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MPP 2025/26 (0077A, 9CFU)

Models for Programming Paradigms

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18c - More on bisimulaton

Properties of guarded processes

Guarded variables

TH. $G(p, X \cup \{x\}) \Rightarrow G(p, X)$

TH. $G(p, X) \wedge x \notin \mathsf{fv}(p) \Rightarrow G(p, X \cup \{x\})$

proofs by structural induction on p (omitted)

Substitutions preserve guardedness

TH. $G(p, X) \wedge \bigwedge_{i \in [1,n]} G(p_i, X) \Rightarrow G(p[{}^{p_1}/_{x_1}, \cdots, {}^{p_n}/_{x_n}], X)$

proof by structural induction on p (omitted)

Transitions preserve guardedness

TH. $G(p, X) \wedge p \xrightarrow{\mu} q \Rightarrow G(q, \emptyset)$

proof by rule induction on $p \xrightarrow{\mu} q$ (omitted)

Guarded processes are finitely branching

TH. $G(p,\emptyset) \Rightarrow \forall \mu.\ \{q \mid p \xrightarrow{\mu} q\}$ is a finite set

we prove a stronger property:

TH. $X = \{x_1, ..., x_n\}$ $\sigma = [{}^{p_1}/_{x_1}, \cdots, {}^{p_n}/_{x_n}]$

$$G(p,X) \wedge \bigwedge_{i \in [1,n]} G(p_i, X) \Rightarrow \{q \mid \exists \mu.\ p\sigma \xrightarrow{\mu} q\} \text{ is a finite set}$$

proof: by structural induction on p

we only show some cases (and omit some details)

(see next slides)

TH. $X = \{x_1, ..., x_n\}$ $\sigma = [{}^{p_1}/_{x_1}, \cdots, {}^{p_n}/_{x_n}]$ (continue)

$$G(p, X) \wedge \bigwedge_{i \in [1,n]} G(p_i, X) \Rightarrow \{q \mid \exists \mu.\ p\sigma \xrightarrow{\mu} q\}$$ is a finite set

proof: by structural induction on p

$p = \mathbf{nil}$ $\mathbf{nil}\sigma = \mathbf{nil}$ $\{q \mid \exists \mu.\ \mathbf{nil} \xrightarrow{\mu} q\} = \emptyset$

$p = x$ $x \in X$ $G(x, X) = \mathbf{false}$

$x \notin X$ $x\sigma = x$ $\{q \mid \exists \mu.\ x \xrightarrow{\mu} q\} = \emptyset$

$p = \mu'.p'$ $\{q \mid \exists \mu.\ p\sigma \xrightarrow{\mu} q\} = \{p'\sigma\}$

TH. $X = \{x_1, ..., x_n\}$ $\sigma = [^{p_1}/_{x_1}, \cdots, ^{p_n}/_{x_n}]$ (continue)

$G(p, X) \wedge \bigwedge_{i \in [1,n]} G(p_i, X) \Rightarrow \{q \mid \exists \mu.\ p\sigma \xrightarrow{\mu} q\}$ is a finite set

proof: by structural induction on p

$p = p' \backslash \alpha$ $p\sigma = p'\sigma \backslash \alpha$ $G(p, X) = G(p', X)$

$$\{q \mid \exists \mu.\ p\sigma \xrightarrow{\mu} q\} = \{q' \backslash \alpha \mid \exists \mu \notin \{\alpha, \overline{\alpha}\}. p'\sigma \xrightarrow{\mu} q'\}$$

finite, by inductive hypothesis

$p = p'_0 + p'_1$ $p\sigma = p'_0\sigma + p'_1\sigma$ $G(p, X) = G(p'_0, X) \wedge G(p'_1, X)$

$$\{q \mid \exists \mu.\ p\sigma \xrightarrow{\mu} q\} = \{q'_0 \mid \exists \mu.\ p'_0\sigma \xrightarrow{\mu} q'_0\} \cup \{q'_1 \mid \exists \mu.\ p'_1\sigma \xrightarrow{\mu} q'_1\}$$

finite, by inductive hypotheses

TH. $X = \{x_1, ..., x_n\}$ $\sigma = [^{p_1}/_{x_1}, \cdots, ^{p_n}/_{x_n}]$ (continue)

$$G(p, X) \wedge \bigwedge_{i \in [1,n]} G(p_i, X) \Rightarrow \{q \mid \exists \mu.\ p\sigma \xrightarrow{\mu} q\}$$ is a finite set

proof: by structural induction on p

$p = \mathbf{rec}\ x.p'$ without loss of generality $x \notin X \cup \bigcup_{i \in [1,n]} \mathsf{fv}(p_i)$

$$p\sigma = \mathbf{rec}\ x.\ p'\sigma$$

$$G(p, X) = G(p', X \cup \{x\})$$

$$\{q \mid \exists \mu.\ p\sigma \xrightarrow{\mu} q\} = \{q' \mid \exists \mu.\ p'\sigma[^{\mathbf{rec}\ x.\ p'\sigma}/_x] \xrightarrow{\mu} q'\}$$

finite, by inductive hypotheses

Example

Guarded!

$$P \triangleq \alpha.(\beta.P + \gamma.P)$$

$$P \overset{?}{\simeq} Q$$

Guarded!

$$Q \triangleq \alpha.\beta.Q + \alpha.\gamma.Q$$

$$\begin{tikzcd}
P \arrow[d, "\alpha"] \\
\beta.P + \gamma.P \arrow[u, bend left=50, "\beta"] \arrow[u, bend right=50, "\gamma"']
\end{tikzcd}$$

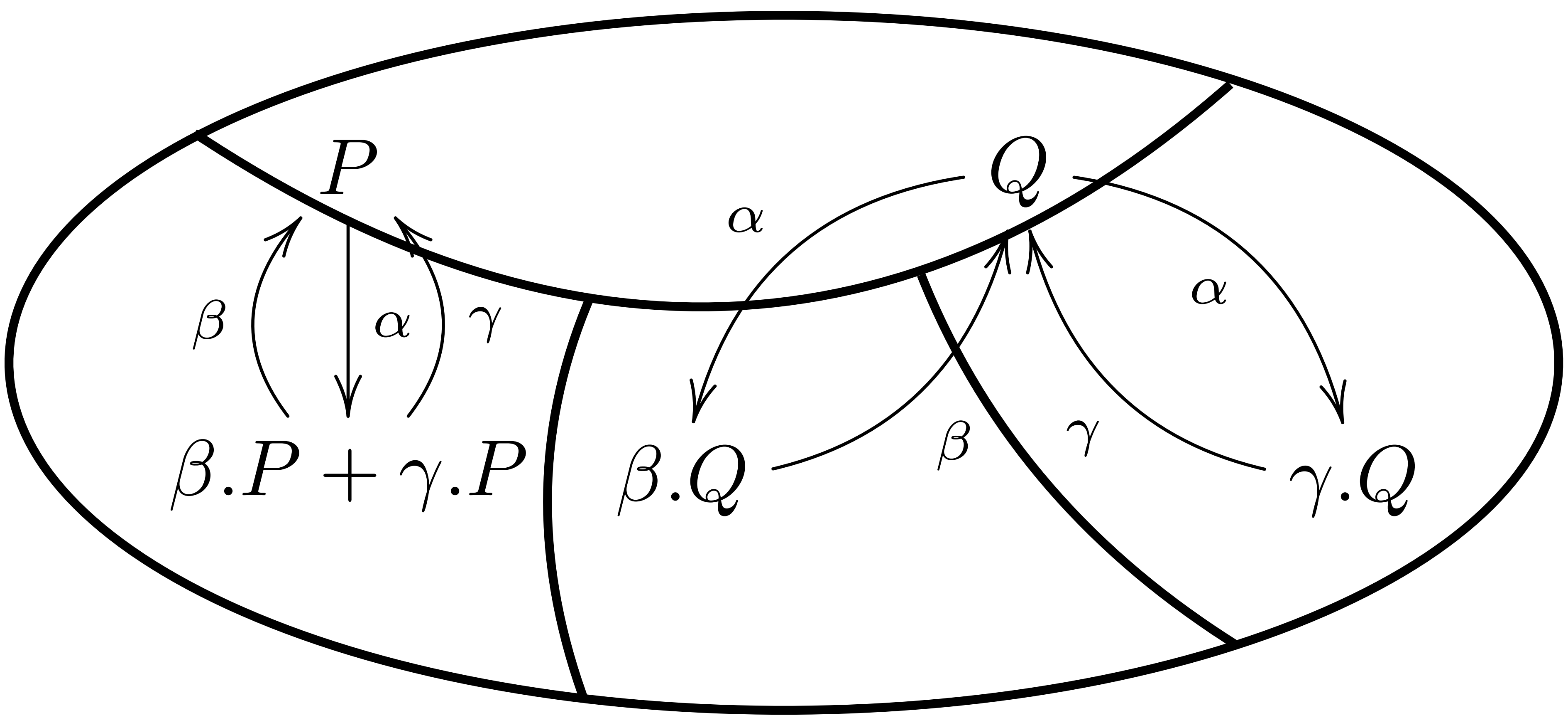
$$\begin{tikzcd}
 & Q \arrow[dl, bend right, "\alpha"'] \arrow[dr, bend left, "\alpha"] & \\
\beta.Q \arrow[ur, bend right, "\beta"'] & & \gamma.Q \arrow[ul, bend left, "\gamma"]
\end{tikzcd}$$

$$\mathbf{R}_0 = \{\ \{P\ ,\ Q\ ,\ \beta.P + \gamma.P\ ,\ \beta.Q\ ,\ \gamma.Q\}\ \}$$

