MPP 2025/26 (0077A, 9CFU)

Models for Programming Paradigms

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08a - Complete Partial Orders

Consistency of expressions

Consistency?

$$\forall a, \sigma, n$$

$$\langle a, \sigma \rangle \longrightarrow n \qquad \overset{?}{\Leftrightarrow} \qquad \mathcal{A}[\![a]\!]\sigma = n$$

$$P(a) \triangleq \forall \sigma.\ \langle a, \sigma \rangle \longrightarrow \mathcal{A}[\![a]\!]\sigma$$

$$\forall a \in \textsf{Aexp}.\ P(a)\,?$$

structural induction!

$$\frac{\forall x \in \textsf{Ide}.\ P(x) \qquad \forall n \in \mathbb{Z}.\ P(n) \qquad \forall a_0, a_1.\ P(a_0) \wedge P(a_1) \Rightarrow P(a_0\ \textsf{op}\ a_1)}{\forall a.\ P(a)}$$

Base cases

$\forall x \in \mathsf{Ide}.\ P(x)$ take $x \in \mathsf{Ide}$

we need to prove $P(x) \triangleq \forall \sigma.\ \langle x, \sigma \rangle \longrightarrow \mathcal{A}[\![x]\!]\sigma = \sigma(x)$

taken a generic σ we conclude by rule $\dfrac{}{\langle x, \sigma \rangle \longrightarrow \sigma(x)}$

$\forall n \in \mathbb{Z}.\ P(n)$ take $n \in \mathbb{Z}$

we need to prove $P(n) \triangleq \forall \sigma.\ \langle n, \sigma \rangle \longrightarrow \mathcal{A}[\![n]\!]\sigma = n$

taken a generic σ we conclude by rule $\dfrac{}{\langle n, \sigma \rangle \longrightarrow n}$

Inductive case

$\forall a_0, a_1.\ P(a_0) \wedge P(a_1) \Rightarrow P(a_0 \ \mathsf{op}\ a_1)$ Take generic a_0, a_1

we assume $P(a_i) \triangleq \forall \sigma.\ \langle a_i, \sigma \rangle \longrightarrow \mathcal{A}[\![a_i]\!]\sigma$

we need to prove $P(a_0\, \mathsf{op}\, a_1) \triangleq \forall \sigma.\ \langle a_0\, \mathsf{op}\, a_1, \sigma \rangle \longrightarrow \mathcal{A}[\![a_0\, \mathsf{op}\, a_1]\!]\sigma$

$$= \mathcal{A}[\![a_0]\!]\sigma\ op\ \mathcal{A}[\![a_1]\!]\sigma$$

take a generic σ

$\langle a_0\, \mathsf{op}\, a_1, \sigma \rangle \longrightarrow n$

$\nwarrow_{n = n_0\ op\ n_1}\ \langle a_0, \sigma \rangle \longrightarrow n_0, \langle a_1, \sigma \rangle \longrightarrow n_1$

by inductive hypotheses, $n_i = \mathcal{A}[\![a_i]\!]\sigma$

and thus $n = n_0\ op\ n_1 = \mathcal{A}[\![a_0]\!]\sigma\ op\ \mathcal{A}[\![a_1]\!]\sigma$

Denotational semantics of commands?

Recursive definitions

for divergence

$$\mathcal{C}[\![\cdot]\!] : \mathsf{Com} \to \mathbb{M} \to \mathbb{M} \cup \{\bot\}$$

$$\begin{aligned}
\mathcal{C}[\![\mathbf{skip}]\!]\sigma &\triangleq \sigma \\
\mathcal{C}[\![x := a]\!]\sigma &\triangleq \sigma[\mathcal{A}[\![a]\!]\sigma/x] \\
\mathcal{C}[\![c_0; c_1]\!]\sigma &\triangleq \mathcal{C}[\![c_1]\!](\mathcal{C}[\![c_0]\!]\sigma)
\end{aligned}$$

almost…

$$\mathcal{C}[\![\mathbf{if}\ b\ \mathbf{then}\ c_0\ \mathbf{else}\ c_1]\!]\sigma \triangleq \begin{cases} \mathcal{C}[\![c_0]\!]\sigma & \text{if } \mathcal{B}[\![b]\!]\sigma \\ \mathcal{C}[\![c_1]\!]\sigma & \text{otherwise} \end{cases}$$

$$\mathcal{C}[\![\mathbf{while}\ b\ \mathbf{do}\ c]\!]\sigma \triangleq \begin{cases} \sigma & \text{if } \neg\mathcal{B}[\![b]\!]\sigma \\ \mathcal{C}[\![\mathbf{while}\ b\ \mathbf{do}\ c]\!](\mathcal{C}[\![c]\!]\sigma) & \text{otherwise} \end{cases}$$

almost…

not well-founded recursion!

Recursive definitions

for divergence

$\mathcal{C}[\![c]\!]^{*}\bot = \bot$ $\qquad \mathcal{C}[\![\cdot]\!] : \mathsf{Com} \to \mathbb{M} \to \mathbb{M} \cup \{\bot\}$

$\mathcal{C}[\![c]\!]^{*}\sigma = \mathcal{C}[\![c]\!]$ $\qquad \mathcal{C}[\![\cdot]\!]^{*} : \mathsf{Com} \to \mathbb{M} \cup \{\bot\} \to \mathbb{M} \cup \{\bot\}$

$$\sigma \quad \mathcal{C}[\![\mathbf{skip}]\!]\sigma \triangleq \sigma$$

$$\mathcal{C}[\![x := a]\!]\sigma \triangleq \sigma[\mathcal{A}[\![a]\!]\sigma/x]$$

$$\mathcal{C}[\![c_0; c_1]\!]\sigma \triangleq \mathcal{C}[\![c_1]\!]^{*}(\mathcal{C}[\![c_0]\!]\sigma)$$

$$\mathcal{C}[\![\mathbf{if}\ b\ \mathbf{then}\ c_0\ \mathbf{else}\ c_1]\!]\sigma \triangleq \begin{cases} \mathcal{C}[\![c_0]\!]\sigma & \text{if } \mathcal{B}[\![b]\!]\sigma \\ \mathcal{C}[\![c_1]\!]\sigma & \text{otherwise} \end{cases}$$

$$\mathcal{C}[\![\mathbf{while}\ b\ \mathbf{do}\ c]\!]\sigma \triangleq \begin{cases} \sigma & \text{if } \neg\mathcal{B}[\![b]\!]\sigma \\ \mathcal{C}[\![\mathbf{while}\ b\ \mathbf{do}\ c]\!]^{*}(\mathcal{C}[\![c]\!]\sigma) & \text{otherwise} \end{cases}$$

not well-founded recursion!

how do we know one solution exists? how do we know it is unique?

The general problem

$$f : D \to D$$

a fixed point of f is $d \in D$ such that $d = f(d)$

This is the set of all fixed points

let $F_f \triangleq \{d \in D \mid d = f(d)\} \subseteq D$

three questions:

Does a solution exists?

- under which hypotheses $F_f \neq \varnothing$?

Is there a "best" solution?

- if $F_f \neq \varnothing$, can we select a preferred element $\mathit{fix}(f) \in F_f$?

Can we compute the "best" solution?

- and can we compute $\mathit{fix}(f)$?

