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MPP 2025/26 (0077A, 9CFU)

Models for Programming Paradigms

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19 - Hennessy-Milner Logic

Bisimilarity

graph isomorphism distinguishes too many processes

trace equivalence identifies too many processes

we need some notion of equivalence in between the two

we introduce the notion of *strong bisimilarity*

as a game

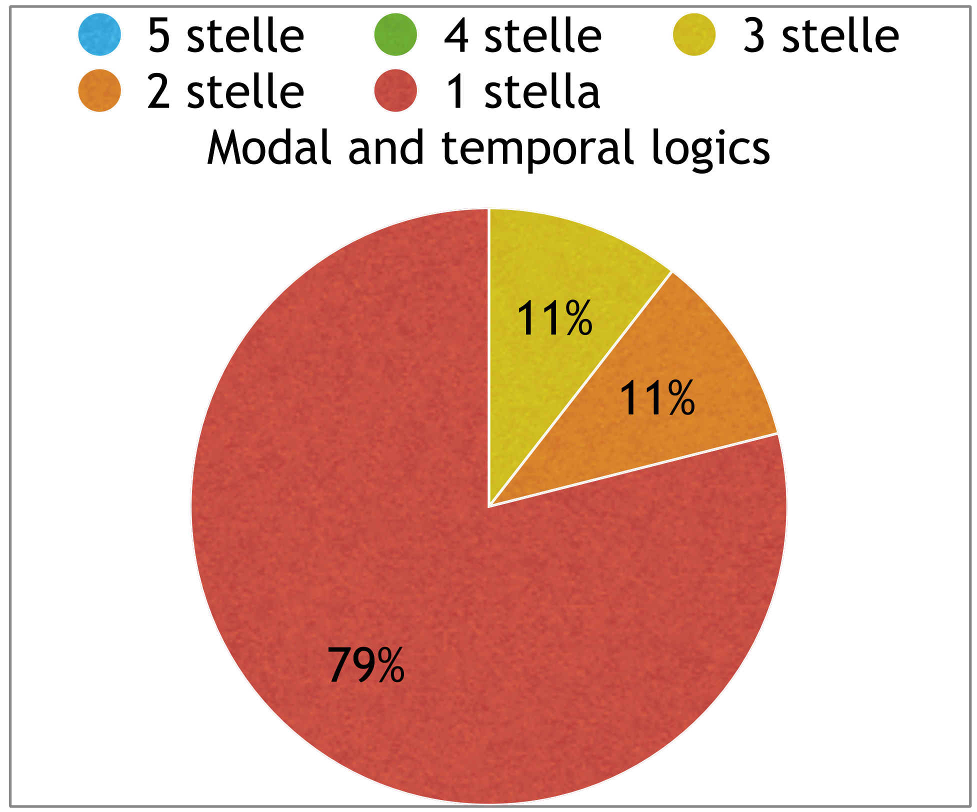
as a fixpoint

as a logical equivalence

to keep in mind: two processes are equivalent unless we have some good reasons to distinguish them

HML
Hennessy-Milner Logic

From your forms



(over 19 answers)

Logical equivalence

Let us take another approach to equivalence

we define some logic (set of formulas)

a process may or may not satisfy a formula

two processes are (logically) equivalent
when they satisfy exactly the same formulas

formulas must describe behavioural properties of processes

the ability / inability to perform transitions
(modal logic: possibly, necessarily)

then, we can compose formulas with usual operators

Hennessy-Milner Logic

We present the core operators

multi-modal:
modal operators are parameterised by actions

no negation:
the converse of a formula can also be written as a formula

no recursion:
each formula express properties about finite steps ahead

denotational semantics of a formula (postponed):
set of processes that satisfy the formula

HML: syntax

$$
\begin{array}{llll}
F, G & ::= & \mathbf{tt} & \text{true} \\
& | & \mathbf{ff} & \text{false} \\
& | & \bigwedge_{i \in I} F_i & \text{conjunction} \\
& | & \bigvee_{i \in I} F_i & \text{disjunction} \\
& | & \Diamond_\mu F & \text{diamond operator} \quad \langle \mu \rangle F \\
& | & \Box_\mu F & \text{box operator} \quad [\mu] F
\end{array}
$$