Example

$$P \triangleq \alpha.(\beta.P + \gamma.P) \qquad P \overset{?}{\simeq} Q \qquad Q \triangleq \alpha.\beta.Q + \alpha.\gamma.Q$$

$$\mathbf{R}_0 = \{\ \{P\ ,\ Q\ ,\ \beta.P + \gamma.P\ ,\ \beta.Q\ ,\ \gamma.Q\}\ \}$$



$$P, Q \xrightarrow{\alpha}$$

$$\beta.P + \gamma.P \xrightarrow{\beta,\gamma}$$

$$\beta.Q \xrightarrow{\beta}$$

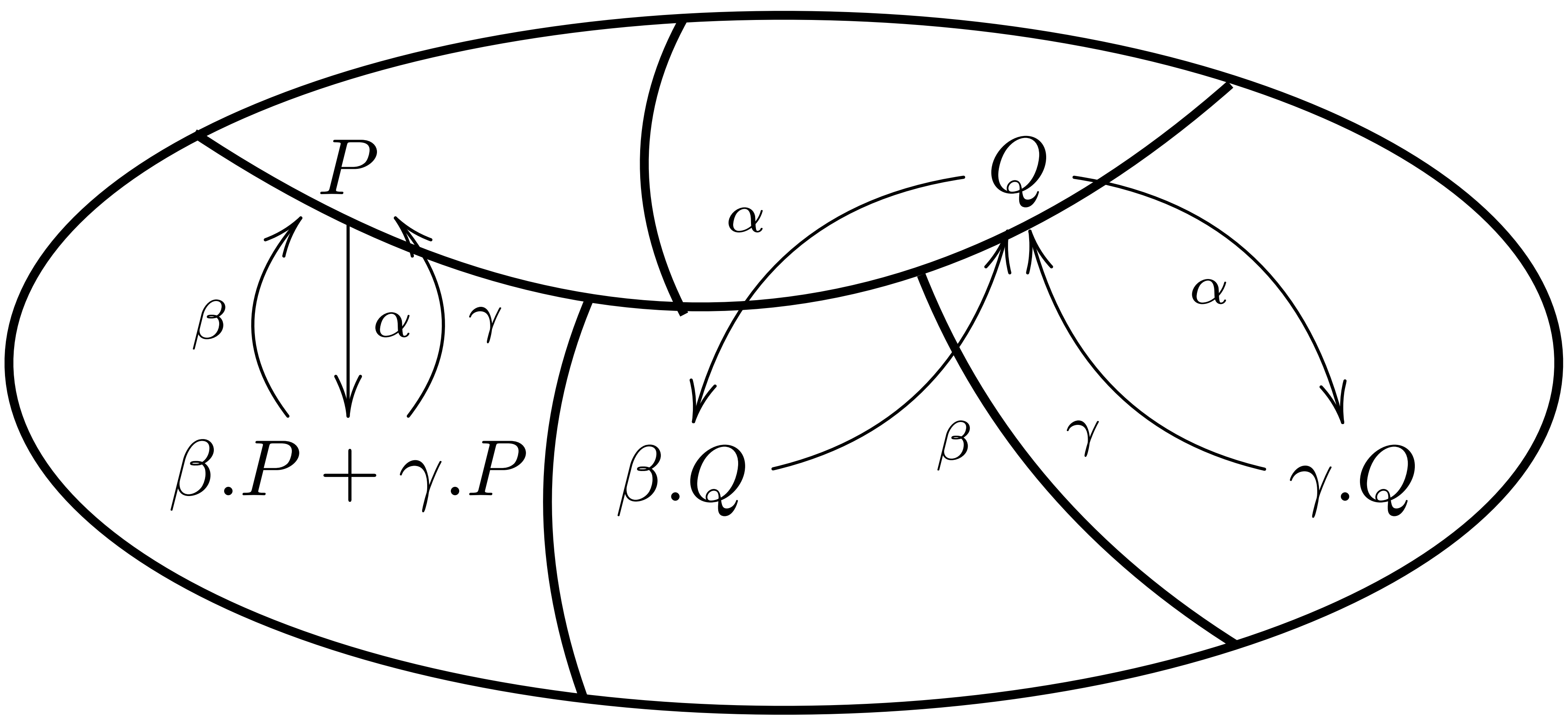
$$\gamma.Q \xrightarrow{\gamma}$$

Processes with different capabilities must be distinguished

Example

$$P \triangleq \alpha.(\beta.P + \gamma.P) \qquad P \overset{?}{\simeq} Q \qquad Q \triangleq \alpha.\beta.Q + \alpha.\gamma.Q$$

$$\mathbf{R}_1 = \{\ \{P\ ,\ Q\}\ ,\ \{\beta.P + \gamma.P\}\ ,\ \{\beta.Q\}\ ,\ \{\gamma.Q\}\ \}$$



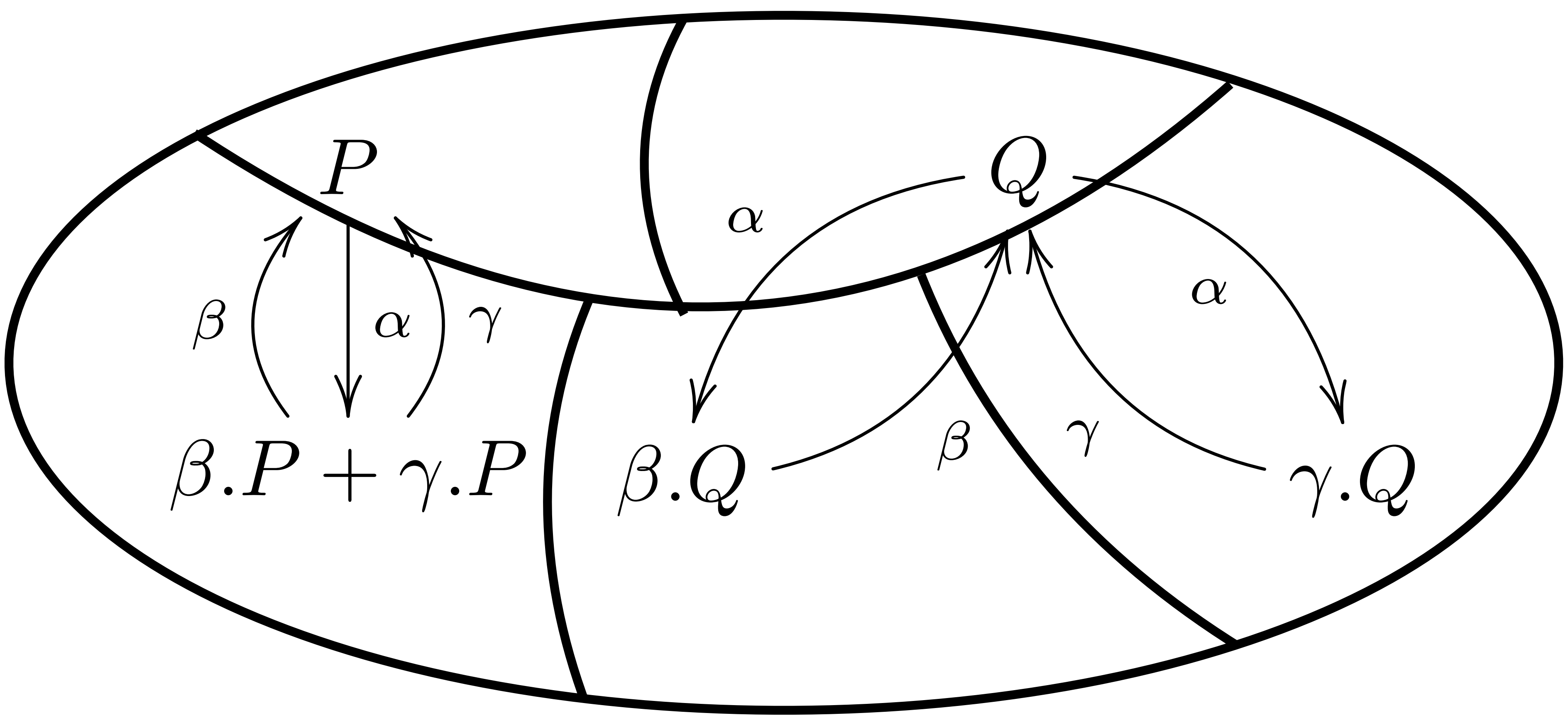
$$P \xrightarrow{\alpha} [\beta.P + \gamma.Q]$$
$$Q \xrightarrow{\alpha} [\beta.Q], [\gamma.Q]$$

The α transitions of P and Q ends in different partitions

Example

$$P \triangleq \alpha.(\beta.P + \gamma.P) \qquad P \overset{?}{\simeq} Q \qquad Q \triangleq \alpha.\beta.Q + \alpha.\gamma.Q$$

$$\mathbf{R}_2 = \{\ \{P\}\ ,\ \{Q\}\ ,\ \{\beta.P + \gamma.P\}\ ,\ \{\beta.Q\}\ ,\ \{\gamma.Q\}\ \}$$



Example

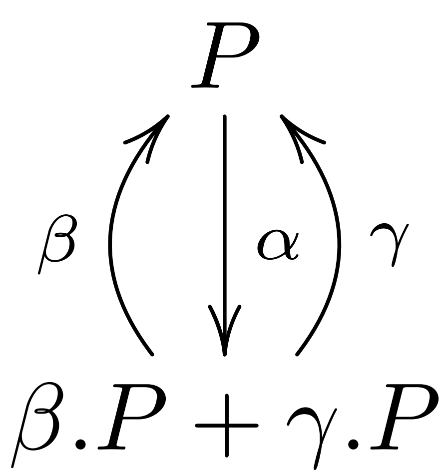
Guarded!