Example

$D = \mathbb{N}$	F_f ($\{n \mid n = f(n)\}$)	$\mathit{fix}(f)$
$f(n) \triangleq n + 1$	$\varnothing$	
$f(n) \triangleq n/2$	$\{0\}$	0
$f(n) \triangleq n^2 - 5n + 8$	$\{2, 4\}$	2
$f(n) \triangleq n\%5$	$\{0, 1, 2, 3, 4\}$	0
$f(n) \triangleq \sum_{i \in \mathsf{div}(n)} i$	$\{6, 28, 496, ...\}$ perfect numbers	6

where $\mathsf{div}(x) \triangleq \{1\} \cup \{d \mid 1 < d < x, x\%d = 0\}$

Example

$D = \wp(\mathbb{N})$	$\overset{\{S \mid S = f(S)\}}{F_f}$	$\mathit{fix}(f)$
$f(S) \triangleq S \cap \{1\}$	$\{\, \varnothing \,,\ \{1\} \,\}$	$\varnothing$
$f(S) \triangleq \mathbb{N} \setminus S$	$\varnothing$	
$f(S) \triangleq S \cup \{1\}$	$\{T \mid 1 \in T\}$	$\{1\}$
$f(S) \triangleq \{n \mid \exists m \in S, n \leq m\}$	$\{[0,k] \mid k \in \mathbb{N}\} \cup \{\varnothing, \mathbb{N}\}$	$\varnothing$

Ingredients

a partial order (to compare elements)

order preserving functions

iterative approximations

a base case

a limit solution

limit preserving functions

Partial orders

Partially ordered set (Poset or just PO)

a set

$(P, \sqsubseteq)$ a binary relation $\sqsubseteq \subseteq P \times P$

reflexive	$\forall p \in P.$	$p \sqsubseteq p$
antisymmetric	$\forall p, q \in P.$	$p \sqsubseteq q \;\wedge\; q \sqsubseteq p \;\Rightarrow\; p = q$
transitive	$\forall p, q, r \in P.$	$p \sqsubseteq q \;\wedge\; q \sqsubseteq r \;\Rightarrow\; p \sqsubseteq r$

q
↑
p

$p \sqsubseteq q$ means that p and q are **comparable** and that p is less than (or equal to) q

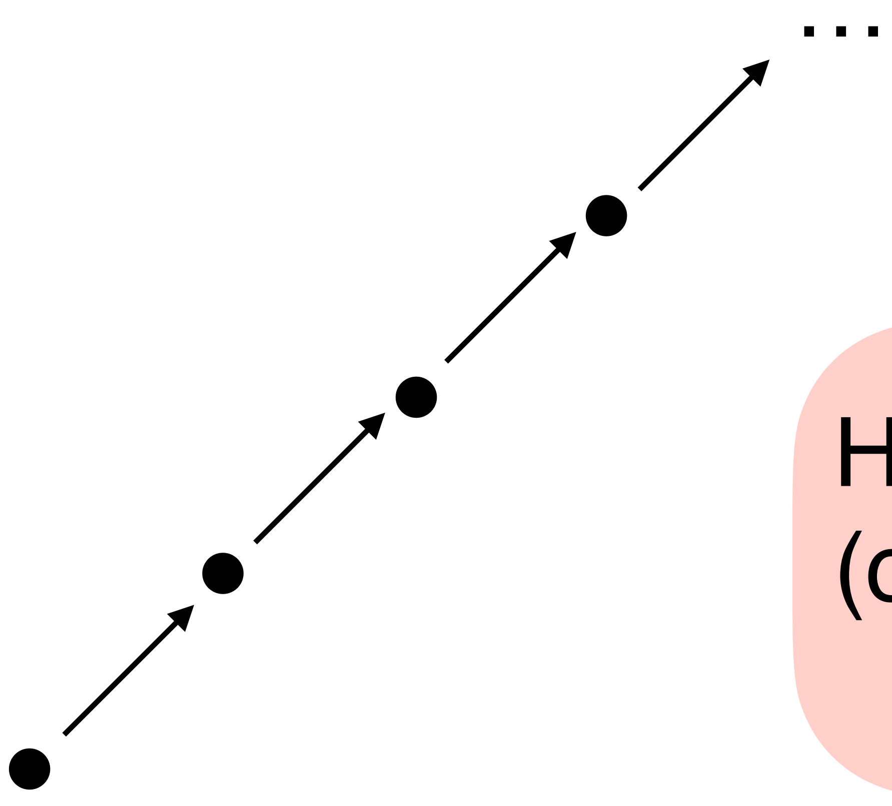
$p \sqsubset q$ means $p \sqsubseteq q \;\wedge\; p \neq q$

Total orders

$(P, \sqsubseteq)$ PO

total $\quad \forall p, q \in P. \quad p \sqsubseteq q \;\vee\; q \sqsubseteq p$

a PO where any two elements are **comparable**



Hasse diagram notation
(omit: reflexive arcs,
transitive arcs)

Discrete orders

$(P, \sqsubseteq)$ PO

discrete $\quad \forall p, q \in P. \quad p \sqsubseteq q \Leftrightarrow p = q$

each element is **comparable** only to itself

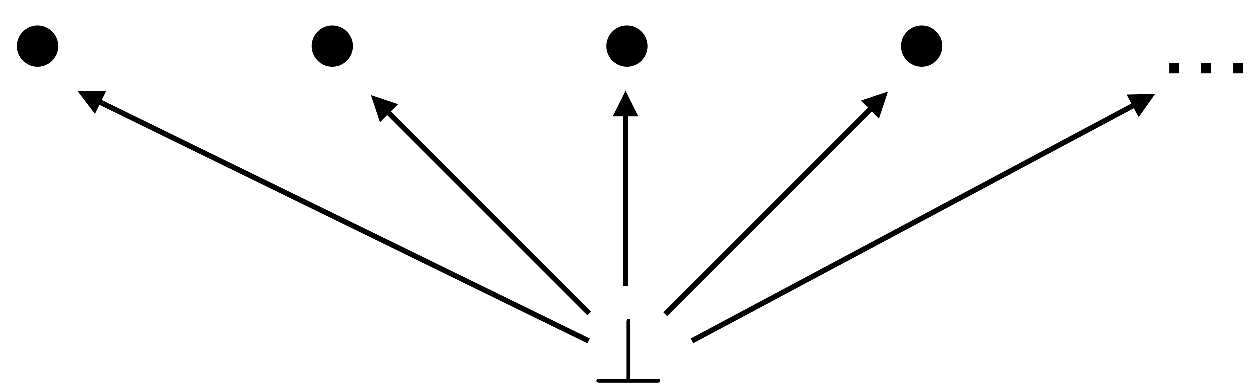
• • • • …

Flat orders

$(P, \sqsubseteq)$ PO

flat $\forall p, q \in P. \quad p \sqsubseteq q \ \Leftrightarrow \ p = q \ \vee \ p = \bot$

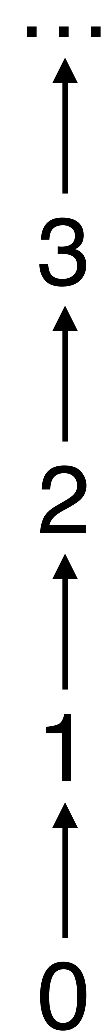
each element is **comparable** only to itself
and with a distinguished (smaller) element $\bot$