$\mathcal{L}$ set of all formulas

HML: semantics

$p \models F$ reads " p satisfies F "

defined inductively on the structure of the formula

$p \models \mathbf{tt}$ any process satisfies true (no process satisfies false)

$p \models \bigwedge_{i \in I} F_i$ iff $\forall i \in I.\ p \models F_i$ p satisfies all F_i

$p \models \bigvee_{i \in I} F_i$ iff $\exists i \in I.\ p \models F_i$ p satisfies one of the F_i

$p \models \Diamond_\mu F$ iff $\exists p'.\ p \xrightarrow{\mu} p' \wedge p' \models F$ p can make one μ-step and then satisfy F

$p \models \Box_\mu F$ iff $\forall p'.\ p \xrightarrow{\mu} p' \Rightarrow p' \models F$ F is satisfied after any μ-step of p

Examples

$\Diamond_\alpha \mathbf{tt}$ satisfied by any process that can make an *α*-step

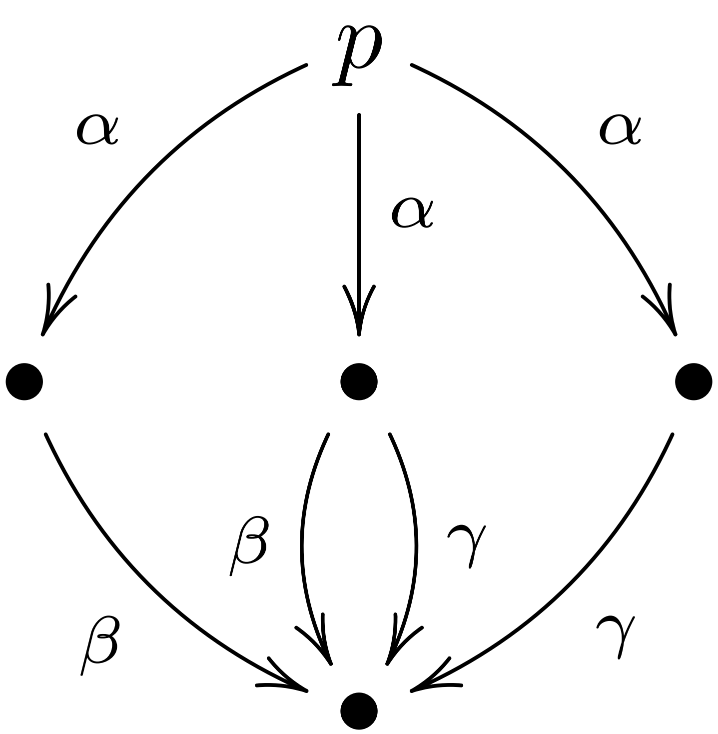
$\Box_\beta \mathbf{ff}$ satisfied by any process that cannot make a *β*-step

$\Diamond_\alpha \mathbf{ff}$ same as **ff**
if a process cannot do *α* the modality is missed
if a process can do *α* its continuation cannot satisfy **ff**

$\Box_\beta \mathbf{tt}$ same as **tt**
if a process cannot do *β* the modality holds trivially
if a process does *β* its continuation will satisfy **tt**

$\Diamond_\alpha(\Diamond_\beta \mathbf{tt} \wedge \Box_\gamma \mathbf{ff})$ satisfied by any process the can do *α*
and reach a process that can do *β* but not *γ*

Examples



$$p \overset{?}{\models} \Diamond_\alpha \mathbf{tt}$$ ✓

$$p \overset{?}{\models} \Box_\alpha \Diamond_\beta \mathbf{tt}$$ ✗

$$p \overset{?}{\models} \Diamond_\alpha \Box_\beta \mathbf{ff} \ \wedge \ \Diamond_\alpha \Box_\gamma \mathbf{ff}$$ ✓

$$p \overset{?}{\models} \Box_\alpha (\Diamond_\beta \mathbf{tt} \vee \Diamond_\gamma \mathbf{tt})$$ ✓

$$p \overset{?}{\models} \Box_\alpha (\Diamond_\beta \mathbf{tt} \wedge \Diamond_\gamma \mathbf{tt})$$ ✗

$$p \overset{?}{\models} \Diamond_\alpha (\Diamond_\beta \mathbf{tt} \wedge \Diamond_\gamma \mathbf{tt})$$ ✓