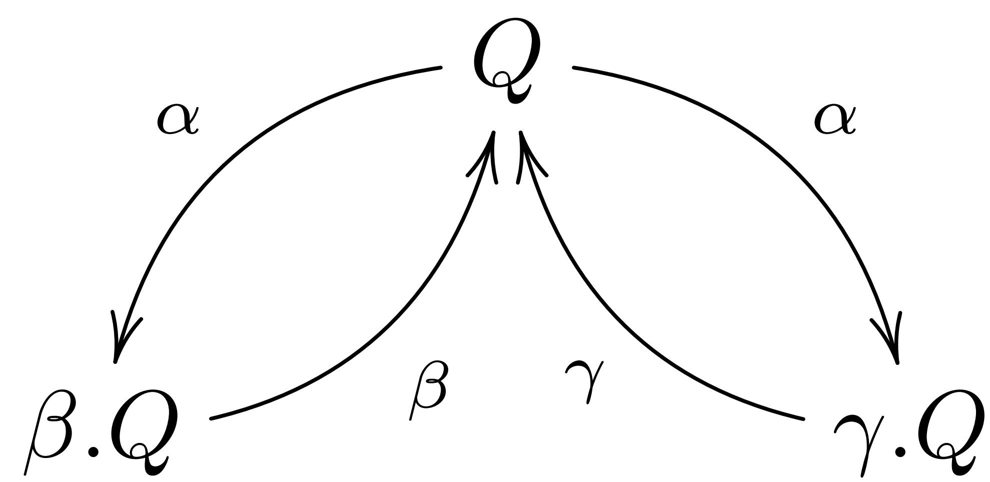
$$P \triangleq \alpha.(\beta.P + \gamma.P)$$

$$P \overset{?}{\simeq} Q$$

Guarded!

$$Q \triangleq \alpha.\beta.Q + \alpha.\gamma.Q$$





$$\mathbf{R}_0 = \{\ \{P\ ,\ Q\ ,\ \beta.P + \gamma.P\ ,\ \beta.Q\ ,\ \gamma.Q\}\ \}$$

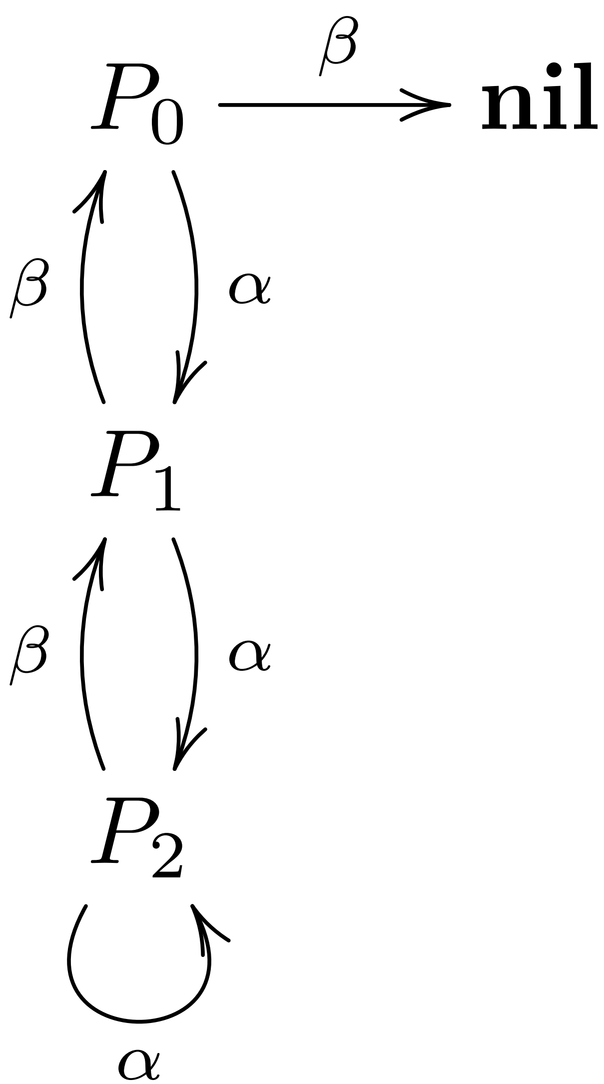
$$\mathbf{R}_1 = \{\ \{P\ ,\ Q\}\ ,\ \{\beta.P + \gamma.P\}\ ,\ \{\beta.Q\}\ ,\ \{\gamma.Q\}\ \}$$

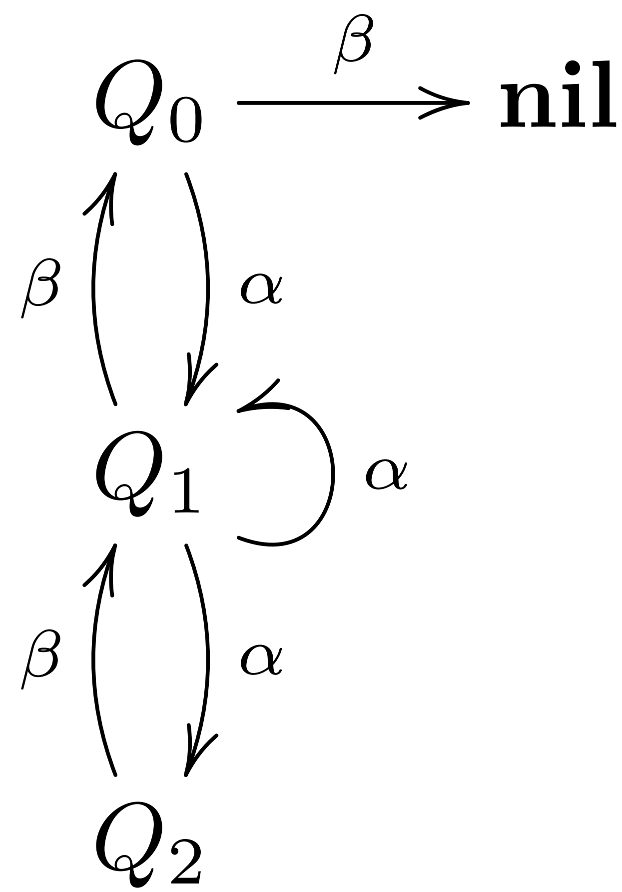
$$\mathbf{R}_2 = \{\ \{P\}\ ,\ \{Q\}\ ,\ \{\beta.P + \gamma.P\}\ ,\ \{\beta.Q\}\ ,\ \{\gamma.Q\}\ \}$$

Only singletons partitions, we can stop $P \not\simeq Q$

Exercise

finitely branching! $\qquad P_0 \overset{?}{\simeq} Q_0 \qquad$ finitely branching!





$\mathbf{nil} \not\rightarrow$

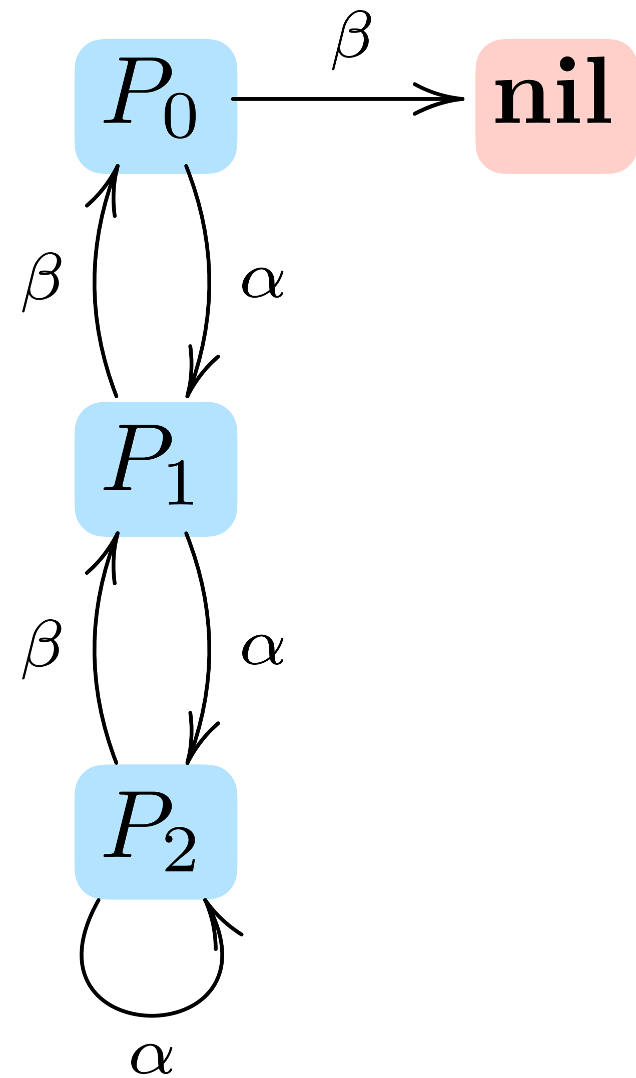
$Q_2 \xrightarrow{\beta}$

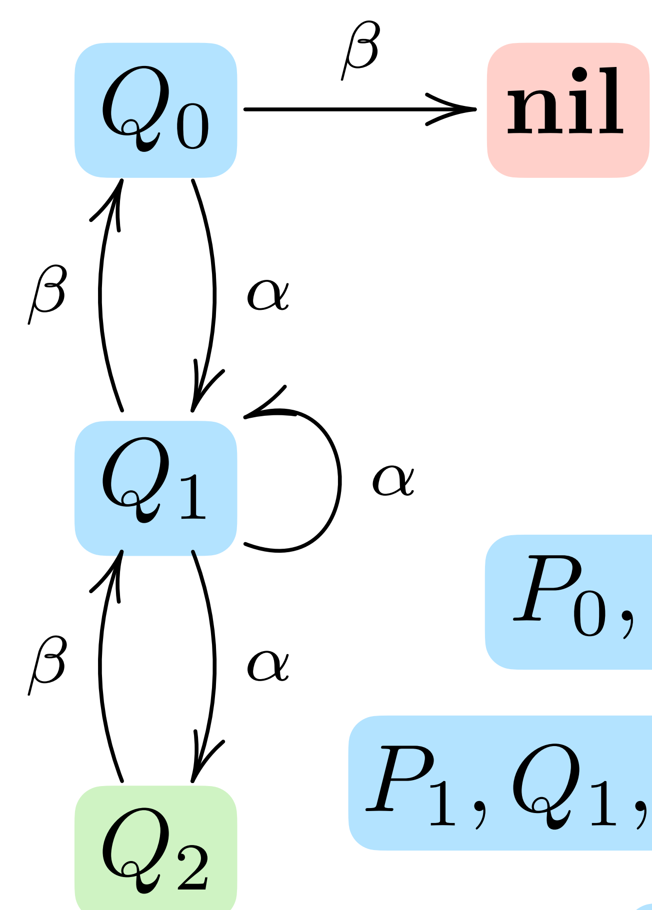
$P_0, Q_0, P_1, Q_1, P_2 \xrightarrow{\alpha,\beta}$