Exercise

$(\mathbb{N}, \leq)$

PO?	Total?	Discrete?	Flat?
✓	✓	✗	✗



Exercise

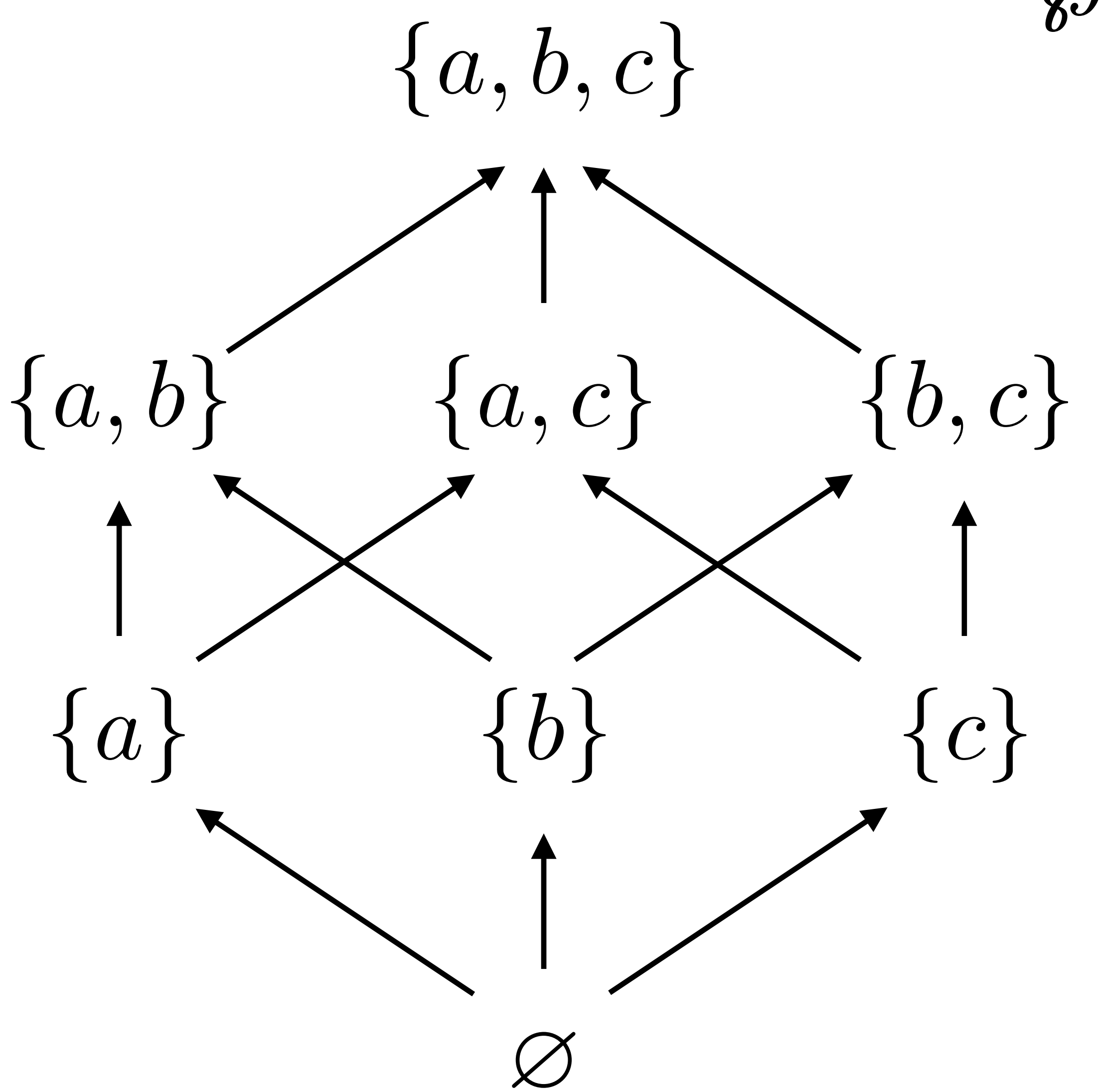
$(\wp(S), \subseteq)$

PO?	Total?	Discrete?	Flat?
✔	$\lvert S \rvert < 2$	$S = \varnothing$	$\lvert S \rvert < 2$

example: $S = \{a, b, c\}$

$\wp(\varnothing) = \{\varnothing\}$

$\wp(\{a\}) = \{\varnothing, \{a\}\}$



$\{a, b\} \quad \not\subseteq \quad \{b, c\}$
$\qquad\quad\;\; \not\supseteq$

$\{a\} \quad \not\subseteq \quad \{b\}$
$\qquad \not\supseteq$

Exercise

$(\mathbb{N}, =)$

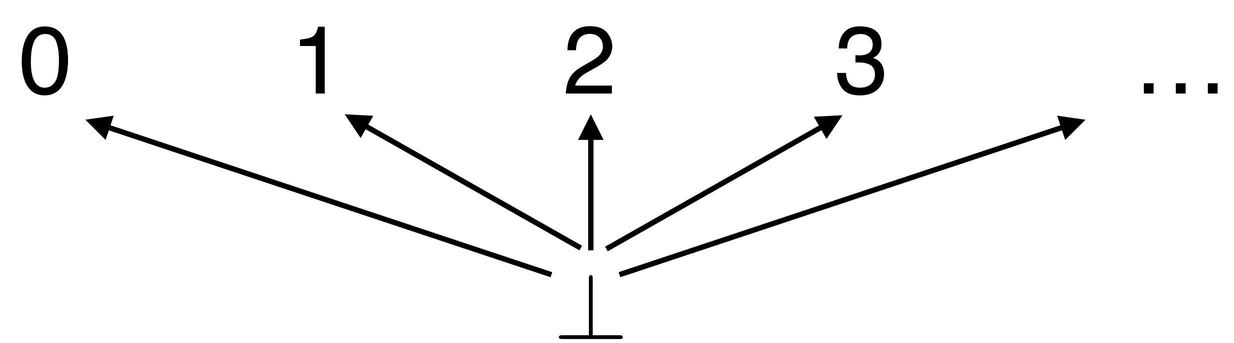
PO?	Total?	Discrete?	Flat?
✓	✗	✓	✗

0 1 2 3 …

Exercise

$$(\mathbb{N} \cup \{\bot\}, \{(\bot, n) \mid n \in \mathbb{N}\}^*)$$

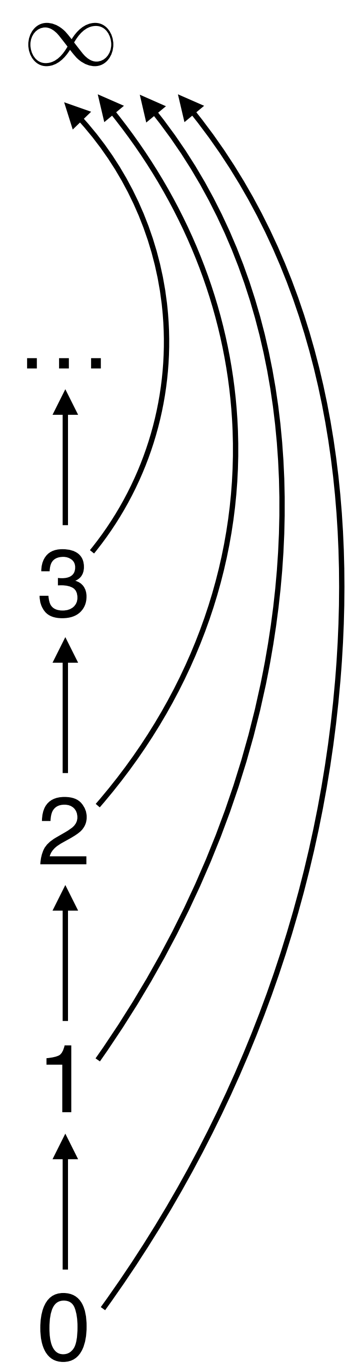
PO?	Total?	Discrete?	Flat?
✓	✗	✗	✓



Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

PO?	Total?	Discrete?	Flat?
✓	✓	✗	✗



Exercise

	PO?	Total?	Discrete?	Flat?
$(\mathbb{N}, <)$	✗	✗	✗	✗
$(\mathbb{Z}, \leq)$	✓	✓	✗	✗
$(\mathbb{Z} \cup \{-\infty, \infty\}, \leq)$	✓	✓	✗	✗
$(T_\Sigma, \prec)$	✗	✗	✗	✗
$(\mathbb{N}, \neq)$	✗	✗	✗	✗