Box/diamond duality

$$\neg \Diamond_\mu F$$

$$\equiv \neg(\exists p'.\ p \xrightarrow{\mu} p' \ \wedge\ p' \models F)$$

$$\equiv \forall p'.\ \neg(p \xrightarrow{\mu} p' \ \wedge\ p' \models F)$$

$$\equiv \forall p'.\ \neg(p \xrightarrow{\mu} p') \ \vee\ \neg(p' \models F)$$

$$\equiv \forall p'.\ p \xrightarrow{\mu} p' \ \Rightarrow\ (p' \models \neg F)$$

$$\equiv \Box_\mu \neg F$$

Negation

not present in the syntax, but not needed

any formula *F* has a *converse* formula F^c such that

$$\forall p.\ p \models F \quad \text{iff} \quad p \not\models F^c$$

F^c can be defined by structural induction

$$\mathbf{tt}^c \triangleq \mathbf{ff} \qquad \mathbf{ff}^c \triangleq \mathbf{tt}$$

$$(\bigwedge_{i\in I} F_i)^c \triangleq \bigvee_{i\in I} F_i^c \qquad (\bigvee_{i\in I} F_i)^c \triangleq \bigwedge_{i\in I} F_i^c$$

$$(\Diamond_\mu F)^c \triangleq \Box_\mu F^c \qquad (\Box_\mu F)^c \triangleq \Diamond_\mu F^c$$

example

$(\Diamond_\alpha \mathbf{tt})^c = \Box_\alpha \mathbf{tt}^c = \Box_\alpha \mathbf{ff}$ (can do *a*)^c = cannot do *a*

Extended syntax

$$A = \{\mu_1, ..., \mu_n\}$$

$$\Diamond_A F \triangleq \Diamond_{\mu_1} F \vee \cdots \vee \Diamond_{\mu_n} F \qquad \Box_A F \triangleq \Box_{\mu_1} F \wedge \cdots \wedge \Box_{\mu_n} F$$

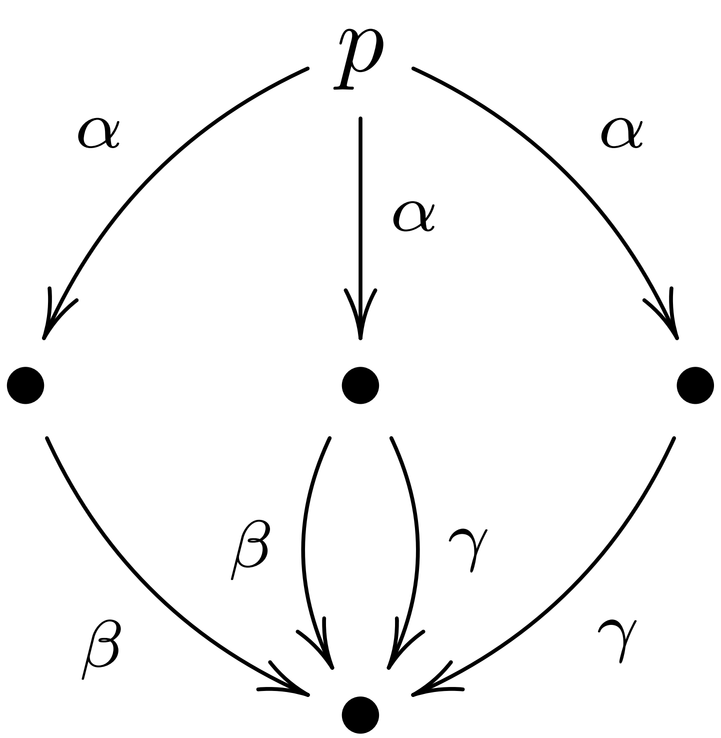
$$= \bigvee_{i \in [1,n]} \Diamond_{\mu_i} F \qquad = \bigwedge_{i \in [1,n]} \Box_{\mu_i} F$$

$$\Diamond_\emptyset F \triangleq \mathbf{ff} \qquad \Box_\emptyset F \triangleq \mathbf{tt}$$