$$\mathbf{R}_0 = \{\ \{P_0, Q_0, P_1, Q_1, P_2, Q_2, \mathbf{nil}\}\ \}$$

Exercise

$$P_0 \stackrel{?}{\simeq} Q_0$$





$$P_0, Q_0 \xrightarrow{\beta} [\mathbf{nil}]$$

$$P_1, Q_1, P_2 \stackrel{\beta}{\not\to} [\mathbf{nil}]$$

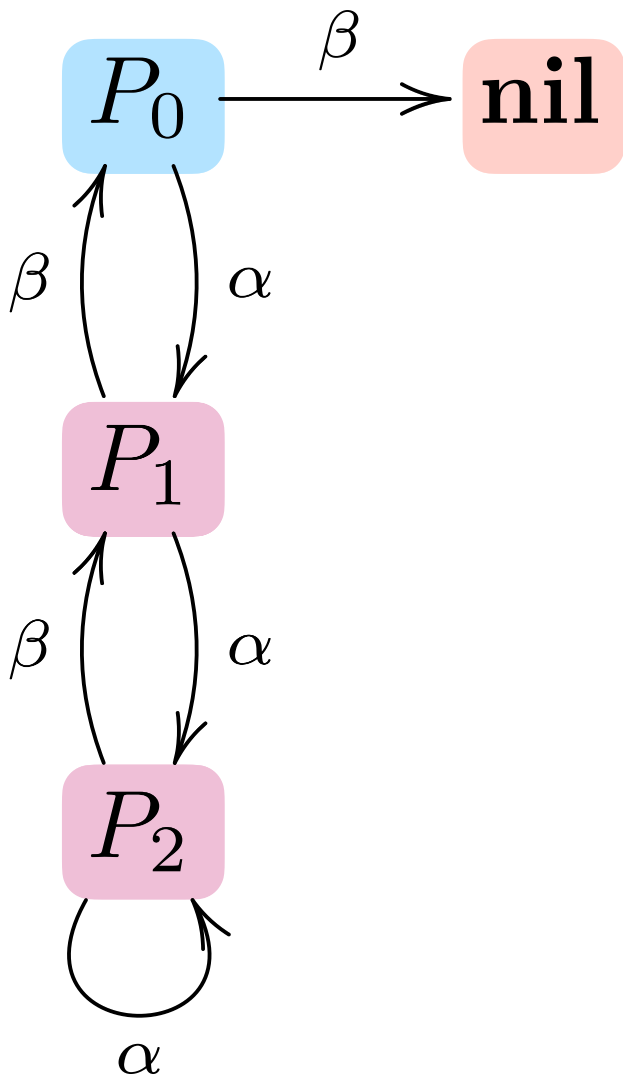
$$Q_1 \xrightarrow{\alpha} [Q_2]$$

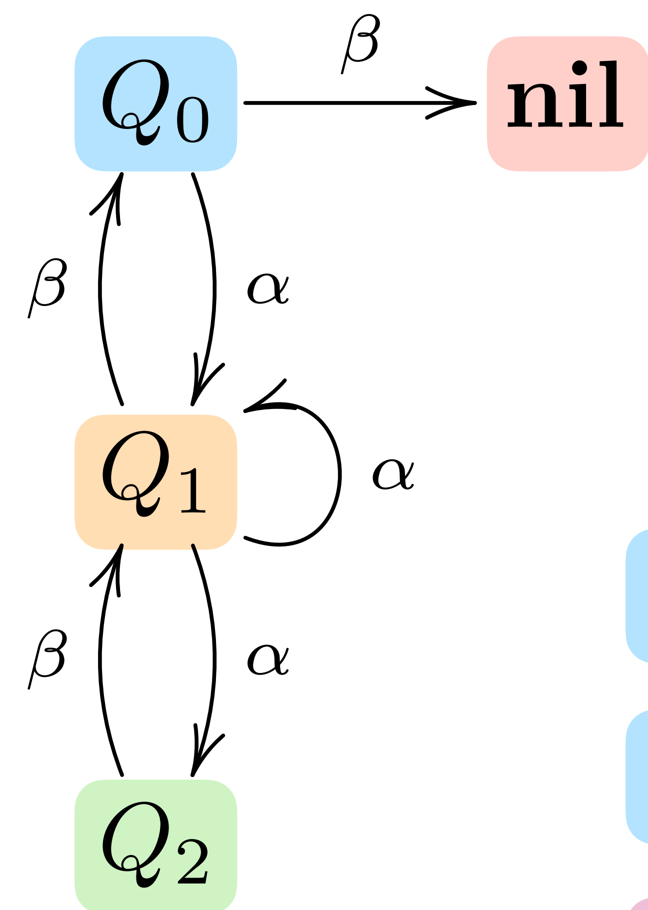
$$P_1, P_2 \stackrel{\alpha}{\not\to} [Q_2]$$

$$\mathbf{R}_1 = \{ \{P_0, Q_0, P_1, Q_1, P_2\}, \{Q_2\}, \{\mathbf{nil}\} \}$$

Exercise

$$P_0 \overset{?}{\simeq} Q_0$$





$P_0 \xrightarrow{\alpha} [P_1]$

$Q_0 \not\xrightarrow{\alpha} [P_1]$

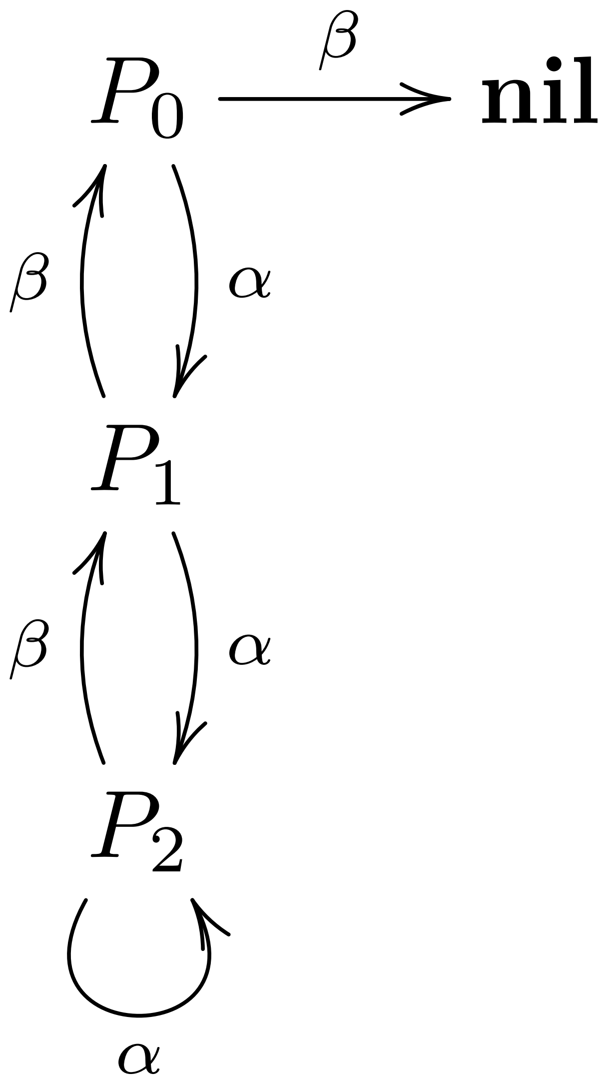
$P_1 \xrightarrow{\beta} [P_0]$

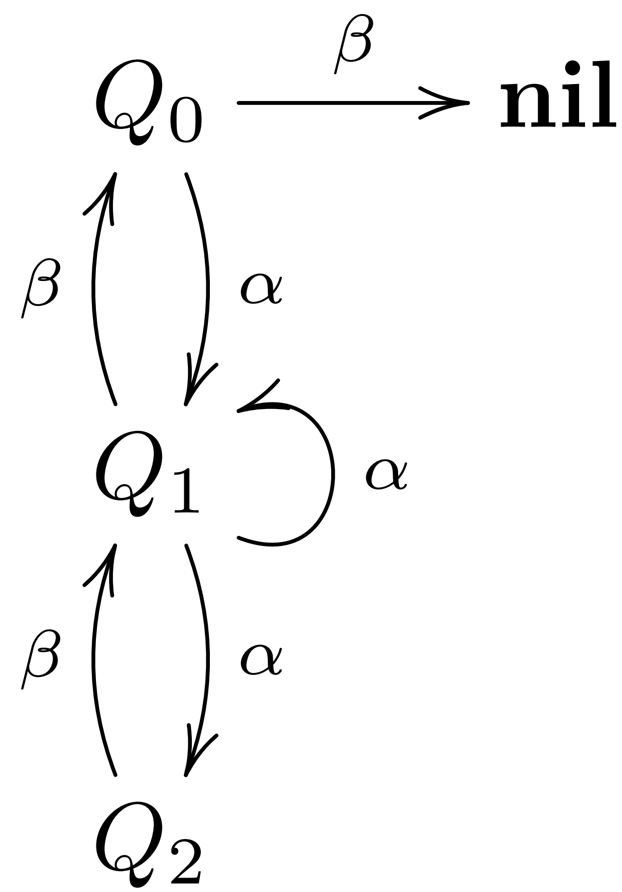
$P_2 \not\xrightarrow{\beta} [P_0]$

$$\mathbf{R}_2 = \{\ \{P_0, Q_0\}\ ,\ \{P_1, P_2\}\ ,\ \{Q_1\}\ ,\ \{Q_2\}\ ,\ \{\mathbf{nil}\}\ \}$$

Exercise

$$P_0 \overset{?}{\simeq} Q_0$$





$$P_0 \not\simeq Q_0$$

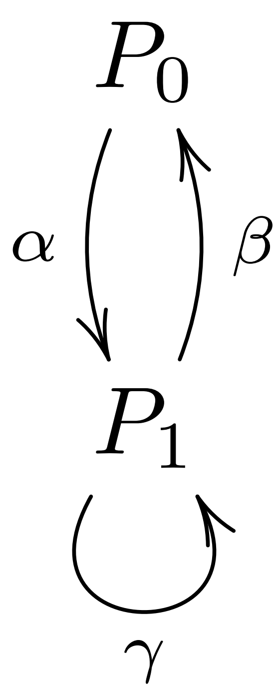
$$\mathbf{R}_3 = \{\ \{P_0\}\ ,\ \{Q_0\}\ ,\ \{P_1\}\ ,\ \{P_2\}\ ,\ \{Q_1\}\ ,\ \{Q_2\}\ ,\ \{\mathbf{nil}\}\ \}$$