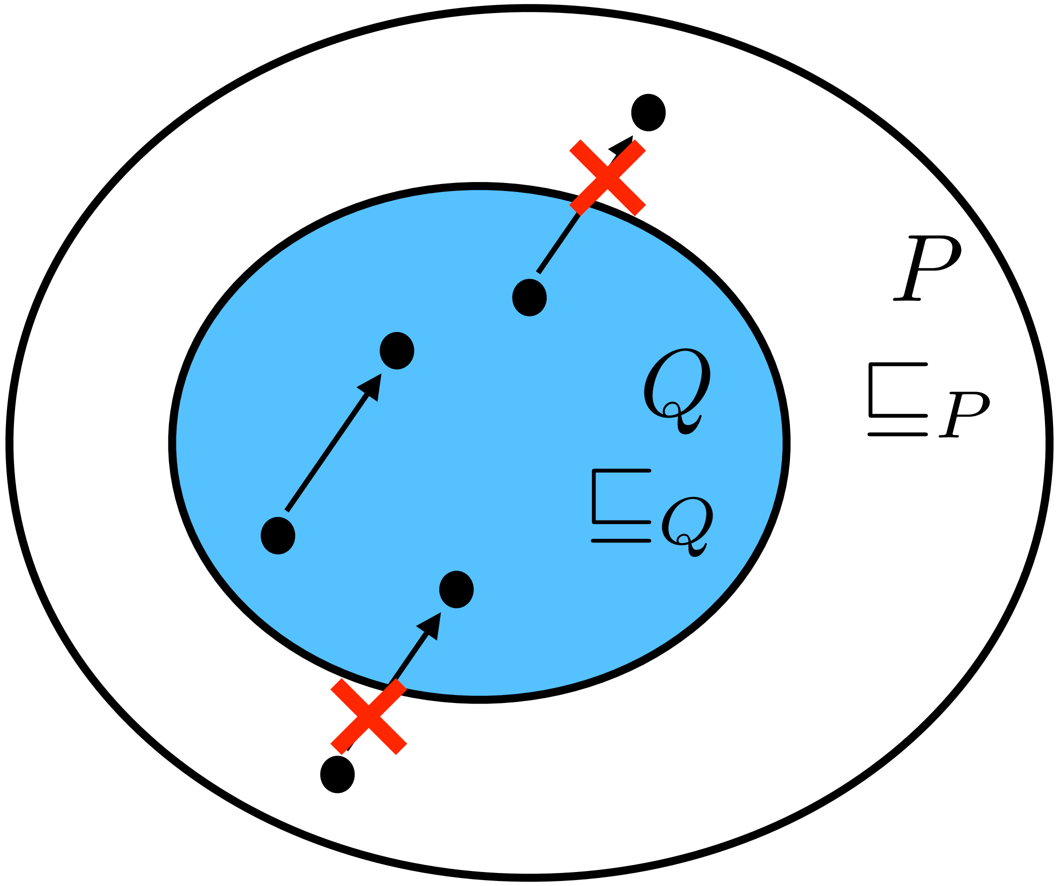
Subset of a PO

$(P, \sqsubseteq_P)$ PO $\qquad Q \subseteq P$

let $\sqsubseteq_Q \triangleq \sqsubseteq_P \cap (Q \times Q)$

TH. $(Q, \sqsubseteq_Q)$ is a PO

TH. if $(P, \sqsubseteq_P)$ is total, then $(Q, \sqsubseteq_Q)$ is total



PO $\sqsubseteq$	w.f. $\prec$
reflexive	not reflexive (otherwise cycle!)
antisymmetric	antisymmetric (otherwise cycle!) $p \prec q \wedge q \prec p$ is always false
transitive	can be transitive ($\prec^+$ w.f.)
has infinite descending chains (if nonempty)	no infinite descending chain
	$\prec^*$ is always a PO
$\sqsubset$ can be w.f.	

Element properties (least, minimal, ...)

Least element

$(P, \sqsubseteq)$ PO $\quad Q \subseteq P \quad \ell \in Q$

ℓ is a least element of Q if $\forall q \in Q. \quad \ell \sqsubseteq q$

TH. (uniqueness of least element)

$(P, \sqsubseteq)$ PO $\quad Q \subseteq P \quad \ell_1, \ell_2$ least elements of Q implies $\ell_1 = \ell_2$

$$\left.\begin{array}{lll} \ell_1 \text{ least element of } Q & \Rightarrow & \ell_1 \sqsubseteq \ell_2 \\ \ell_2 \text{ least element of } Q & \Rightarrow & \ell_2 \sqsubseteq \ell_1 \end{array}\right\} \Rightarrow \quad \ell_1 = \ell_2$$

