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MPP 2025/26 (0077A, 9CFU)

Models for Programming Paradigms

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20 - Weak semantics

CCS
Weak transitions

Sequential buffer

$$B_0^2 \triangleq in.B_1^2$$

$$B_1^2 \triangleq in.B_2^2 + \overline{out}.B_0^2$$

$$B_2^2 \triangleq \overline{out}.B_1^2$$

$$\begin{tikzcd}
B_0^2 \arrow[d, bend left, "in"] \\
B_1^2 \arrow[u, bend left, "\overline{out}"] \arrow[d, bend left, "in"] \\
B_2^2 \arrow[u, bend left, "\overline{out}"]
\end{tikzcd}$$

Parallel buffer

$$B_0^1 \triangleq in.B_1^1$$

$$B_1^1 \triangleq \overline{out}.B_0^1$$

$$\begin{tikzcd}
 & B_0^1|B_0^1 \arrow[dl, bend left, "in"] \arrow[dr, bend right, "in"'] & \\
B_1^1|B_0^1 \arrow[ur, bend left, "\overline{out}"] \arrow[dr, bend left, "in"] & & B_0^1|B_1^1 \arrow[ul, bend right, "\overline{out}"'] \arrow[dl, bend right, "in"'] \\
 & B_1^1|B_1^1 \arrow[ul, bend left, "\overline{out}"] \arrow[ur, bend right, "\overline{out}"'] &
\end{tikzcd}$$

Linked buffer

$$B_0^1 \triangleq in.B_1^1 \qquad \eta(out) = c$$

$$p \frown q \triangleq (p[\eta] | q[\phi]) \backslash c$$

$$B_1^1 \triangleq \overline{out}.B_0^1 \qquad \phi(in) = c$$

$$\begin{tikzcd}
 & B_0^1 \frown B_0^1 \arrow[dl, "in"'] & \\
B_1^1 \frown B_0^1 \arrow[rr, "\tau"] & & B_0^1 \frown B_1^1 \arrow[ul, "\overline{out}"'] \arrow[dl, "in"] \\
 & B_1^1 \frown B_1^1 \arrow[ul, "\overline{out}"] &
\end{tikzcd}$$

Comparing buffers

$$
\begin{tikzcd}
B_0^2 \arrow[d, bend left, "in"] \\
B_1^2 \arrow[u, bend left, "\overline{out}"] \arrow[d, bend left, "in"] \\
B_2^2 \arrow[u, bend left, "\overline{out}"]
\end{tikzcd}
$$

are they all
strong bisimilar?

$$
\begin{tikzcd}
 & B_0^1|B_0^1 \arrow[dl, bend right, "in"'] \arrow[dr, bend left, "in"] & \\
B_1^1|B_0^1 \arrow[ur, bend left, "\overline{out}"] \arrow[dr, bend left, "in"] & & B_0^1|B_1^1 \arrow[ul, bend right, "\overline{out}"'] \arrow[dl, bend right, "in"'] \\
 & B_1^1|B_1^1 \arrow[ul, bend left, "\overline{out}"] \arrow[ur, bend right, "\overline{out}"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & B_0^1 \frown B_0^1 \arrow[dl, "in"'] & \\
B_1^1 \frown B_0^1 \arrow[rr, "\tau"] & & B_0^1 \frown B_1^1 \arrow[ul, "\overline{out}"'] \arrow[dl, "in"] \\
 & B_1^1 \frown B_1^1 \arrow[ul, "\overline{out}"] &
\end{tikzcd}
$$

$$F \triangleq \Diamond_{in} \Diamond_{\tau} \mathbf{tt}$$

$$B_0^2 \not\models F$$

$$B_0^1|B_0^1 \not\models F$$

$$B_0^1 \frown B_0^1 \models F$$

Silent transitions

τ-transitions are silent, non observable

they represent internal steps of the system

they can be used just for bookkeeping

can we abstract away from them?

can we find a broader equivalence?

necessary to relate an abstract specification (little use of τ)
with a concrete implementation (lots/tons of τ)

Weak bisimulation game

coarser equivalence: more power to the defender!

Alice picks a process and an ordinary transition

Bob replies possibly using many additional silent transitions

arbitrarily many, but finitely many

such sequences are called *weak* transitions

$$p \overset{\mu}{\Rightarrow} q$$

what if Alice picks a silent transition?

Bob can just leave the other process idle

i.e. can choose not to move

Weak transitions

$$p \overset{\tau}{\Rightarrow} q \quad \text{iff} \quad p\ (\overset{\tau}{\rightarrow})^* q$$

$$p = q \ \vee \ p \overset{\tau}{\rightarrow} \ldots \overset{\tau}{\rightarrow} q$$

p can reach *q* via a (possibly empty) finite sequence of τ-transitions

$$p \overset{\lambda}{\Rightarrow} q \quad \text{iff} \quad \exists p', q'.\ p \overset{\tau}{\Rightarrow} p' \overset{\lambda}{\rightarrow} q' \overset{\tau}{\Rightarrow} q$$

p can reach *q* via a λ-transition possibly preceded and followed by empty/finite sequences of τ-transitions