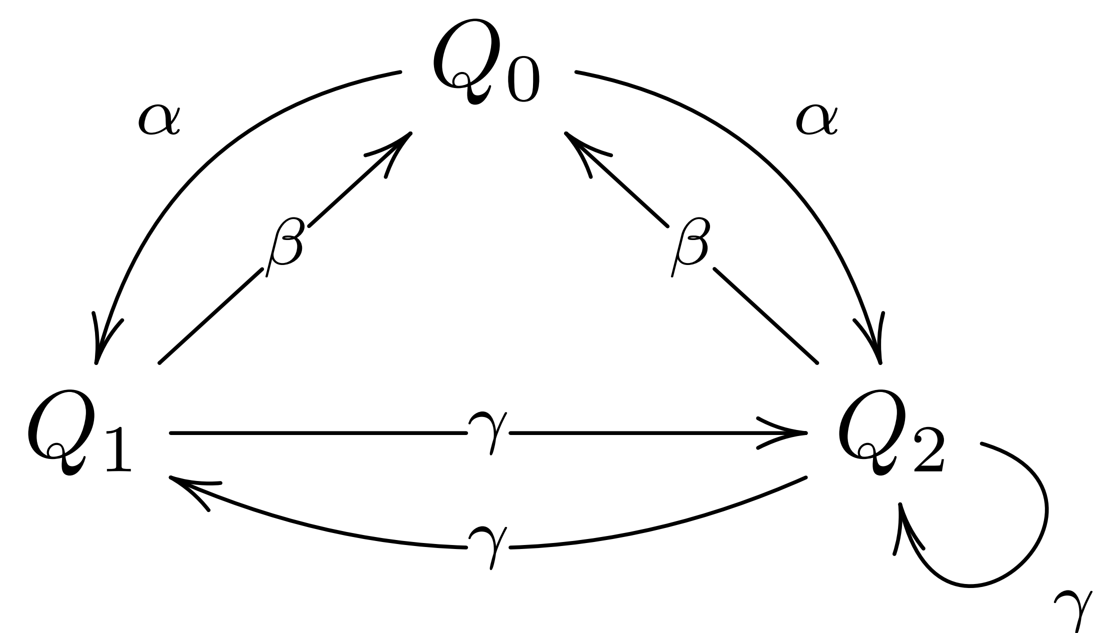
Exercise

finitely branching!

$$P_0 \overset{?}{\simeq} Q_0$$

finitely branching!





$$\mathbf{R}_0 = \{\ \{P_0, Q_0, P_1, Q_1, Q_2\}\ \}$$

$$P_0, Q_0 \xrightarrow{\alpha}$$

$$P_1, Q_1, Q_2 \xrightarrow{\beta,\gamma}$$

$$\mathbf{R}_1 = \{\ \{P_0, Q_0\}\ ,\ \{P_1, Q_1, Q_2\}\ \}$$

No more reasons to discriminate!

Unguarded processes?

What about the general case? (unguarded processes)

any powerset ordered by inclusion defines a complete lattice

Complete lattice: $(D, \sqsubseteq)$ PO such that

any $X \subseteq D$ has a least upper bound $\bigsqcup X$

any $X \subseteq D$ has a greatest lower bound $\bigsqcap X$

it has bottom and top elements $\bot = \bigsqcap D$ $\top = \bigsqcup D$

TH. [Knaster-Tarski] $(D, \sqsubseteq)$ complete lattice

$f : D \to D$ monotone

has least and greatest fixpoint

$$\underbrace{d_{\mathsf{min}} \triangleq \sqcap}_{\text{glb}} \underbrace{\{d \in D \mid f(d) \sqsubseteq d\}}_{\text{pre-fixpoints}}$$ is the least fixpoint

$$\underbrace{d_{\mathsf{max}} \triangleq \sqcup}_{\text{lub}} \underbrace{\{d \in D \mid d \sqsubseteq f(d)\}}_{\text{post-fixpoints}}$$ is the greatest fixpoint

least and greatest fixpoint exist… but how to compute them?

CCS
Compositionality

Compositionality

recall that an equivalence $\equiv$ is a congruence when

$$\forall \mathbb{C}[\cdot].\ \forall p, q.\ \boxed{p \equiv q \ \Rightarrow\ \mathbb{C}[p] \equiv \mathbb{C}[q]}$$

we can replace equivalent processes in any context without changing the abstract semantics

TH. Strong bisimilarity is a congruence

1. $\forall p, q.\ p \simeq q \;\Rightarrow\; \forall \mu.\ \mu.p \simeq \mu.q$

2. $\forall p, q.\ p \simeq q \;\Rightarrow\; \forall \alpha.\ p \backslash \alpha \simeq q \backslash \alpha$

3. $\forall p, q.\ p \simeq q \;\Rightarrow\; \forall \phi.\ p[\phi] \simeq q[\phi]$

4. $\forall p_0, q_0, p_1, q_1.\ p_0 \simeq q_0 \wedge p_1 \simeq q_1 \;\Rightarrow\; p_0 + p_1 \simeq q_0 + q_1$

5. $\forall p_0, q_0, p_1, q_1.\ p_0 \simeq q_0 \wedge p_1 \simeq q_1 \;\Rightarrow\; p_0 | p_1 \simeq q_0 | q_1$

let us omit quantification to make the statement more readable

TH. Strong bisimilarity is a congruence

1. $p \simeq q \ \Rightarrow\ \mu.p \simeq \mu.q$
2. $p \simeq q \ \Rightarrow\ p\backslash\alpha \simeq q\backslash\alpha$
3. $p \simeq q \ \Rightarrow\ p[\phi] \simeq q[\phi]$
4. $p_0 \simeq q_0 \wedge p_1 \simeq q_1 \ \Rightarrow\ p_0 + p_1 \simeq q_0 + q_1$
5. $p_0 \simeq q_0 \wedge p_1 \simeq q_1 \ \Rightarrow\ p_0|p_1 \simeq q_0|q_1$

proof technique:
“guess” a relation large enough to contain all pairs of interest;
show that it is a bisimulation relation;
then it is contained in the strong bisimilarity relation

TH. Strong bisimilarity is a congruence (3)

take $\mathbf{R} \triangleq \{(p[\phi], q[\phi]) \mid p \simeq q\}$

we show that $\mathbf{R}$ is a strong bisimulation relation

take $(p[\phi], q[\phi]) \in \mathbf{R}$ (i.e. with $p \simeq q$)

take $p[\phi] \xrightarrow{\mu} p'$ we want to find $q[\phi] \xrightarrow{\mu} q'$ with $(p', q') \in \mathbf{R}$

by rule rel) it must be $p \xrightarrow{\mu'} p''$ $\mu = \phi(\mu')$ $p' = p''[\phi]$

since $p \simeq q$ then $q \xrightarrow{\mu'} q''$ with $p'' \simeq q''$

by rule rel) $q[\phi] \xrightarrow{\phi(\mu')} q''[\phi]$

take $q' = q''[\phi]$ so that $(p', q') = (p''[\phi], q''[\phi]) \in \mathbf{R}$

take $q[\phi] \xrightarrow{\mu} q'$ we want to find $p[\phi] \xrightarrow{\mu} p'$ with $(p', q') \in \mathbf{R}$

analogous to the previous case

TH. Strong bisimilarity is a congruence (4)

take $\mathbf{R} \triangleq \{(p_0 + p_1, q_0 + q_1) \mid p_0 \simeq q_0 \wedge p_1 \simeq q_1\}$

we show that $\mathbf{R}$ is a strong bisimulation relation

take $(p_0 + p_1, q_0 + q_1) \in \mathbf{R}$ (i.e. with $p_0 \simeq q_0$ $p_1 \simeq q_1$)

take $p_0 + p_1 \xrightarrow{\mu} p'$ we need $q_0 + q_1 \xrightarrow{\mu} q'$ with $(p', q') \in \mathbf{R}$

if rule suml) was used: $p_0 \xrightarrow{\mu} p'$

since $p_0 \simeq q_0$ then $q_0 \xrightarrow{\mu} q'$ with $p' \simeq q'$

by rule suml) $q_0 + q_1 \xrightarrow{\mu} q'$

but unfortunately $(p', q') \in \simeq$ not necessarily $(p', q') \in \mathbf{R}$

how can we repair the proof?

TH. Strong bisimilarity is a congruence (4)

take $\mathbf{R} \triangleq \{(p_0 + p_1, q_0 + q_1) \mid p_0 \simeq q_0 \wedge p_1 \simeq q_1\} \boxed{\cup \simeq}$

we show that $\mathbf{R}$ is a strong bisimulation relation

take $(p_0 + p_1, q_0 + q_1) \in \mathbf{R}$ (i.e. with $p_0 \simeq q_0$ $p_1 \simeq q_1$)

take $p_0 + p_1 \xrightarrow{\mu} p'$ we need $q_0 + q_1 \xrightarrow{\mu} q'$ with $(p', q') \in \mathbf{R}$

if rule suml) was used: $p_0 \xrightarrow{\mu} p'$

since $p_0 \simeq q_0$ then $q_0 \xrightarrow{\mu} q'$ with $p' \simeq q'$

by rule suml) $q_0 + q_1 \xrightarrow{\mu} q'$

then $(p', q') \in \simeq \subseteq \mathbf{R}$

how can we repair the proof?

(no need to check the pairs in $\simeq$)

fill in the missing details

- sumr)
- $q_0 + q_1$ moves

CCS: some laws

$$p + \mathbf{nil} \simeq p$$
$$p + q \simeq q + p$$
$$p + (q + r) \simeq (p + q) + r$$
$$p + p \simeq p$$

$$p|\mathbf{nil} \simeq p$$
$$p|q \simeq q|p$$
$$p|(q|r) \simeq (p|q)|r$$

how to prove them? find a strong bisimulation for each of them

$$\mathbf{nil}\backslash\alpha \simeq \mathbf{nil}$$
$$(\mu.p)\backslash\alpha \simeq \mathbf{nil} \quad \text{if } \mu \in \{\alpha, \overline{\alpha}\}$$
$$(\mu.p)\backslash\alpha \simeq \mu.(p\backslash\alpha) \quad \text{if } \mu \notin \{\alpha, \overline{\alpha}\}$$
$$(p + q)\backslash\alpha \simeq (p\backslash\alpha) + (q\backslash\alpha)$$
$$p\backslash\alpha\backslash\alpha \simeq p\backslash\alpha$$
$$p\backslash\alpha\backslash\beta \simeq p\backslash\beta\backslash\alpha$$

$$\mathbf{nil}[\phi] \simeq \mathbf{nil}$$
$$(\mu.p)[\phi] \simeq \phi(\mu).(p[\phi])$$
$$(p + q)[\phi] \simeq (p[\phi]) + (q[\phi])$$
$$p[\phi][\eta] \simeq p[\eta \circ \phi]$$

Bisimulation revisited

Strong bisimilarity

$\mathcal{P}$ set of processes $\qquad \mathbf{R} \subseteq \mathcal{P} \times \mathcal{P}$ a binary relation