by antisymmetry

Bottom

$(P, \sqsubseteq)$ PO

the least element of P (if it exists) is called bottom and denoted $\bot$

sometimes written $\bot_P$

Examples

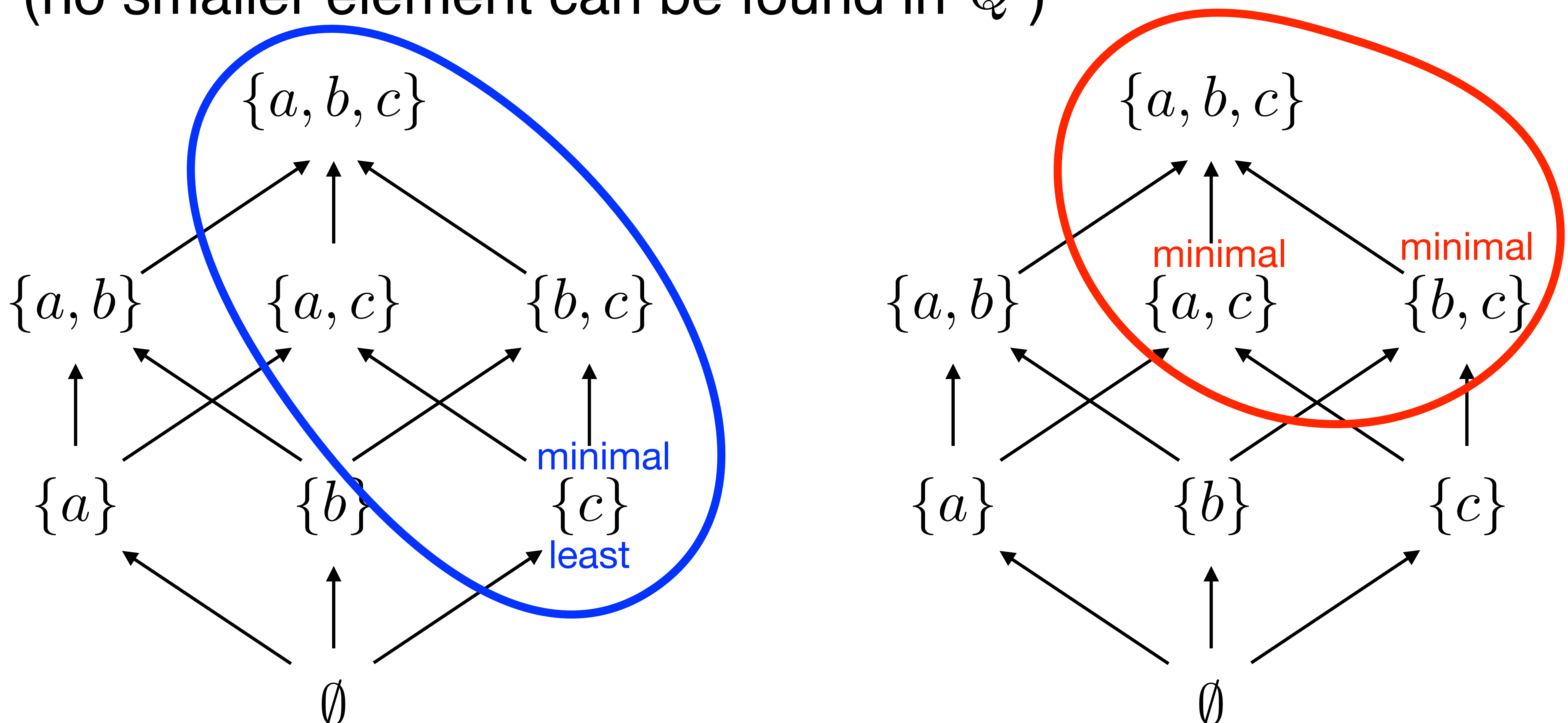
PO	bottom?
$(\mathbb{N} \cup \{\infty\}, \leq)$	0
$(\wp(S), \subseteq)$	$\varnothing$
$(\mathbb{Z}, \leq)$	
$(\mathbb{Z} \cup \{-\infty, \infty\}, \leq)$	$-\infty$

Minimal element

$(P, \sqsubseteq)$ PO $\quad Q \subseteq P \quad m \in Q$

m is a minimal element of Q if $\forall q \in Q. \quad q \sqsubseteq m \Rightarrow q = m$

(no smaller element can be found in Q)



Least vs minimal

least $\forall q \in Q. \quad \ell \sqsubseteq q$	minimal $\forall q \in Q. \quad q \sqsubseteq m \Rightarrow q = m$
unique	not necessarily unique
minimal	
	not necessarily least
	can be least

Reverse order

TH. $(P, \sqsubseteq)$ PO implies $(P, \sqsupseteq)$ PO

note:
$\sqsupseteq = \sqsubseteq^{-1}$

proof. it is immediate to check that $\sqsupseteq$
is reflexive
is antisymmetric
is transitive