Example

$$\begin{tikzcd}
 & B_0^1 \frown B_0^1 \arrow[dl, "in"'] & \\
B_1^1 \frown B_0^1 \arrow[rr, "\tau"] & & B_0^1 \frown B_1^1 \arrow[ul, "\overline{out}"'] \arrow[dl, "in"] \\
 & B_1^1 \frown B_1^1 \arrow[ul, "\overline{out}"] &
\end{tikzcd}$$

$$B_0^1 \frown B_0^1 \overset{\tau}{\Rightarrow} B_0^1 \frown B_0^1$$

$$B_0^1 \frown B_0^1 \overset{in}{\Longrightarrow} B_0^1 \frown B_1^1$$

$$B_1^1 \frown B_0^1 \overset{\overline{out}}{\Longrightarrow} B_0^1 \frown B_0^1$$

CCS
weak bisimulation

Weak bisimulation

$\mathbf{R}$ is a *weak* bisimulation if

$$\forall p, q.\ (p, q) \in \mathbf{R} \Rightarrow \begin{cases} \forall \mu, p'.\ p \xrightarrow{\mu} p' & \Rightarrow \quad \exists q'.\ \boxed{q \overset{\mu}{\Rightarrow} q'} \wedge p'\ \mathbf{R}\ q' \\ \wedge & \\ \forall \mu, q'.\ q \xrightarrow{\mu} q' & \Rightarrow \quad \exists p'.\ \boxed{p \overset{\mu}{\Rightarrow} p'} \wedge p'\ \mathbf{R}\ q' \end{cases}$$

Alice plays

Bob replies

weak transitions

Weak bisimilariity

weak bisimilarity:

$p \approx q \quad \text{iff} \quad \exists \mathbf{R}$ a weak bisimulation with $(p, q) \in \mathbf{R}$

TH. weak bisimilarity is an equivalence relation

TH. any strong bisimulation is a weak bisimulation

Cor. strong bisimilarity implies weak bisimilarity

Weaker bisimilarity?

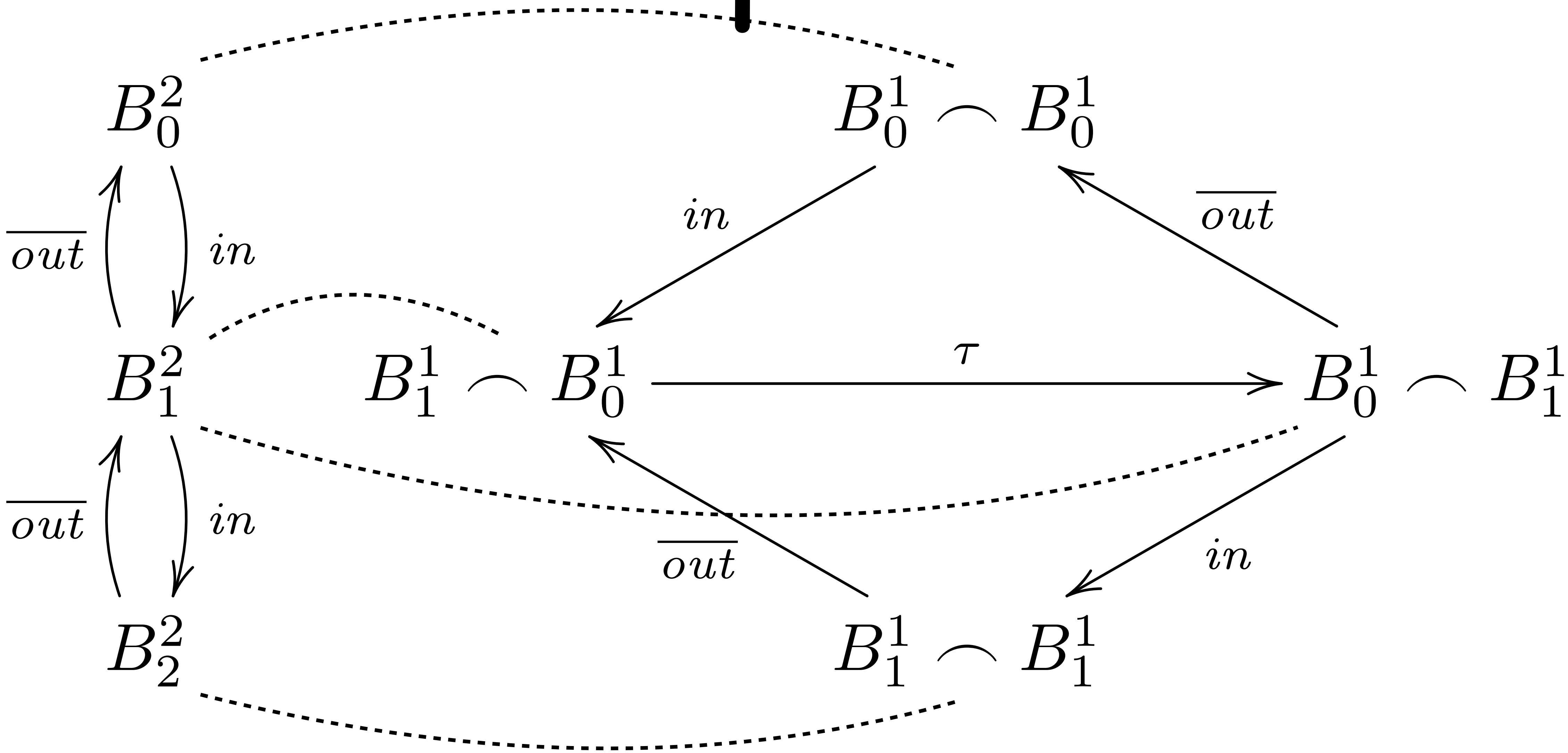
what if we give extra power to Alice as well?