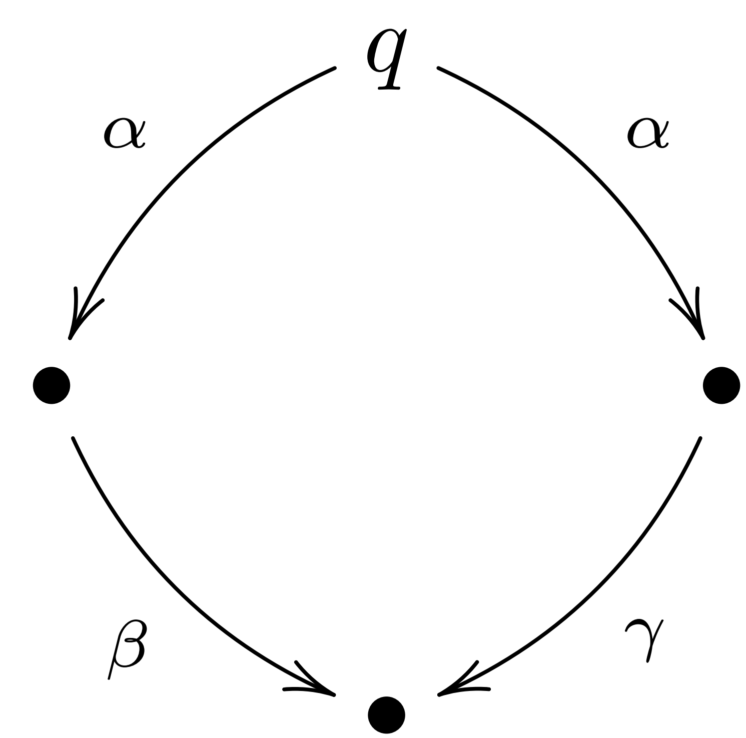
HML: logical equivalence

two processes are equivalent iff they satisfy the same formulas

$$p \equiv_{\mathrm{HM}} q \quad \text{iff} \quad \forall F \in \mathcal{L}.\ (p \models F \ \Leftrightarrow\ q \models F)$$



$$p \overset{?}{\equiv}_{\mathrm{HM}} q$$



$p \models F$ $\qquad F \triangleq \Diamond_\alpha(\Diamond_\beta \mathbf{tt} \wedge \Diamond_\gamma \mathbf{tt})$ $\qquad q \not\models F$

$p \not\models F^c$ $\qquad F^c \triangleq \Box_\alpha(\Box_\beta \mathbf{ff} \vee \Box_\gamma \mathbf{ff})$ $\qquad q \models F^c$

Strong bis as logic equiv

TH. for any finitely branching processes *p,q*

$$p \simeq q \quad \text{iff} \quad p \equiv_{\mathrm{HM}} q$$

(proof omitted)

consequences:

to show that two processes are strong bisimilar:
exhibit a strong bisimulation relation that relates them

to show that two processes are not strong bisimilar:
exhibit a HML formula that distinguishes between them

Exercise

find a HML formula that distinguishes the two processes

$$\begin{tikzcd} P_0 \arrow[r, "\beta"] \arrow[d, bend left, "\alpha"] & \mathbf{nil} \\ P_1 \arrow[u, bend left, "\beta"] \arrow[d, bend left, "\alpha"] \\ P_2 \arrow[u, bend left, "\beta"] \arrow[loop below, "\alpha"] \end{tikzcd}$$