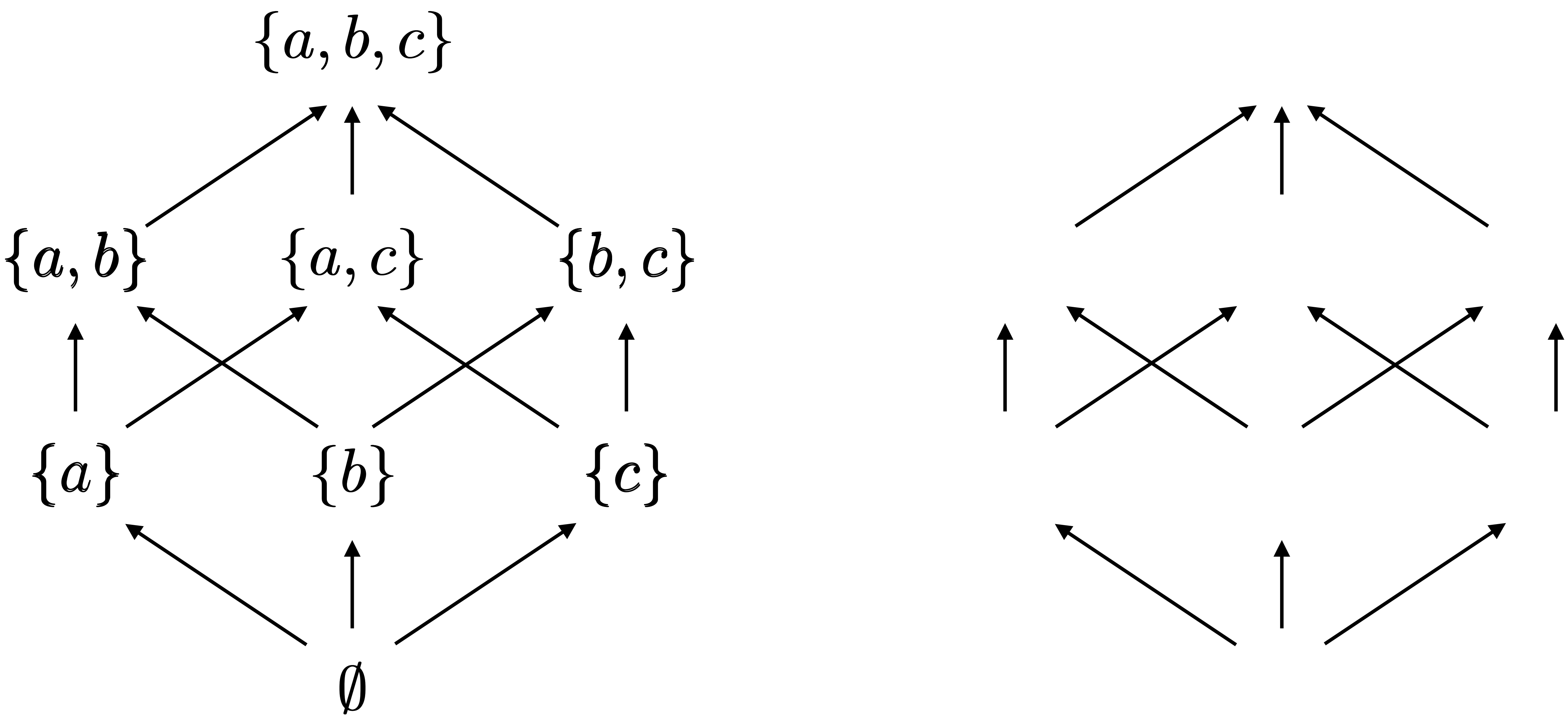
$(P, \sqsubseteq)$ PO $Q \subseteq P$

greatest element: least element of Q w.r.t. $(P, \sqsupseteq)$

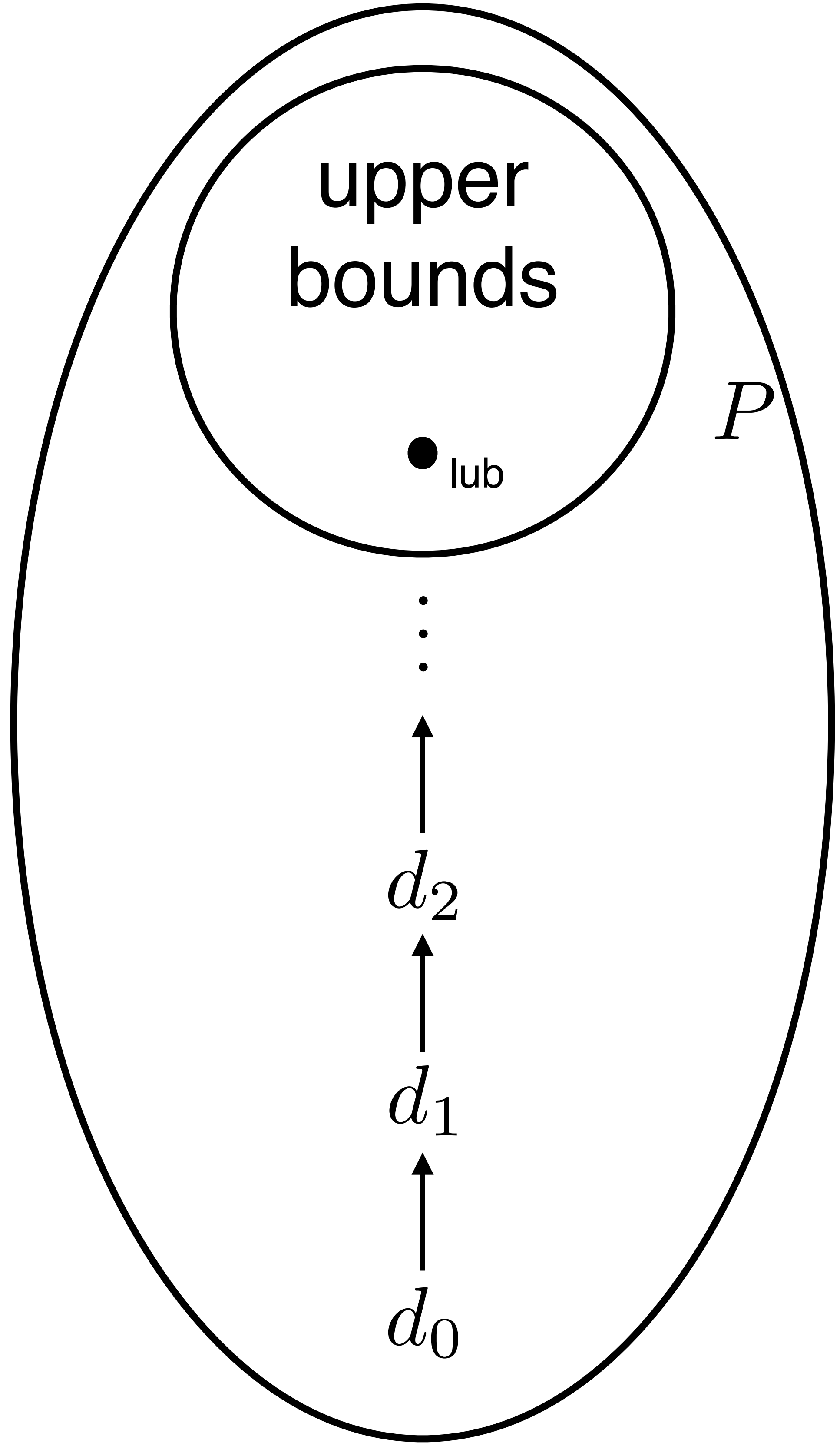
top element: $\top$ greatest element of P (if it exists)