$$\forall p,q.\ (p,q) \in \mathbf{R} \Rightarrow \begin{cases} \forall \mu, p'.\ \boxed{p \stackrel{\mu}{\Rightarrow} p'} & \Rightarrow \quad \exists q'.\ q \stackrel{\mu}{\Rightarrow} q' \wedge p' \mathrel{\mathbf{R}} q' \\ \wedge \quad \text{Alice plays} & \qquad\qquad \text{Bob replies} \\ \forall \mu, q'.\ \boxed{q \stackrel{\mu}{\Rightarrow} q'} & \Rightarrow \quad \exists p'.\ p \stackrel{\mu}{\Rightarrow} p' \wedge p' \mathrel{\mathbf{R}} q' \end{cases}$$

weak transitions

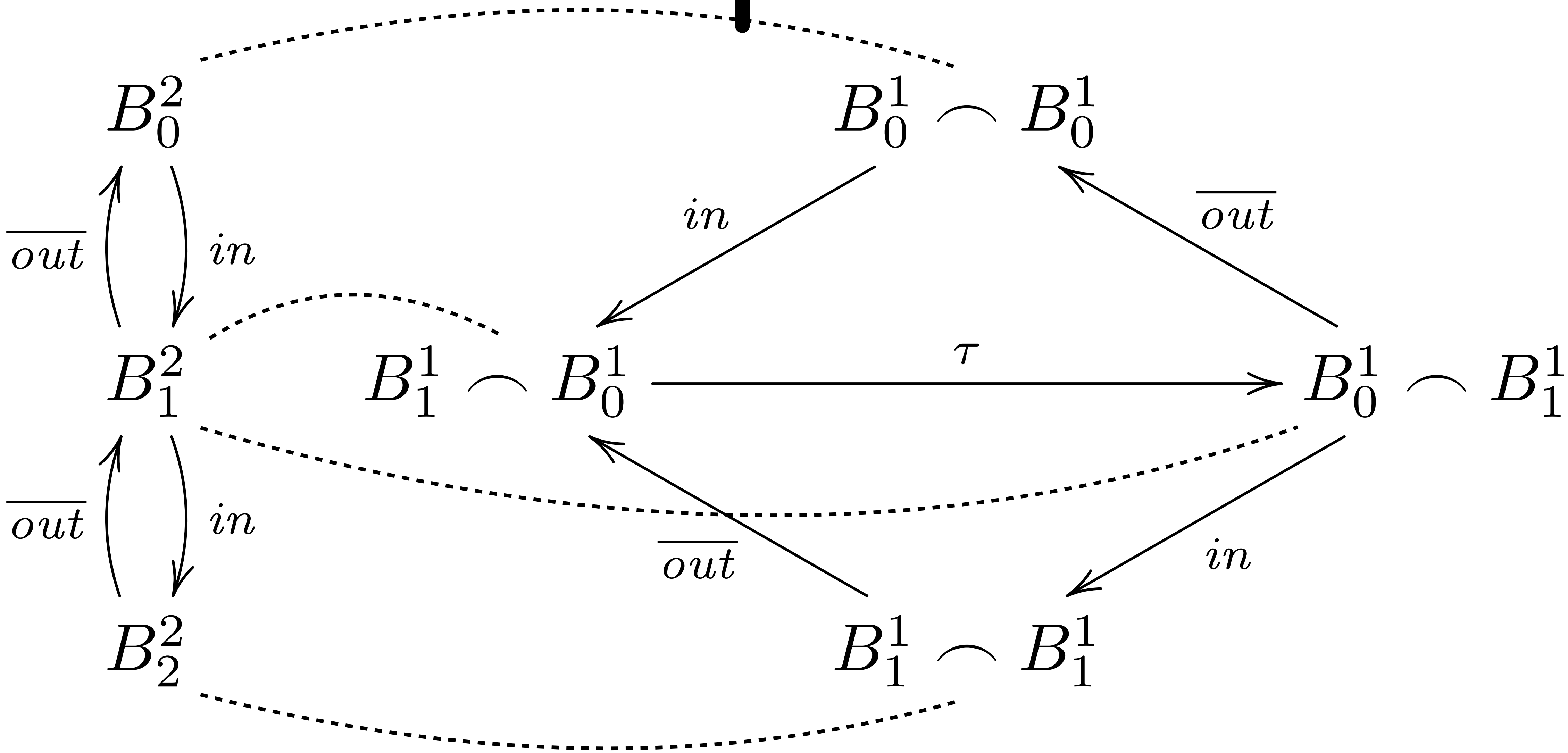
nothing changes: we still get the same weak bisimilarity