$\mathbf{R}$ is a strong bisimulation if

$$\forall p, q.\ (p, q) \in \mathbf{R} \boxed{\Rightarrow} \begin{cases} \forall \mu, p'.\ p \xrightarrow{\mu} p' \quad \Rightarrow \quad \exists q'.\ q \xrightarrow{\mu} q' \wedge p' \ \mathbf{R}\ q' \\ \wedge \qquad \text{Alice plays} \qquad\qquad \text{Bob replies} \\ \forall \mu, q'.\ q \xrightarrow{\mu} q' \quad \Rightarrow \quad \exists p'.\ p \xrightarrow{\mu} p' \wedge p' \ \mathbf{R}\ q' \end{cases}$$

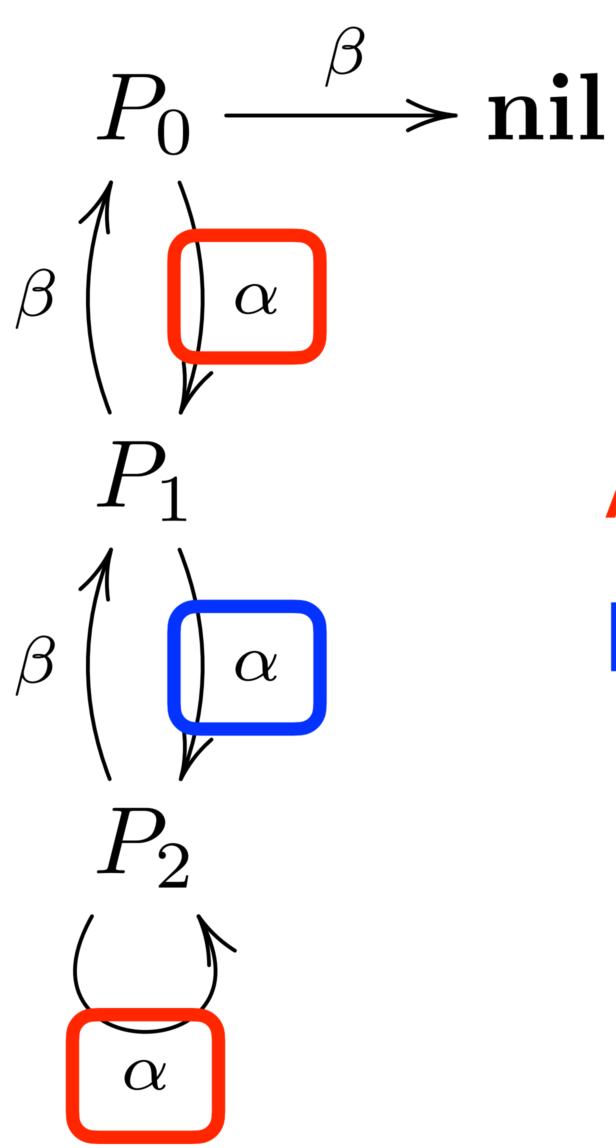
strong bisimilarity $\quad \simeq \ \triangleq \ \bigcup_{\mathbf{R}\ \text{s.b.}} \mathbf{R}$

is an equivalence

is a strong bisimulation

$$\forall p, q.\ p \simeq q \boxed{\Leftrightarrow} \begin{cases} \forall \mu, p'.\ p \xrightarrow{\mu} p' \quad \Rightarrow \quad \exists q'.\ q \xrightarrow{\mu} q' \wedge p' \simeq q' \\ \wedge \\ \forall \mu, q'.\ q \xrightarrow{\mu} q' \quad \Rightarrow \quad \exists p'.\ p \xrightarrow{\mu} p' \wedge p' \simeq q' \end{cases}$$

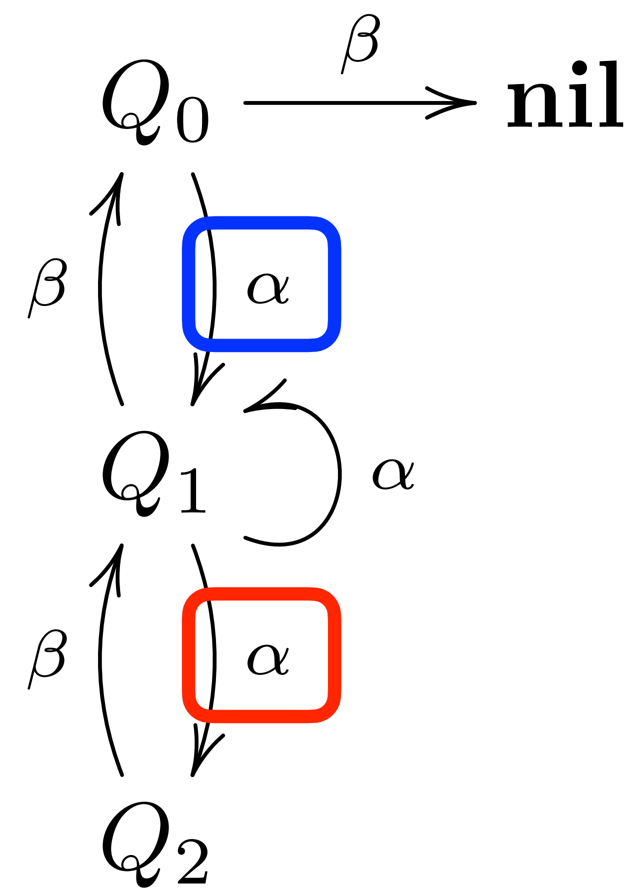
Bisimulation game



$P_0 \overset{?}{\simeq} Q_0$

Alice plays

Bob replies



Bisimulation revisited

$(S, L, \rightarrow)$

labelled transitions

labels

states

alternative presentation of transitions

$\alpha : S \rightarrow S \rightarrow \wp(L)$ $\qquad$ $\alpha\ p\ q = \{\mu \mid p \xrightarrow{\mu} q\}$

generalization to sets of targets

$\gamma : S \times \wp(S) \rightarrow \wp(L)$ $\qquad$ $\gamma(p, I) = \{\mu \mid \exists q \in I.\ p \xrightarrow{\mu} q\}$

$$\gamma(p, I) = \bigcup_{q \in I} \alpha\ p\ q$$

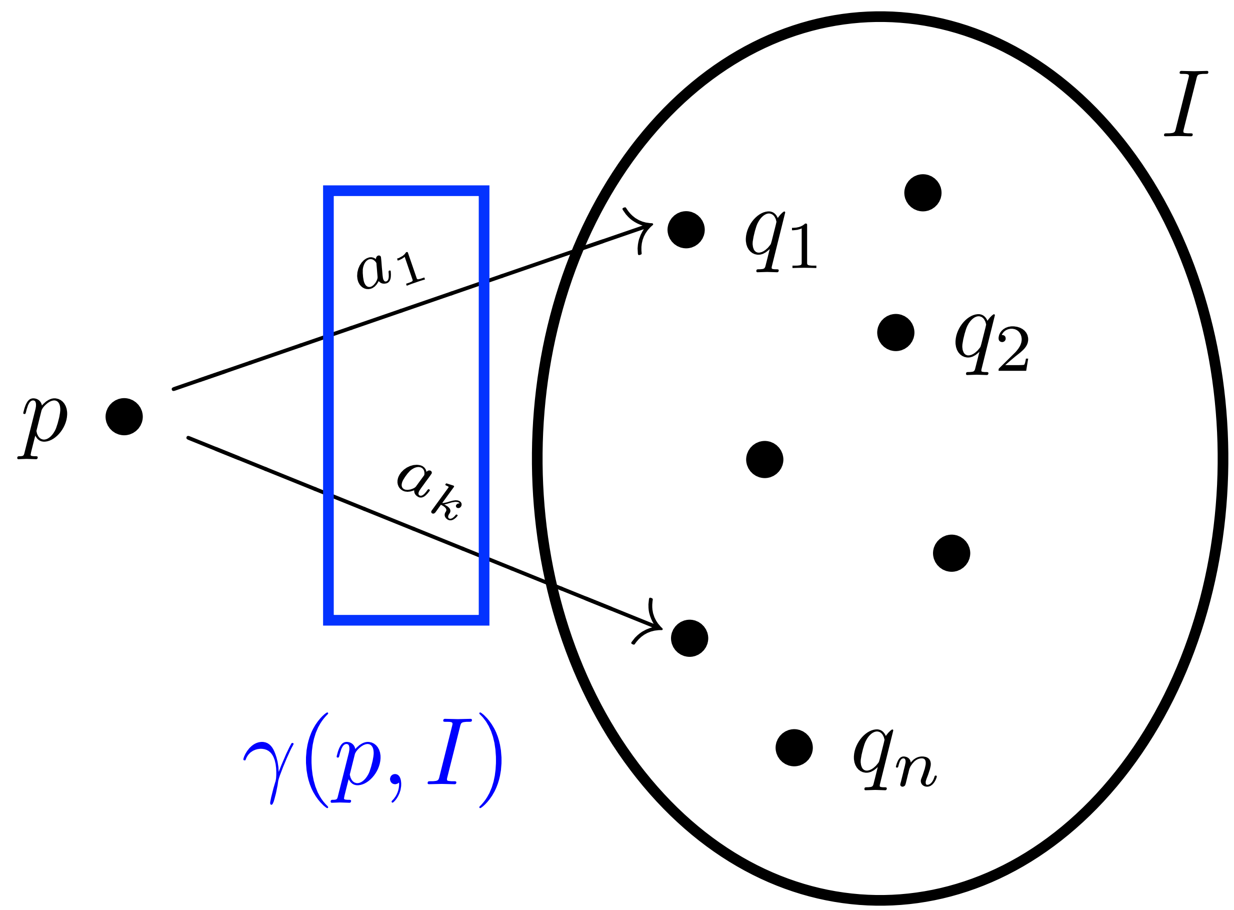
set of targets

source

Bisimulation revisited

$$\alpha : S \to S \to \wp(L) \qquad \alpha\ p\ q = \{\mu \mid p \xrightarrow{\mu} q\}$$

$$\gamma : S \times \wp(S) \to \wp(L) \qquad \gamma(p, I) = \bigcup_{q \in I} \alpha\ p\ q$$



$$p \xrightarrow{\gamma(p,I)} I$$

Bisimulation revisited

$\alpha\ p\ q = \{\mu \mid p \xrightarrow{\mu} q\}$ $\qquad \gamma(p, I) = \bigcup_{q \in I} \alpha\ p\ q$

take the equivalence $\equiv_{\mathbf{R}}$ induced by a relation $\mathbf{R}$
(if $\mathbf{R}$ is a bisimulation, then $\equiv_{\mathbf{R}}$ is a bisimulation)

take the set of equivalence classes induced by $\equiv_{\mathbf{R}}$: $S_{|\equiv_{\mathbf{R}}}$

take $J \in S_{|\equiv_{\mathbf{R}}}$ and $p, q \in J$ (i.e., $p \equiv_{\mathbf{R}} q$)

if $p \xrightarrow{\mu} p'$ for some μ, p' then $q \xrightarrow{\mu} q'$ for some q' with $p' \equiv_{\mathbf{R}} q'$
(and vice versa) (i.e. $\exists I \in S_{|\equiv_{\mathbf{R}}}.\ p', q' \in I$)

now consider the function $\Phi : \wp(S \times S) \rightarrow \wp(S \times S)$

$$p\ \Phi(\mathbf{R})\ q \triangleq \forall I \in S_{|\equiv_{\mathbf{R}}}.\ \gamma(p, I) = \gamma(q, I)$$

by the above argument, a bisimulation is such if $\mathbf{R} \subseteq \Phi(\mathbf{R})$