maximal element: minimal element of Q w.r.t. $(P, \sqsupseteq)$

Reversed powerset



Limit: idea

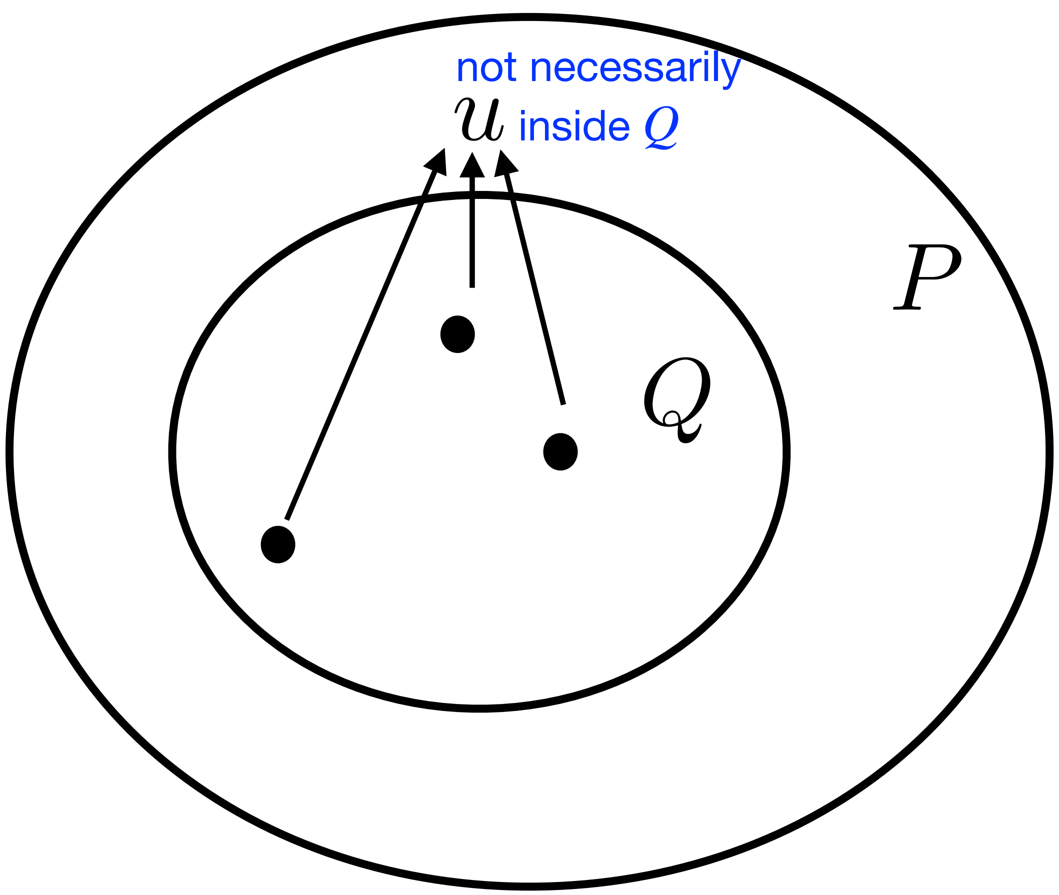


Upper bound

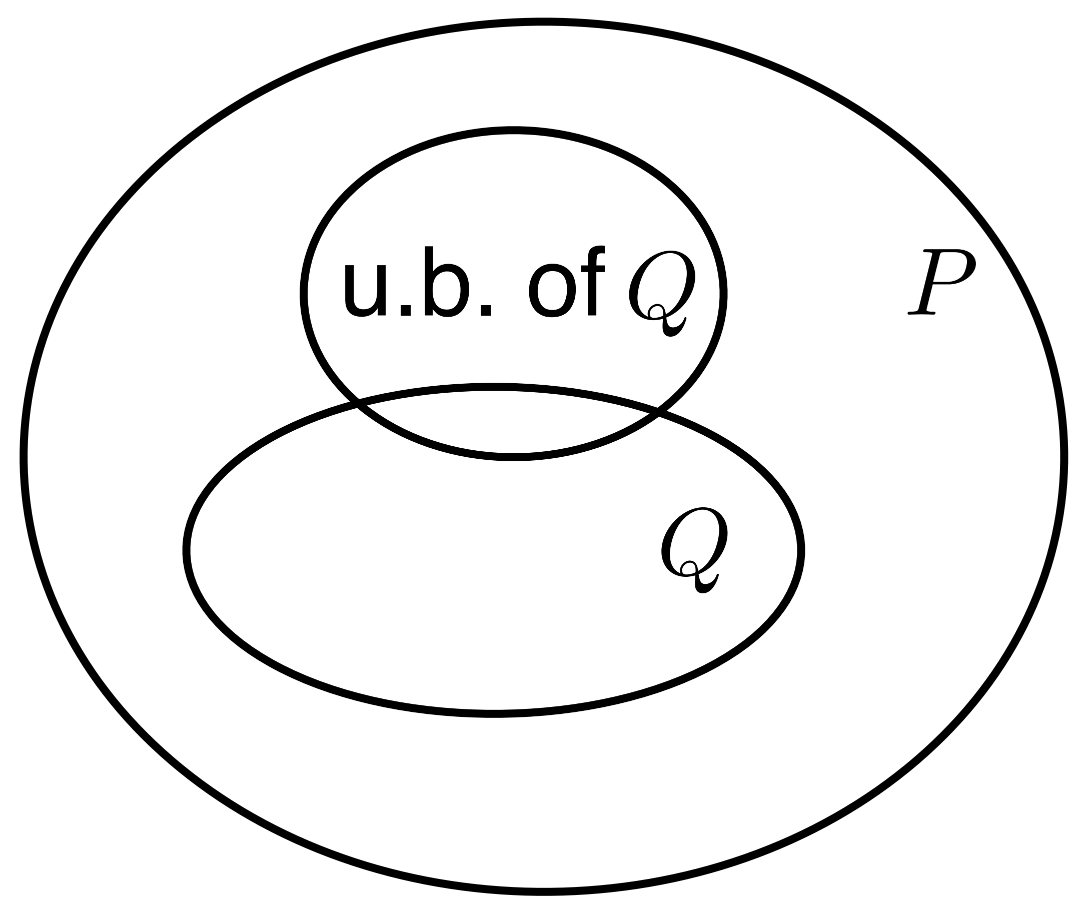
$(P, \sqsubseteq)$ PO $\qquad Q \subseteq P \qquad u \in P$

u is an upper bound of Q if $\forall q \in Q.\ \ q \sqsubseteq u$

(all the elements of Q are smaller than u)



Q may have many upper bounds



Least upper bound

$(P, \sqsubseteq)$ PO $\quad Q \subseteq P \quad p \in P$

p is the **least upper bound** (**lub**) of Q if

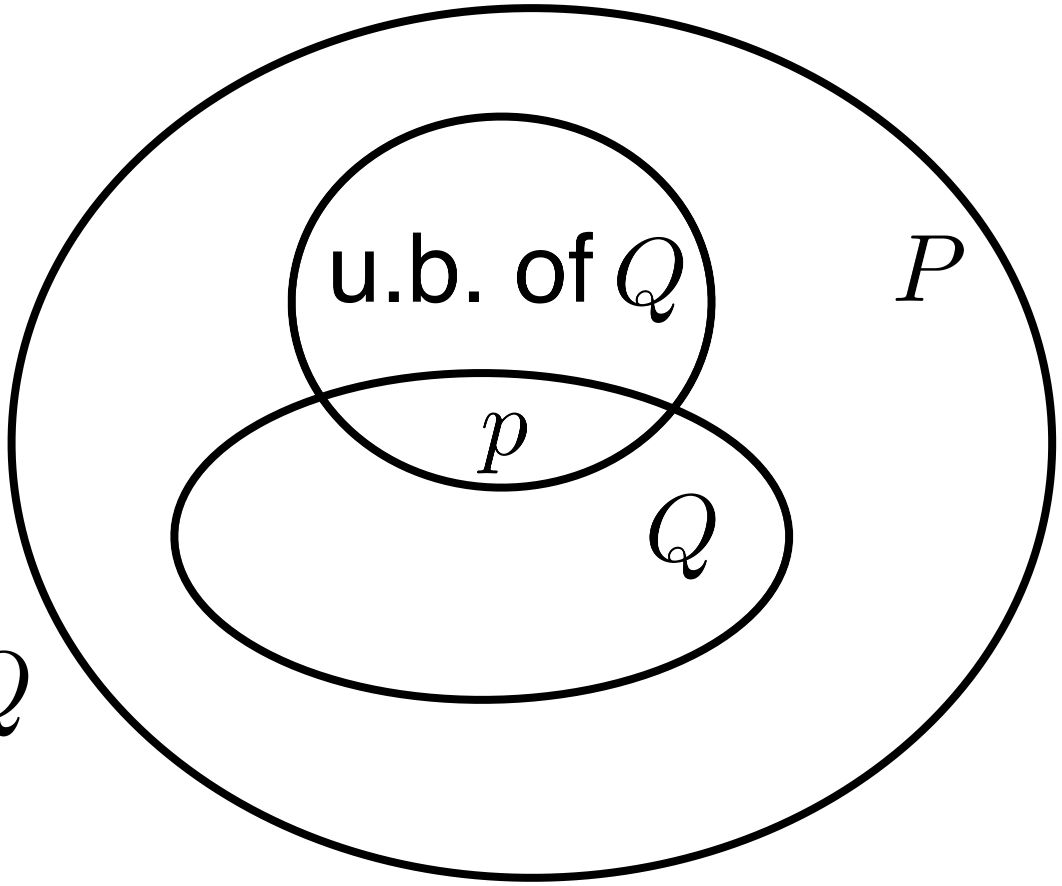
1. it is an upper bound of Q $\quad \forall q \in Q. \quad q \sqsubseteq p$
2. it is smaller than any other upper bound of Q

$$\forall u \in P. \quad (\forall q \in Q. q \sqsubseteq u) \Rightarrow p \sqsubseteq u$$