Example



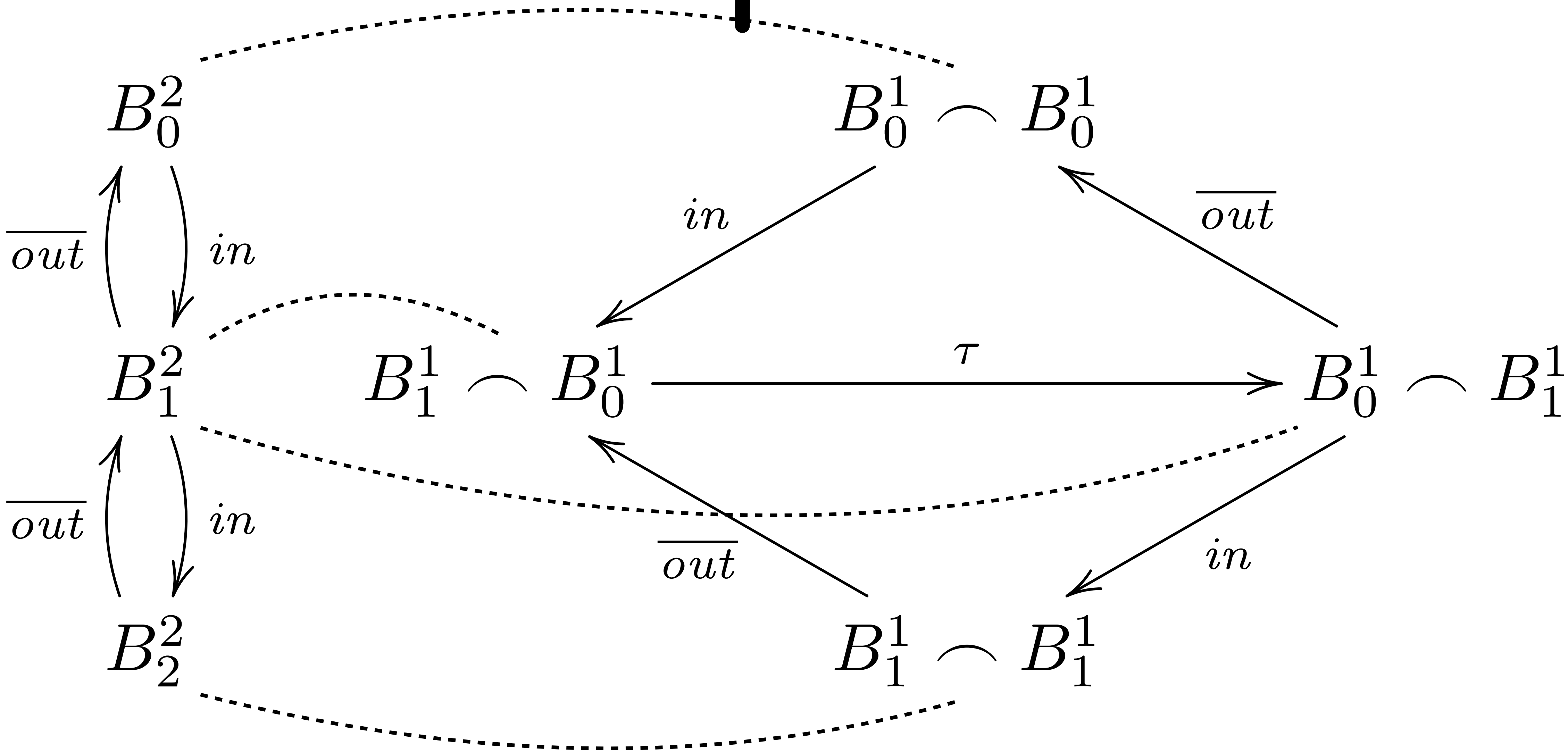
$$\mathbf{R} \triangleq \left\{ \begin{array}{l} (B_0^2, B_0^1 \frown B_0^1), \\ (B_1^2, B_1^1 \frown B_0^1), \\ (B_1^2, B_0^1 \frown B_1^1), \\ (B_2^2, B_1^1 \frown B_1^1) \end{array} \right\}$$ is a weak bisimulation relation