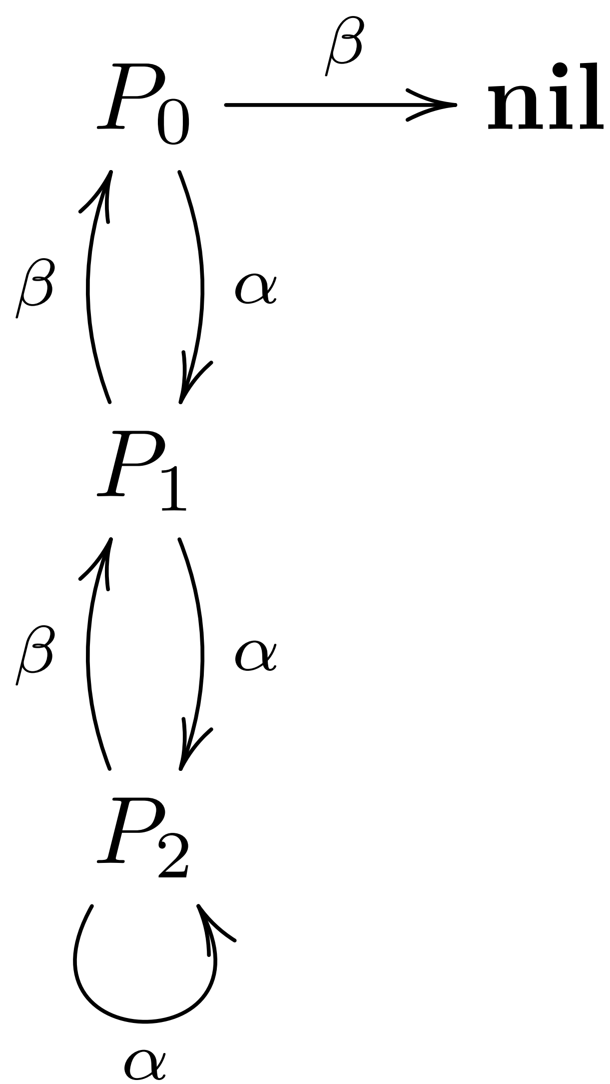
$$P_0 \not\simeq R_0$$

$$\begin{tikzcd} R_0 \arrow[r, "\beta"] \arrow[d, bend left, "\alpha"] & \mathbf{nil} \\ R_1 \arrow[u, bend left, "\beta"] \arrow[d, bend left, "\alpha"] \\ R_2 \arrow[u, bend left, "\beta"] \end{tikzcd}$$

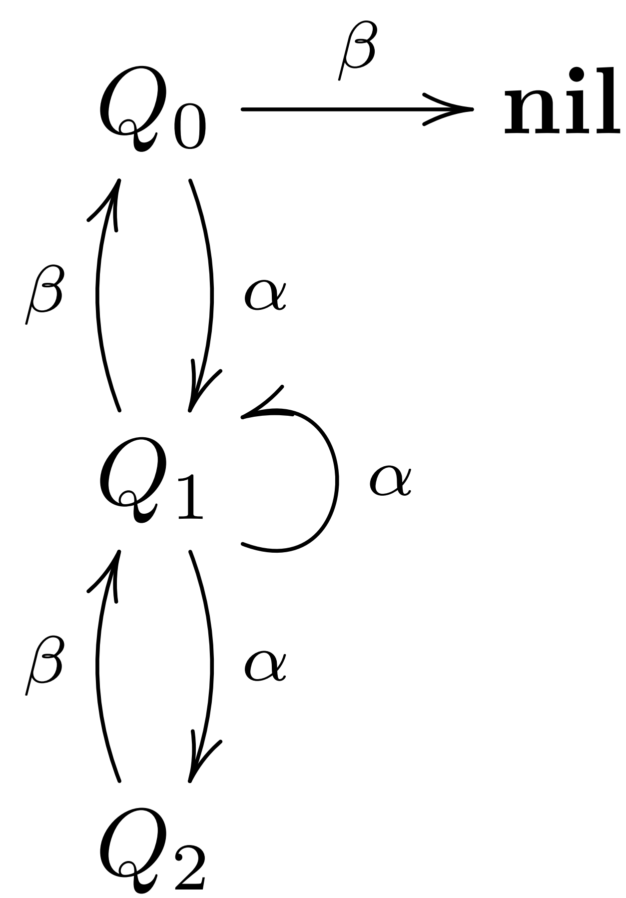
$P_0 \models F$ $\qquad F \triangleq \Diamond_\alpha \Diamond_\alpha \Diamond_\alpha \mathbf{tt}$ $\qquad R_0 \not\models F$

