqsubseteq f^1(\perp)$

$$f^0(\perp) = \perp \sqsubseteq f^1(\perp)$$

$\forall n \in \mathbb{N}.\ P(n) \Rightarrow P(n+1)$

take a generic n

assume $P(n) \triangleq f^n(\perp) \sqsubseteq f^{n+1}(\perp)$

we want to prove $P(n+1) \triangleq f^{n+1}(\perp) \sqsubseteq f^{n+2}(\perp)$

$$f^n(\perp) \sqsubseteq f^{n+1}(\perp)$$

$$\Downarrow$$

$$f^{n+1}(\perp) = f(f^n(\perp)) \sqsubseteq f(f^{n+1}(\perp)) = f^{n+2}(\perp)$$

Towards Kleene's theo.

when $(D, \sqsubseteq)$ is a $\mathrm{CPO}_\perp$

then $\{f^n(\perp)\}_{n\in\mathbb{N}}$ is a chain

it must have a limit

$\{f^n(d)\}_{n\in\mathbb{N}}$
not necessarily
a chain!

Kleene's fix point theorem states that
if f is continuous, then the limit of the above chain
is the least fixpoint of f

Pre-fixpoints

$(D, \sqsubseteq)$ PO $\qquad f : D \to D$ monotone

fixpoint $\qquad p \in D \qquad f(p) = p$

pre-fixpoint $\quad p \in D \qquad f(p) \sqsubseteq p$

Clearly any fixpoint is also a pre-fixpoint

Kleene's theorem

$(D, \sqsubseteq)$ $\mathrm{CPO}_\perp$ $\quad f : D \to D$ continuous

let $\mathit{fix}(f) \triangleq \bigsqcup_{n \in \mathbb{N}} f^n(\perp)$

1. $\mathit{fix}(f)$ is a fix point of f

$$f(\mathit{fix}(f)) = \mathit{fix}(f)$$

2. $\mathit{fix}(f)$ is the least pre-fixpoint of f

$$\forall d \in D.\ f(d) \sqsubseteq d \Rightarrow \mathit{fix}(f) \sqsubseteq d$$

if d is a pre-fixpoint then $\mathit{fix}(f)$ is smaller than d

Kleene's theorem: 1

1. $f(\mathit{fix}(f)) = \mathit{fix}(f)$

proof.

$$f(\mathit{fix}(f)) = f\left(\bigsqcup_{n\in\mathbb{N}} f^n(\bot)\right) \quad \text{by def of } \mathit{fix}$$

$$= \bigsqcup_{n\in\mathbb{N}} f(f^n(\bot)) \quad \text{by continuity}$$

$$= \bigsqcup_{n\in\mathbb{N}} f^{n+1}(\bot) \quad \text{by def of } f^n$$

$$= \bigsqcup_{n\in\mathbb{N}} f^n(\bot) \quad \text{by prefix independence of limits}$$

$$= \mathit{fix}(f) \quad \text{by def of } \mathit{fix}$$

Kleene's theorem: 2

2. $\forall d \in D.\ f(d) \sqsubseteq d \Rightarrow \mathit{fix}(f) \sqsubseteq d$

proof.

we prove that any pre-fixpoint is an upper bound of the chain

$$\{f^n(\bot)\}_{n \in \mathbb{N}}$$

by definition $\mathit{fix}(f)$ is the lub of the same chain

and thus smaller than any other upper bound

Kleene's theorem: 2

2. $\forall d \in D.\ f(d) \sqsubseteq d \Rightarrow \mathit{fix}(f) \sqsubseteq d$

take any $d \in D$ such that $f(d) \sqsubseteq d$

we prove $\forall n \in \mathbb{N}.\ f^n(\bot) \sqsubseteq d$ (d is an upper bound)

$P(n) \triangleq f^n(\bot) \sqsubseteq d$ by mathematical induction

$P(0) \triangleq f^0(\bot) \sqsubseteq d$ $\quad f^0(\bot) = \bot \sqsubseteq d$

$\forall n \in \mathbb{N}.\ P(n) \Rightarrow P(n+1)$

take a generic n

assume $P(n) \triangleq f^n(\bot) \sqsubseteq d$

we want to prove $P(n+1) \triangleq f^{n+1}(\bot) \sqsubseteq d$

$$f^n(\bot) \sqsubseteq d$$

$$f^{n+1}(\bot) \overset{\text{(by def)}}{=} f(f^n(\bot)) \overset{\text{(monot.)}\Downarrow}{\sqsubseteq} f(d) \overset{\text{(pre-fixpoint)}}{\sqsubseteq} d$$

Example

$n = 2 \cdot n$ $\quad (\mathbb{N} \cup \{\infty\}, \leq)$ $\quad \bot = 0$ $\quad \mathrm{CPO}_\bot$

$$f(n) = 2 \cdot n$$
$$f(\infty) = \infty$$

monotone? ok
continuous? ok

$$f^0(0) = 0$$

$$f^1(0) = f(0) = 2 \cdot 0 = 0$$

fixpoint reached!

Example

$n = n + 1$ $\quad (\mathbb{N} \cup \{\infty\}, \leq)$ $\quad \bot = 0$ $\quad \mathrm{CPO}_\bot$

$$f(n) = n + 1$$
$$f(\infty) = \infty$$

monotone? ok
continuous? ok

$$f^0(0) = 0$$
$$f^1(0) = f(0) = 0 + 1 = 1$$
$$f^2(0) = f(f^1(0)) = f(1) = 1 + 1 = 2$$
$$f^3(0) = f(f^2(0)) = f(2) = 2 + 1 = 3$$

$$\bigsqcup_{n \in \mathbb{N}} f^n(0) = \bigsqcup_{n \in \mathbb{N}} n = \infty$$ fixpoint

Example

$X = X \cap \{1\}$ $\quad (\wp(\mathbb{N}), \subseteq)$ $\quad \bot = \emptyset$ $\quad \mathrm{CPO}_\bot$

$f(X) = X \cap \{1\}$

monotone? ok
continuous? ok

$f^0(\emptyset) = \emptyset$

$f^1(\emptyset) = f(\emptyset) = \emptyset \cap \{1\} = \emptyset$

fixpoint reached!

Example

$X = \mathbb{N} \setminus X$ $\qquad (\wp(\mathbb{N}), \subseteq)$ $\qquad \bot = \emptyset \qquad \mathrm{CPO}_\bot$

$f(X) = \mathbb{N} \setminus X$

monotone? NO

the larger X the smaller $f(X)$

$$f^0(\emptyset) = \emptyset$$

$$f^1(\emptyset) = f(\emptyset) = \mathbb{N} \setminus \emptyset = \mathbb{N}$$

$$f^2(\emptyset) = f(f^1(\emptyset)) = f(\mathbb{N}) = \mathbb{N} \setminus \mathbb{N} = \emptyset$$

not a chain!

Example

$X = X \cup \{1\}$ $\quad (\wp(\mathbb{N}), \subseteq)$ $\quad \bot = \emptyset$ $\quad \mathrm{CPO}_\bot$

$f(X) = X \cup \{1\}$

monotone? ok
continuous? ok

$$f^0(\emptyset) = \emptyset$$

$$f^1(\emptyset) = f(\emptyset) = \emptyset \cup \{1\} = \{1\}$$

$$f^2(\emptyset) = f(f^1(\emptyset)) = f(\{1\}) = \{1\} \cup \{1\} = \{1\}$$

fixpoint reached!