Example



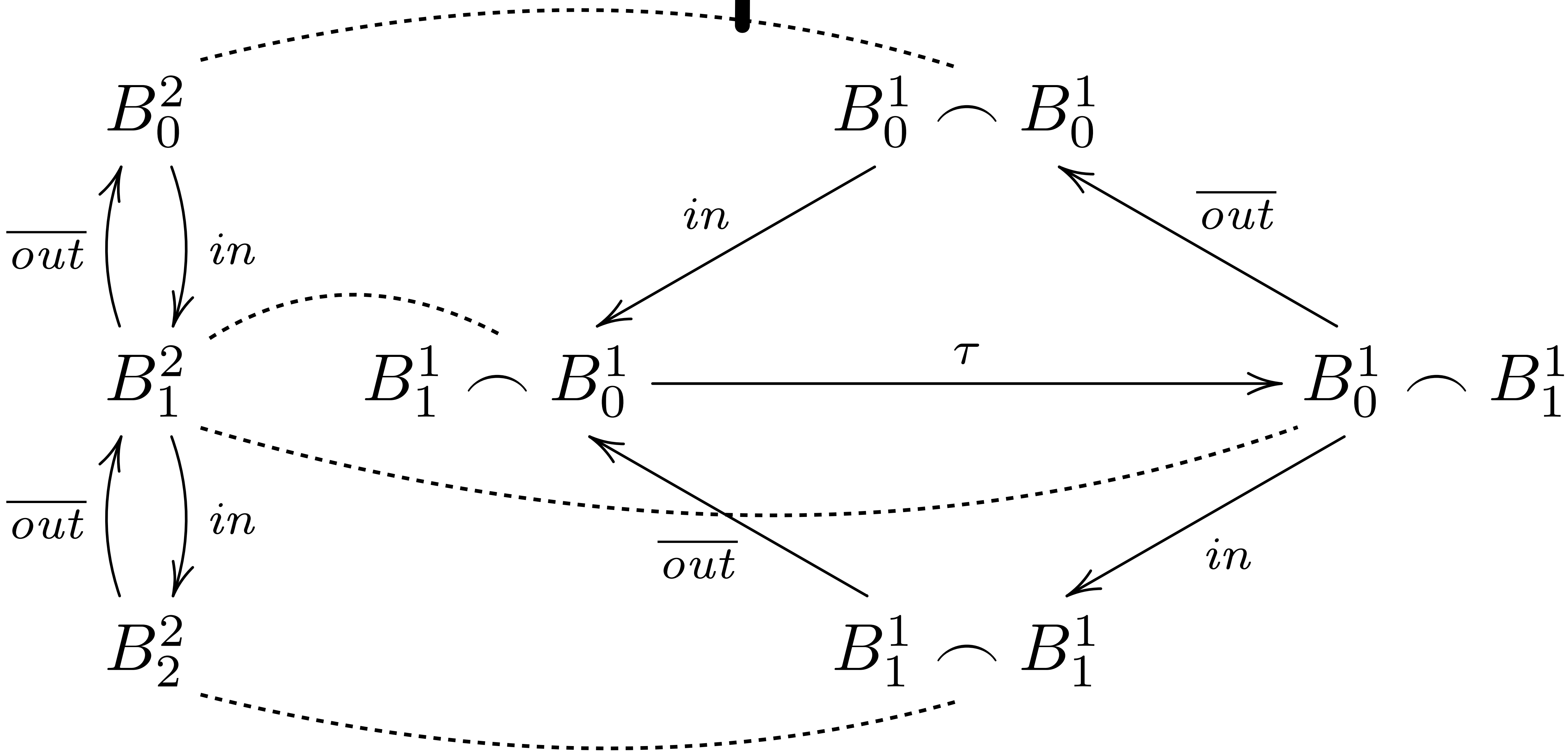
$$\begin{tikzcd}
B_0^2 \arrow[d, "in"] & \mathbf{R} & B_0^1 \frown B_0^1 \arrow[d, Rightarrow, "in"] \\
B_1^2 & \mathbf{R} & B_1^1 \frown B_0^1
\end{tikzcd}
\qquad
\begin{tikzcd}
B_0^2 \arrow[d, Rightarrow, "in"] & \mathbf{R} & B_0^1 \frown B_0^1 \arrow[d, "in"] \\
B_1^2 & \mathbf{R} & B_1^1 \frown B_0^1
\end{tikzcd}$$

Example



$$
\begin{tikzcd}
B^2_1 \arrow[d, "in"] & \mathbf{R} & B^1_1 \frown B^1_0 \arrow[d, Rightarrow, "in"] \\
B^2_2 & \mathbf{R} & B^1_1 \frown B^1_1
\end{tikzcd}
\qquad
\begin{tikzcd}
B^2_1 \arrow[d, "\overline{out}"] & \mathbf{R} & B^1_1 \frown B^1_0 \arrow[d, Rightarrow, "\overline{out}"] \\
B^2_0 & \mathbf{R} & B^1_0 \frown B^1_0
\end{tikzcd}
$$

Example



$$
\begin{tikzcd}
B_1^2 \arrow[d, Rightarrow, "\tau"] & \mathbf{R} & B_1^1 \frown B_0^1 \arrow[d, "\tau"] \\
B_1^2 & \mathbf{R} & B_0^1 \frown B_1^1
\end{tikzcd}
$$

(stays idle!)

(etc. for the other pairs)

Weak bis as a fixpoint

$$\Psi(\mathbf{R}) \triangleq \left\{ (p,q) \;\middle|\; \begin{array}{lcl} \forall \mu, p'.\ p \xrightarrow{\mu} p' & \Rightarrow & \exists q'.\ \boxed{q \overset{\mu}{\Rightarrow} q'} \wedge p' \ \mathbf{R}\ q' \\ \wedge & & \\ \forall \mu, q'.\ q \xrightarrow{\mu} q' & \Rightarrow & \exists p'.\ \boxed{p \overset{\mu}{\Rightarrow} p'} \wedge p' \ \mathbf{R}\ q' \end{array} \right\}$$

$$\Psi : \wp(\mathcal{P} \times \mathcal{P}) \to \wp(\mathcal{P} \times \mathcal{P})$$