Exercise

find a HML formula that distinguishes the two processes



$P_0 \not\sim Q_0$



$P_0 \models F$

$F \triangleq \Diamond_\alpha \Box_\alpha \Diamond_\alpha \mathbf{tt}$

$Q_0 \not\models F$

Non bisimilar processes

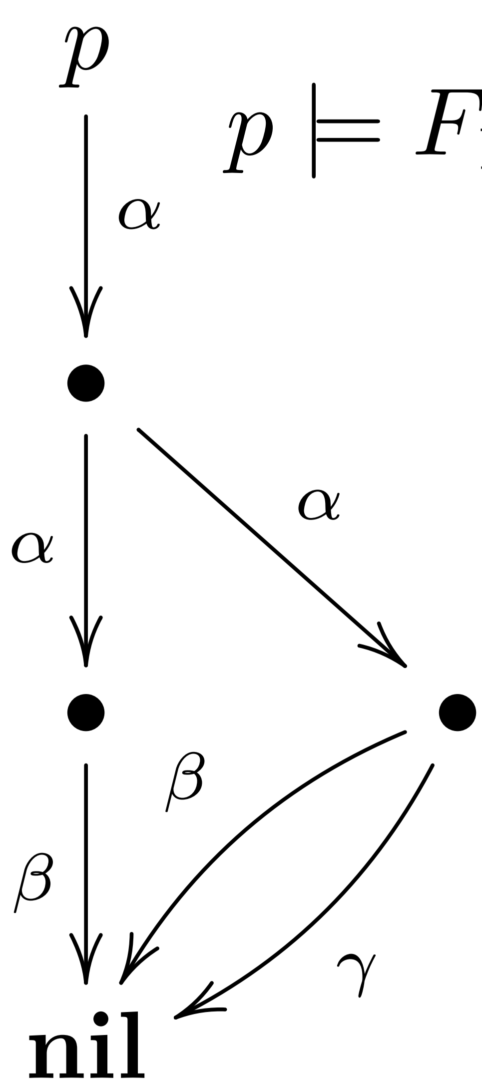
Prove that the CCS agents

$$p \stackrel{\text{def}}{=} \alpha.(\alpha.\beta.\mathbf{nil} + \alpha.(\beta.\mathbf{nil} + \gamma.\mathbf{nil})) \qquad \text{and} \qquad q \stackrel{\text{def}}{=} \alpha.(\alpha.\beta.\mathbf{nil} + \alpha.\gamma.\mathbf{nil})$$

are not strong bisimilar.

Non bisimilar processes

$$p \triangleq \alpha.(\alpha.\beta + \alpha.(\beta + \gamma)) \qquad\qquad q \triangleq \alpha.(\alpha.\beta + \alpha.\gamma)$$



$$F_1 \triangleq \Diamond_\alpha \Diamond_\alpha (\Diamond_\beta \mathbf{tt} \wedge \Diamond_\gamma \mathbf{tt})$$

$p \models F_1$ $\qquad$ $q \not\models F_1$

$$F_2 \triangleq \Box_\alpha \Box_\alpha \Diamond_\beta \mathbf{tt}$$

$p \models F_2$ $\qquad$ $q \not\models F_2$

$$F_2^c \triangleq \Diamond_\alpha \Diamond_\alpha \Box_\beta \mathbf{ff}$$

$p \not\models F_2^c$ $\qquad$ $q \models F_2^c$

$$F_1^c \triangleq \Box_\alpha \Box_\alpha (\Box_\beta \mathbf{ff} \vee \Box_\gamma \mathbf{ff})$$

$p \not\models F_1^c$ $\qquad$ $q \models F_1^c$