$\simeq\ \triangleq \bigcup_{\mathbf{R} \subseteq \Phi(\mathbf{R})} \mathbf{R}$ is the largest bisimulation

Bisimulation revisited

$$\alpha : S \to S \to \wp(L)$$
$$\alpha\ p\ q = \{\mu \mid p \xrightarrow{\mu} q\}$$

$$\gamma : S \times \wp(S) \to \wp(L)$$
$$\gamma(p, I) = \bigcup_{q \in I} \alpha\ p\ q$$

$$\Phi : \wp(S \times S) \to \wp(S \times S)$$
$$p\ \Phi(\mathbf{R})\ q \triangleq \forall I \in S_{|\equiv_{\mathbf{R}}}.\ \gamma(p, I) = \gamma(q, I)$$

bisimulation $\quad \mathbf{R} \subseteq \Phi(\mathbf{R})$

bisimilarity $\quad \simeq \triangleq \bigcup_{\mathbf{R} \subseteq \Phi(\mathbf{R})} \mathbf{R}$

CCS

Infinite LTS

Write a guarded CCS process whose LTS has infinitely many states without using parallel composition.

Infinite LTS

using par $\qquad P \triangleq \mathbf{rec}\ x.\ \alpha.(x|\beta.\mathbf{nil}) \qquad P \triangleq \alpha.(P|\beta)$

$$\begin{tikzcd}
P \arrow[d, "\alpha"] \\
P|\beta \arrow[d, "\alpha"] \arrow[r, "\beta"] & \cdots \\
(P|\beta)|\beta \arrow[d, "\alpha"] \arrow[r, "\beta"] & \cdots \\
((P|\beta)|\beta)|\beta \arrow[d, "\alpha"] \arrow[r, "\beta"] & \cdots \\
\cdots
\end{tikzcd}$$

Infinite LTS

$P \triangleq \mathbf{rec}\ x.\ (\alpha.x)\backslash\beta$ $\qquad$ $P \triangleq (\alpha.P)\backslash\beta$

$$
\begin{array}{c}
P \\
\big\downarrow \alpha \\
P\backslash\beta \\
\big\downarrow \alpha \\
P\backslash\beta\backslash\beta \\
\big\downarrow \alpha \\
P\backslash\beta\backslash\beta\backslash\beta \\
\big\downarrow \alpha \\
\cdots
\end{array}
$$

all states are bisimilar

Infinite LTS

$$P \triangleq \mathbf{rec}\ x.\ \alpha_1.(x[\phi]) \qquad \phi(\alpha_i) = \alpha_{i+1}$$

$$\begin{array}{c} P \\ \big\downarrow{\scriptstyle \alpha_1} \\ P[\phi] \\ \big\downarrow{\scriptstyle \alpha_2} \\ P[\phi][\phi] \\ \big\downarrow{\scriptstyle \alpha_3} \\ P[\phi][\phi][\phi] \\ \big\downarrow{\scriptstyle \alpha_4} \\ \cdots \end{array}$$

Bisimulation

n positions buffers

Define a CCS process B^n_k that represents an in/out buffer with capacity n of which k positions are taken. Show that B^n_0 is strongly bisimilar to n copies of B^1_0 that run in parallel.

n positions buffers

$B_0^n \triangleq in.B_1^n$

$B_k^n \triangleq in.B_{k+1}^n + \overline{out}.B_{k-1}^n$ with $0 < k < n$

$B_n^n \triangleq \overline{out}.B_{n-1}^n$

we want to prove $\quad B_0^n \simeq \underbrace{B_0^1|\cdots|B_0^1}_{n}$

$$\mathbf{R} \triangleq \{(B_k^n, B_{k_1}^1|\cdots|B_{k_n}^1) \mid \forall i.\ k_i \in \{0,1\} \wedge \sum_{i=1}^{n} k_i = k\}$$

we prove that **R** is a strong bisimulation

n positions buffers

$$
\begin{array}{ccc}
B_0^n & \mathbf{R} & B_0^1|\cdots|B_0^1 \\
\downarrow{\scriptstyle in} & & \downarrow{\scriptstyle in} \\
B_1^n & \mathbf{R} & B_1^1|\cdots|B_0^1
\end{array}
$$

$$
\begin{array}{ccc}
B_0^n & \mathbf{R} & B_0^1|\cdots|B_0^1 \\
\downarrow{\scriptstyle in} & & \downarrow{\scriptstyle in} \\
B_1^n & \mathbf{R} & B_0^1|\cdots|B_1^1|\cdots|B_0^1
\end{array}
$$

n positions buffers

$$0 < k < n \qquad \begin{array}{ccc} B^n_k & \mathbf{R} & B^1_{k_1}|\cdots|B^1_{k_n} \\ \downarrow{\scriptstyle in} & & \downarrow{\scriptstyle in} \\ B^n_{k+1} & \mathbf{R} & B^1_{k_1}|\cdots|B^n_{k_i+1}|\cdots|B^1_{k_n} \end{array} \qquad \sum_{i=1}^{n} k_i = k \quad \exists k_i = 0$$

$$\begin{array}{ccc} B^n_k & \mathbf{R} & B^1_{k_1}|\cdots|B^1_{k_n} \\ \downarrow{\scriptstyle in} & & \downarrow{\scriptstyle in} \\ B^n_{k+1} & \mathbf{R} & B^1_{k_1}|\cdots|B^n_{k_i+1}|\cdots|B^1_{k_n} \end{array} \qquad \exists k_i = 0$$

n positions buffers

$$0 < k < n \qquad B_k^n \quad \mathbf{R} \quad B_{k_1}^1 | \cdots | B_{k_n}^1 \qquad \sum_{i=1}^{n} k_i = k$$

$$\begin{array}{ccc} B_k^n & & B_{k_1}^1 | \cdots | B_{k_n}^1 \\ \Big\downarrow \overline{out} & & \Big\downarrow \overline{out} \end{array} \qquad \exists k_i = 1$$

$$B_{k-1}^n \;\; \mathbf{R} \;\; B_{k_1}^1 | \cdots | B_{k_i-1}^n | \cdots | B_{k_n}^1$$

$$B_k^n \quad \mathbf{R} \quad B_{k_1}^1 | \cdots | B_{k_n}^1 \qquad \exists k_i = 1$$

$$\begin{array}{ccc} \Big\downarrow \overline{out} & & \Big\downarrow \overline{out} \end{array}$$

$$B_{k-1}^n \;\; \mathbf{R} \;\; B_{k_1}^1 | \cdots | B_{k_i-1}^n | \cdots | B_{k_n}^1$$

n positions buffers

$$\begin{tikzcd} B_n^n \arrow[d, "\overline{out}"] & \mathbf{R} & B_1^1 | \cdots | B_1^1 \arrow[d, "\overline{out}"] \\ B_{n-1}^n & \mathbf{R} & B_0^1 | \cdots | B_1^1 \end{tikzcd}$$

$$\begin{tikzcd} B_n^n \arrow[d, "\overline{out}"] & \mathbf{R} & B_1^1 | \cdots | B_1^1 \arrow[d, "\overline{out}"] \\ B_{n-1}^n & \mathbf{R} & B_1^1 | \cdots | B_0^1 | \cdots | B_1^1 \end{tikzcd}$$

Congruence for \a

Prove that CCS strong bisimilarity is a congruence w.r.t. restriction, i.e., that for all p, q, α:

$$p \simeq q \;\Rightarrow\; p\backslash\alpha \simeq\; q\backslash\alpha$$

Congruence for \a

$\mathbf{R} \triangleq \{(p\backslash\alpha, q\backslash\alpha) \mid p \simeq q\}$ we prove **R** is a strong bisimulation

take $(p\backslash\alpha, q\backslash\alpha) \in \mathbf{R}$ (with $p \simeq q$)

take $p\backslash\alpha \xrightarrow{\mu} p'$ we want to find $q\backslash\alpha \xrightarrow{\mu} q'$ with $p' \,\mathbf{R}\, q'$

by rule res) it must be $p \xrightarrow{\mu} p''$ with $\mu \not\in \{\alpha, \overline{\alpha}\}$ and $p' = p''\backslash\alpha$

since $p \simeq q$ and $p \xrightarrow{\mu} p''$ we have $q \xrightarrow{\mu} q''$ with $p'' \simeq q''$

take $q' = q''\backslash\alpha$: by rule res) we have $q\backslash\alpha \xrightarrow{\mu} q'$ and $(p', q') \in \mathbf{R}$

take $q\backslash\alpha \xrightarrow{\mu} q'$ we want to find $p\backslash\alpha \xrightarrow{\mu} p'$ with $p' \,\mathbf{R}\, q'$

analogous to the previous case

Bisimilar processes

Let us consider the guarded CCS processes

$$p \stackrel{\text{def}}{=} \mathbf{rec}\ x.(\alpha.x + \beta.x) \quad q \stackrel{\text{def}}{=} \mathbf{rec}\ y.(\overline{\alpha}.\mathbf{nil} + \gamma.y) \quad r \stackrel{\text{def}}{=} \mathbf{rec}\ z.(\overline{\beta}.\mathbf{nil} + \overline{\gamma}.z)$$

1. Draw the LTSs of the processes p, q, r and $s \stackrel{\text{def}}{=} (p|q|r)\backslash\alpha\backslash\beta\backslash\gamma$.