maps relations to relations

$$\mathbf{R} \subseteq \Psi(\mathbf{R})$$

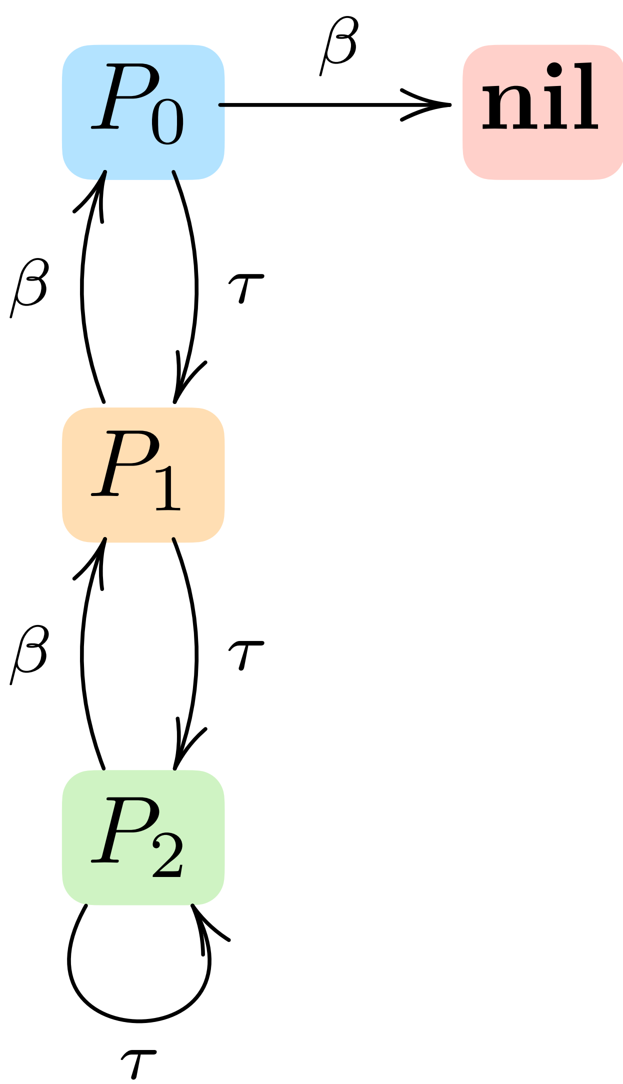
a weak bisimulation

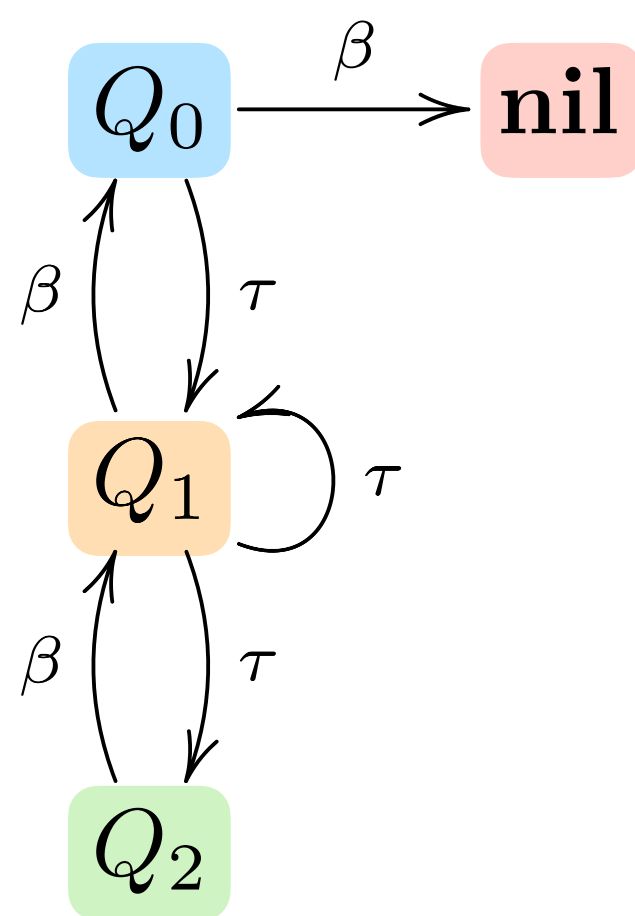
$$\approx \; = \Psi(\approx)$$

weak bisimilarity is a fixpoint

Exercise

$$P_0 \overset{?}{\approx} Q_0$$



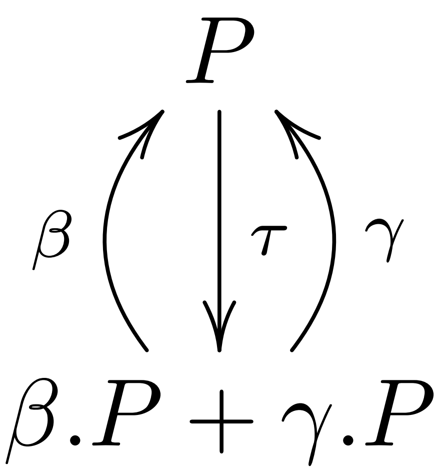


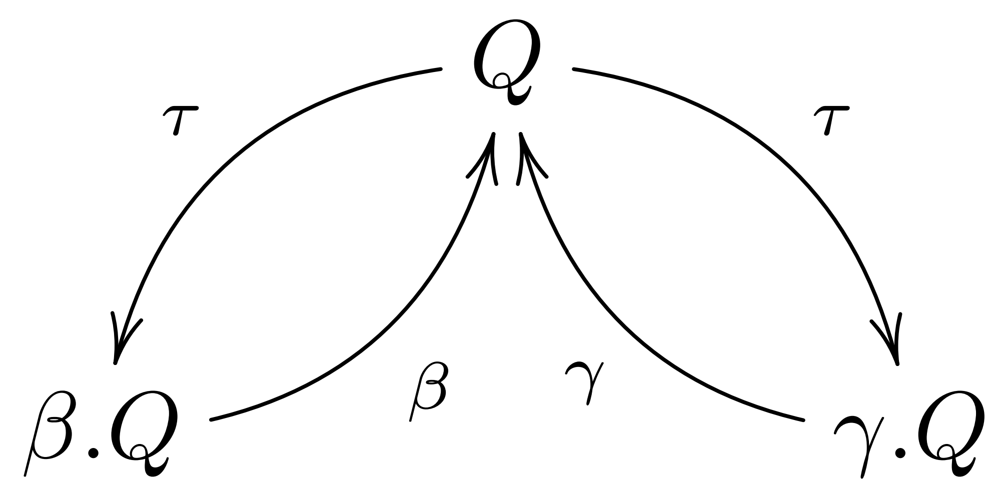
$$\mathbf{R} = \{\ \{P_0, Q_0\}\ ,\ \{P_1, Q_1\}\ ,\ \{P_2, Q_2\}\ ,\ \{\mathbf{nil}\}\ \}$$

is a weak bisimulation!

Exercise

$P \triangleq \tau.(\beta.P + \gamma.P)$ $\qquad P \overset{?}{\approx} Q \qquad$ $Q \triangleq \tau.\beta.Q + \tau.\gamma.Q$





$Q \xrightarrow{\tau} \beta.Q$ $\qquad P \overset{\tau}{\Longrightarrow} P$

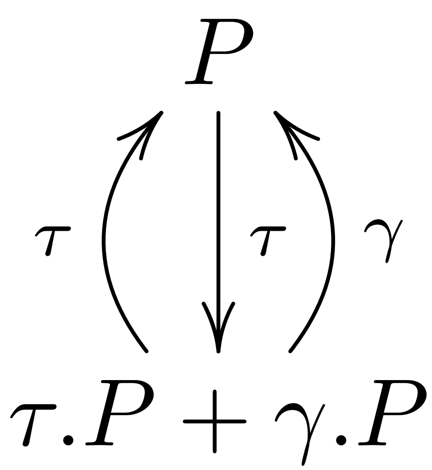
$P \xrightarrow{\tau} \beta.P + \gamma.P$ $\qquad \beta.Q \overset{\tau}{\Longrightarrow} \beta.Q$

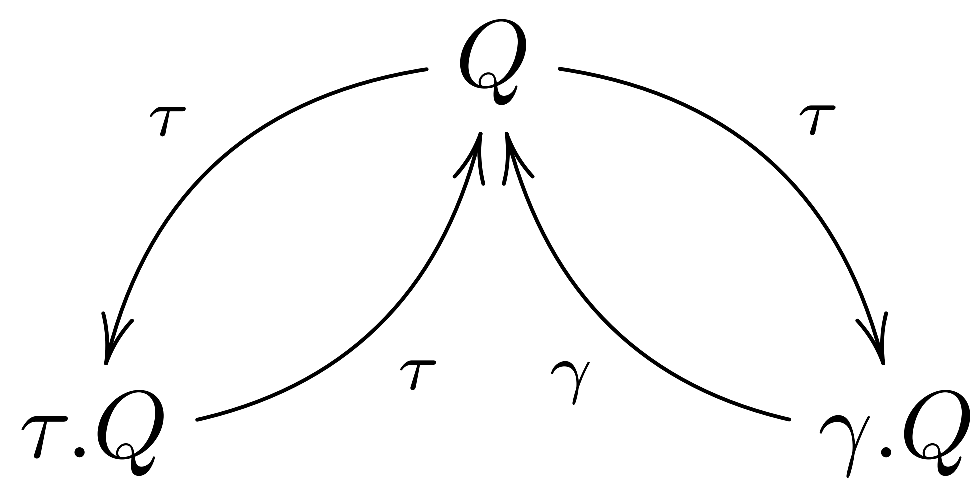
$\beta.P + \gamma.P \xrightarrow{\gamma} P$ $\qquad \beta.Q \overset{\gamma}{\not\Longrightarrow}$

Alice wins!

Exercise

$$P \triangleq \tau.(\tau.P + \gamma.P) \qquad P \overset{?}{\approx} Q \qquad Q \triangleq \tau.\tau.Q + \tau.\gamma.Q$$





$$\mathbf{R}_0 = \{\ \{P\ ,\ Q\ ,\ \tau.P + \gamma.P\ ,\ \tau.Q\ ,\ \gamma.Q\}\ \}$$

is a weak bisimulation!

Exercise

$$P \triangleq \alpha.(\tau.P + \gamma.P) \qquad P \overset{?}{\approx} Q \qquad Q \triangleq \alpha.\tau.Q + \alpha.\gamma.Q$$

$$\begin{tikzcd}
P \arrow[d, "\alpha"] \\
\tau.P + \gamma.P \arrow[u, bend left=40, "\tau"] \arrow[u, bend right=40, "\gamma"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & Q \arrow[dl, bend right, "\alpha"'] \arrow[dr, bend left, "\alpha"] & \\
\tau.Q \arrow[ur, bend right, "\tau"'] & & \gamma.Q \arrow[ul, bend left, "\gamma"]
\end{tikzcd}$$

$$Q \xrightarrow{\alpha} \gamma.Q \qquad P \overset{\alpha}{\Longrightarrow} \tau.P + \gamma.P$$

$$\tau.P + \gamma.P \xrightarrow{\tau} P \qquad \gamma.Q \overset{\tau}{\Longrightarrow} \gamma.Q$$

$$P \xrightarrow{\alpha} \tau.P + \gamma.P \qquad \gamma.Q \overset{\alpha}{\not\Longrightarrow}$$

Alice wins!