2. Show that s is strong bisimilar to the process $t \stackrel{\text{def}}{=} \mathbf{rec}\ w.(\tau.w + \tau.\tau.\mathbf{nil})$.

Bisimilar processes

$$p \triangleq \mathbf{rec}\ x.(\alpha.x + \beta.x)$$

$$\begin{tikzcd} p \arrow[loop left, "\alpha"] \arrow[loop right, "\beta"] \end{tikzcd}$$

$$q \triangleq \mathbf{rec}\ y.(\overline{\alpha} + \gamma.y)$$

$$\begin{tikzcd} q \arrow[loop left, "\gamma"] \arrow[r, "\overline{\alpha}"] & \mathbf{nil} \end{tikzcd}$$

$$r \triangleq \mathbf{rec}\ z.(\overline{\beta} + \overline{\gamma}.z)$$

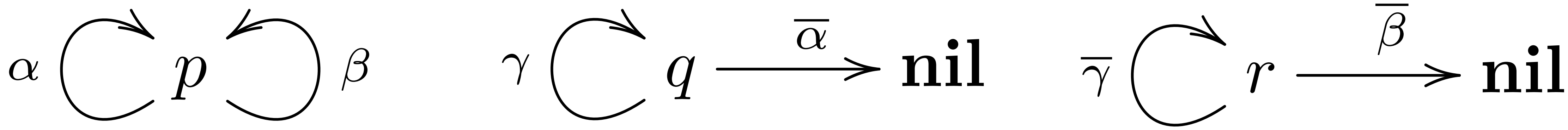
$$\begin{tikzcd} r \arrow[loop left, "\overline{\gamma}"] \arrow[r, "\overline{\beta}"] & \mathbf{nil} \end{tikzcd}$$

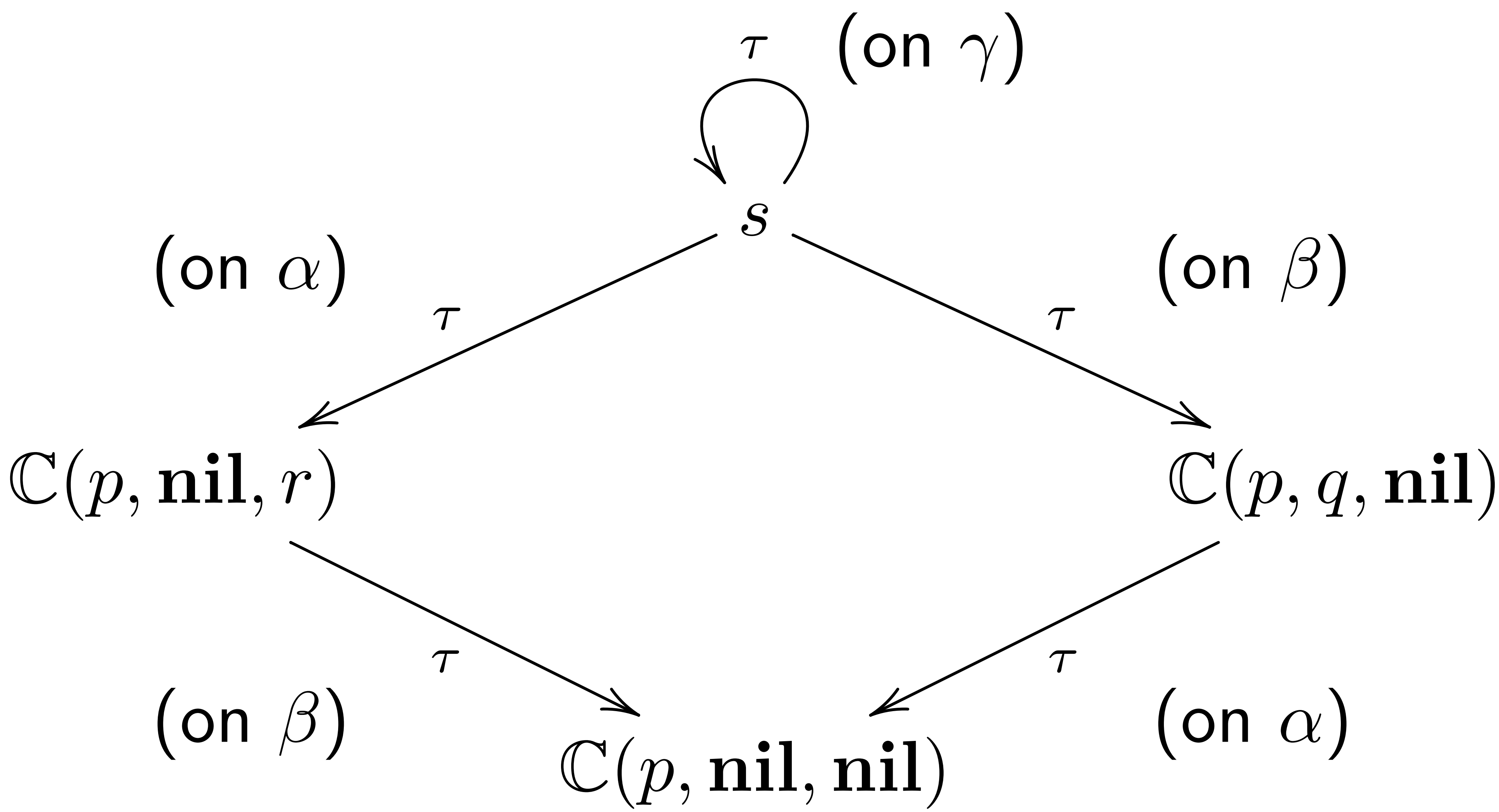
$$s \triangleq (p|q|r)\backslash\alpha\backslash\beta\backslash\gamma$$

$$\mathbb{C}(p,q,r) \triangleq (p|q|r)\backslash\alpha\backslash\beta\backslash\gamma$$

$$s \triangleq \mathbb{C}(p,q,r)$$

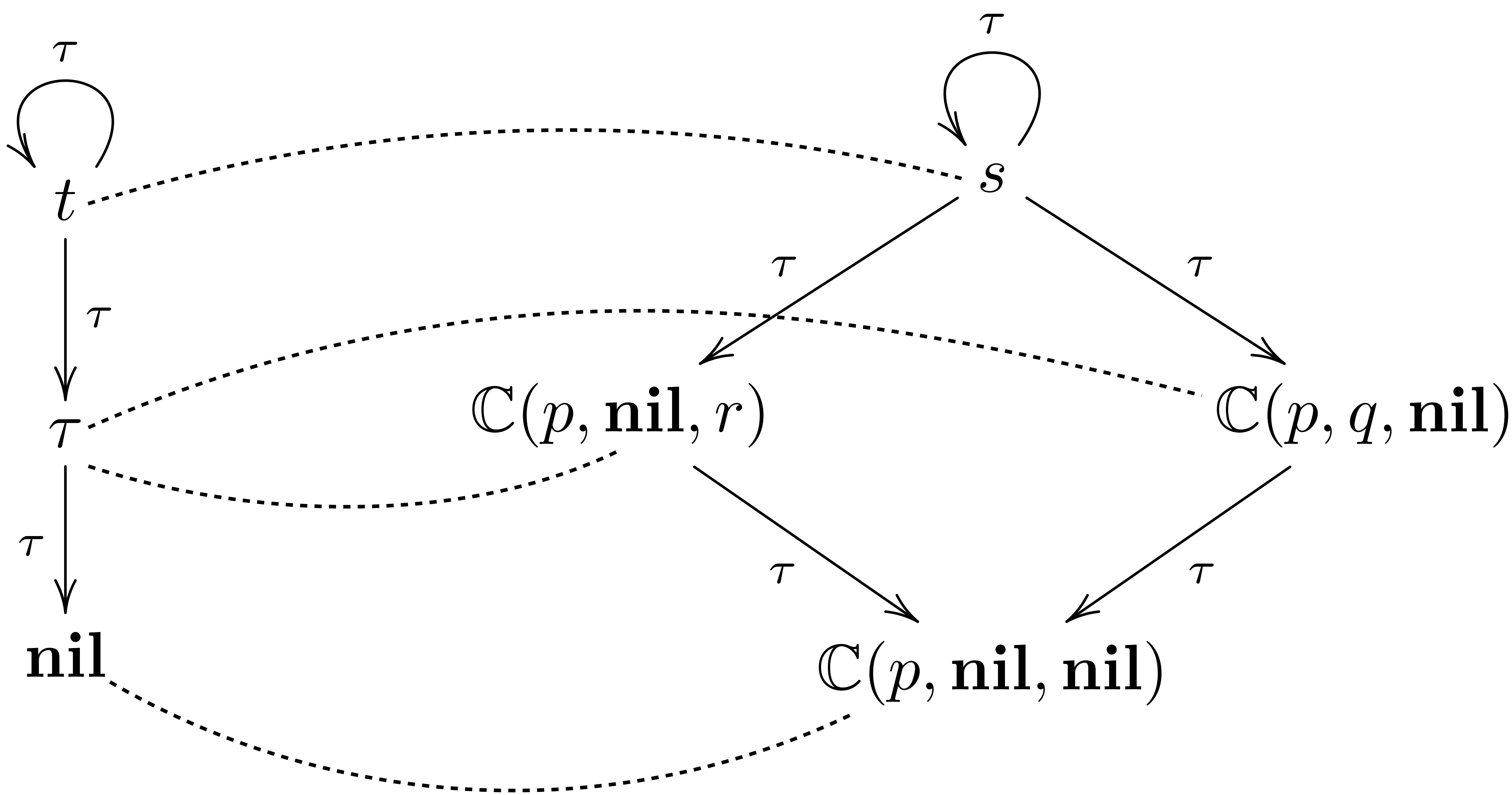
Bisimilar processes





Bisimilar processes

$t \triangleq \mathbf{rec}\ w.(\tau.w + \tau.\tau)$



$$\mathbf{R} \triangleq \{\ \{s, t\}\ ,\ \{\tau, \mathbb{C}(p, \mathbf{nil}, r), \mathbb{C}(p, q, \mathbf{nil})\}\ ,\ \{\mathbf{nil}, \mathbb{C}(p, \mathbf{nil}, \mathbf{nil})\}\ \}$$

R is a strong bisimulation