CCS
problems with weak semantics

Problems with weak bis

with respect to weak transitions,
guarded processes can have infinitely branching LTS

$$P \triangleq \mathbf{rec}\ x.\ \tau.x|\beta$$

$$\begin{tikzcd}
P \arrow[r, "\tau"] \arrow[d, "\beta"'] & P|\beta \arrow[r, "\tau"] \arrow[d, "\beta"'] & P|\beta|\beta \arrow[r, "\tau"] \arrow[d, "\beta"'] & \cdots \\
\tau.P & \tau.P|\beta & \tau.P|\beta|\beta &
\end{tikzcd}$$

many arrows omitted

assume $p|\mathbf{nil} = p$

$$\begin{tikzcd}
P \arrow[r, "\tau"] \arrow[d, Rightarrow, "\beta"'] \arrow[dr, Rightarrow, "\beta"'] \arrow[drr, Rightarrow, "\beta"] & P|\beta \arrow[r, "\tau"] \arrow[d, "\beta"'] & P|\beta|\beta \arrow[r, "\tau"] \arrow[d, "\beta"'] & \cdots \\
\tau.P & \tau.P|\beta & \tau.P|\beta|\beta &
\end{tikzcd}$$

avoid τ-prefixes?

$$P \triangleq \mathbf{rec}\ x.\ (\alpha.x|\overline{\alpha}|\beta)\backslash\alpha$$

Problems with weak bis

weak bisimilarity is not a congruence (w.r.t. +)

take $P \triangleq \alpha$ $\quad Q \triangleq \tau.\alpha$

if $P \xrightarrow{\alpha} \mathbf{nil}$ then $Q \stackrel{\alpha}{\Rightarrow} \mathbf{nil}$

if $Q \xrightarrow{\tau} \alpha$ then $P \stackrel{\tau}{\Rightarrow} P$

$P \approx Q$ $\quad \mathbb{C}[P] \not\approx \mathbb{C}[Q]$

$\mathbb{C}[P] \triangleq \alpha + \beta$

$\mathbb{C}[Q] \triangleq \tau.\alpha + \beta$

take the context $\mathbb{C}[\cdot] \triangleq [\cdot] + \beta$

Alice plays $\mathbb{C}[Q] \xrightarrow{\tau} \alpha$

Bob can only reply $\mathbb{C}[P] \stackrel{\tau}{\Rightarrow} \mathbb{C}[P]$

Alice plays $\mathbb{C}[P] \xrightarrow{\beta} \mathbf{nil}$

Bob cannot reply $\alpha \stackrel{\beta}{\not\Rightarrow}$

Alice wins!

Problems with weak bis

cannot distinguish between deadlock and silent divergence

$$\mathbf{rec}\ x.\ \tau.x \approx \mathbf{nil}$$

$$\mathbf{rec}\ x.\ \tau.x \xrightarrow{\tau} \mathbf{rec}\ x.\ \tau.x \qquad \mathbf{nil} \stackrel{\tau}{\Rightarrow} \mathbf{nil}$$

CCS
weak observational congruence

Weak obs congruence

$$p \cong q \quad \text{iff} \quad p \approx q \wedge \forall r.\ p + r \approx q + r$$

Equivalently

$$p \cong q \quad \text{iff} \quad \begin{cases} \forall p'.\ p \xrightarrow{\tau} p' & \Rightarrow \quad \exists q', q''.\ q \xrightarrow{\tau} q'' \stackrel{\tau}{\Rightarrow} q' \wedge p' \approx q' \\ \forall \lambda, p'.\ p \xrightarrow{\lambda} p' & \Rightarrow \quad \exists q'.\ q \stackrel{\lambda}{\Rightarrow} q' \wedge p' \approx q' \\ \text{and vice versa} & \end{cases}$$

not a recursive definition!
(refers to weak bisimilarity)

at the level of bisimulation game:

Bob is not allowed to use an idle move at the very first turn
(at the following turns, ordinary weak bisimulation game)

TH. $\cong$ is the largest congruence contained in $\approx$

Weak obs congruence

Note: $\approxeq$ is not a weak bisimulation!

$$P \triangleq \alpha \qquad\qquad Q \triangleq \tau.\alpha$$

$$\begin{array}{ccc} \beta.P & \approxeq & \beta.Q \\ \downarrow \beta & & \downarrow \beta \\ P & \approx & Q \end{array}$$

$$P \not\approxeq Q$$

$$\approxeq \not\subseteq \Psi(\approxeq)$$

Weak obs congruence

All the laws for strong bisimilarity are still valid

Additionally: Milner's τ-laws

$$p + \tau.p \approxeq \tau.p$$

$$\mu.(p + \tau.q) \approxeq \mu.(p + \tau.q) + \mu.q$$

$$\mu.\tau.p \approxeq \mu.p$$

Weak bisimilar processes

Prove that the following property is valid for any agent p, where $\approx$ is the weak bisimilarity:

$$p + \tau.p \approx \tau.p$$

Weak bisimilar processes

$$\mathbf{R} \triangleq \{(p + \tau.p, \tau.p) \mid p \in \mathcal{P}\} \cup Id$$

we check that **R** is a weak bisimulation

no need to check pairs in Id

$$\begin{tikzcd} p \arrow[d, "\mu"] & p + \tau.p \arrow[d, "\mu"] & \mathbf{R} & \tau.p \arrow[d, Rightarrow, "\mu"] \\ p' & p' & \mathbf{R} & p' \end{tikzcd}$$

$$\begin{tikzcd} p + \tau.p \arrow[d, "\tau"] & \mathbf{R} & \tau.p \arrow[d, Rightarrow, "\tau"] \\ p & \mathbf{R} & p \end{tikzcd}$$

Weak bisimilar processes

$$\mathbf{R} \triangleq \{(p + \tau.p, \tau.p) \mid p \in \mathcal{P}\} \cup Id$$

we check that **R** is a weak bisimulation

no need to check pairs in Id

$$\begin{array}{ccc} p + \tau.p & \mathbf{R} & \tau.p \\ \Big\Downarrow \tau & & \Big\downarrow \tau \\ p & \mathbf{R} & p \end{array}$$