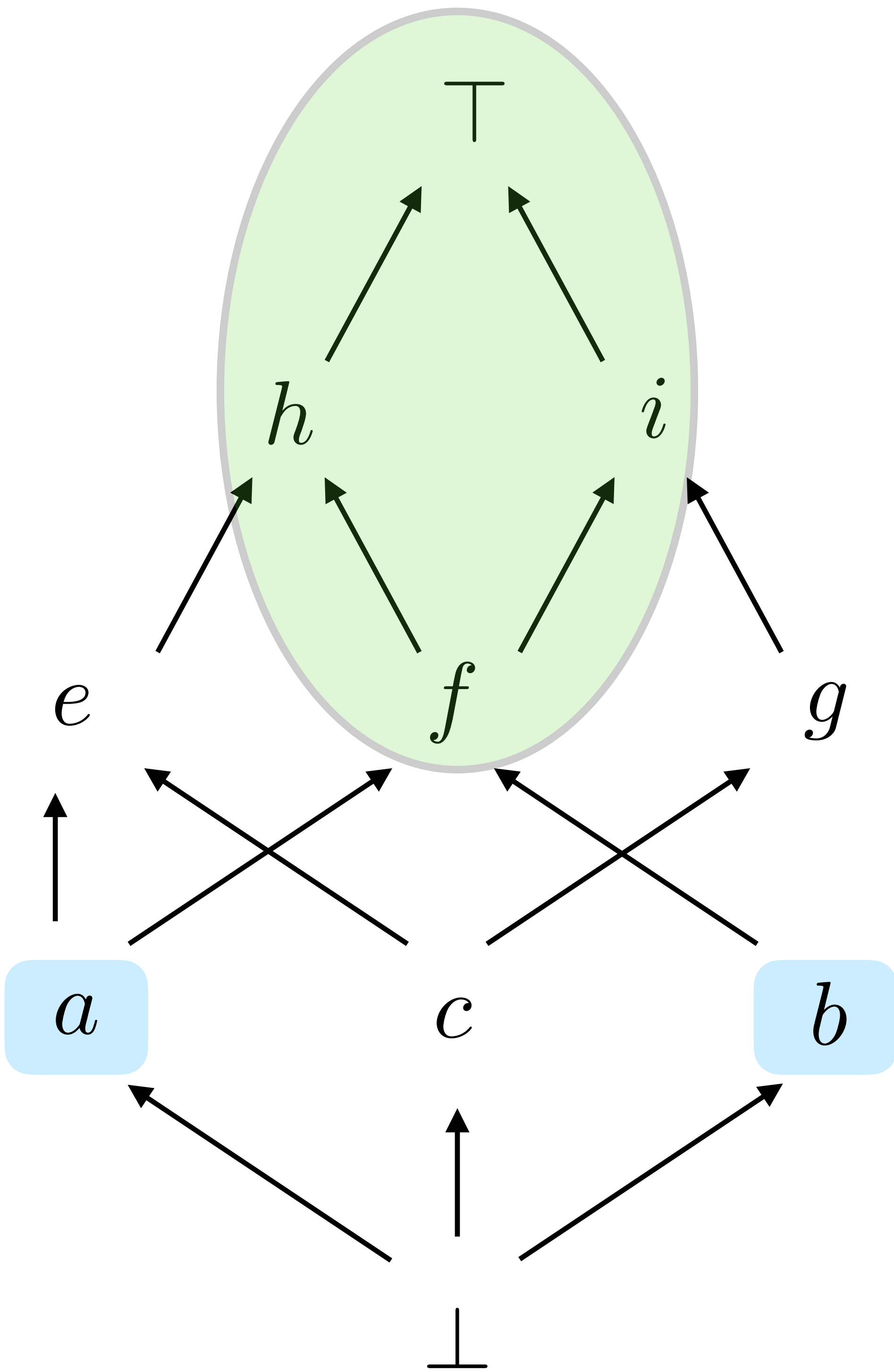
we write $p = lub\ Q$

intuitively, it is the least element that represents all of Q

p not necessarily an element of Q



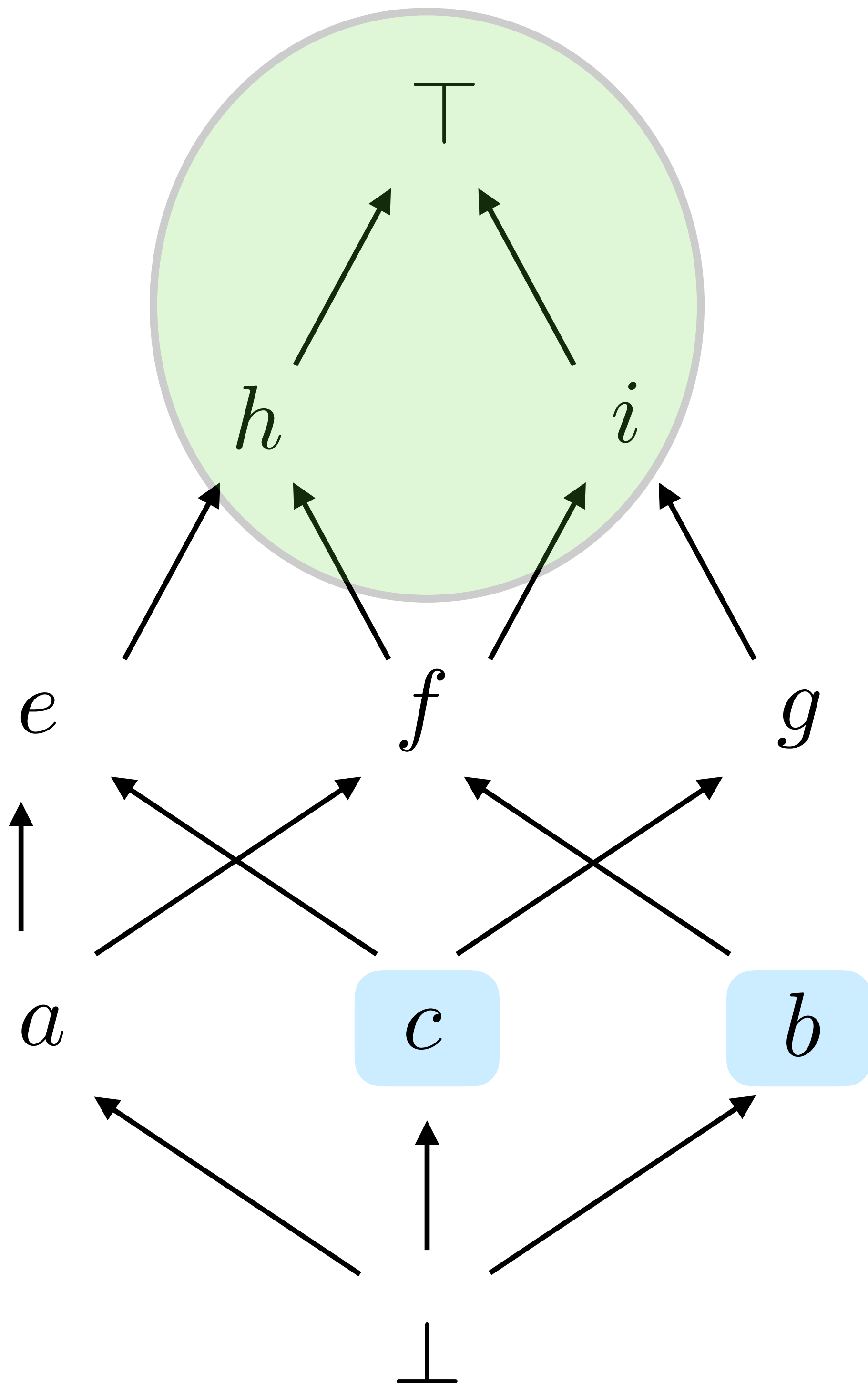
Exercise



Upper bounds of $\{a, b\}$? $\{f, h, i, \top\}$

lub? f

Exercise



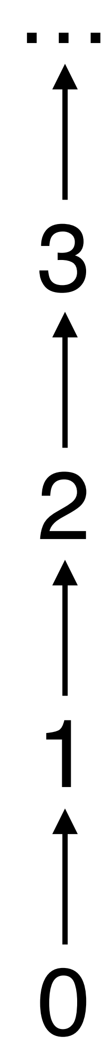
Upper bounds of $\{b, c\}$? $\{h, i, \top\}$

lub? no lub!

Exercise

$(\mathbb{N}, \leq)$ $\quad Q \subseteq \mathbb{N}$ $\quad$ lub? $\quad$ if Q finite $\quad lub\ Q = \max Q$

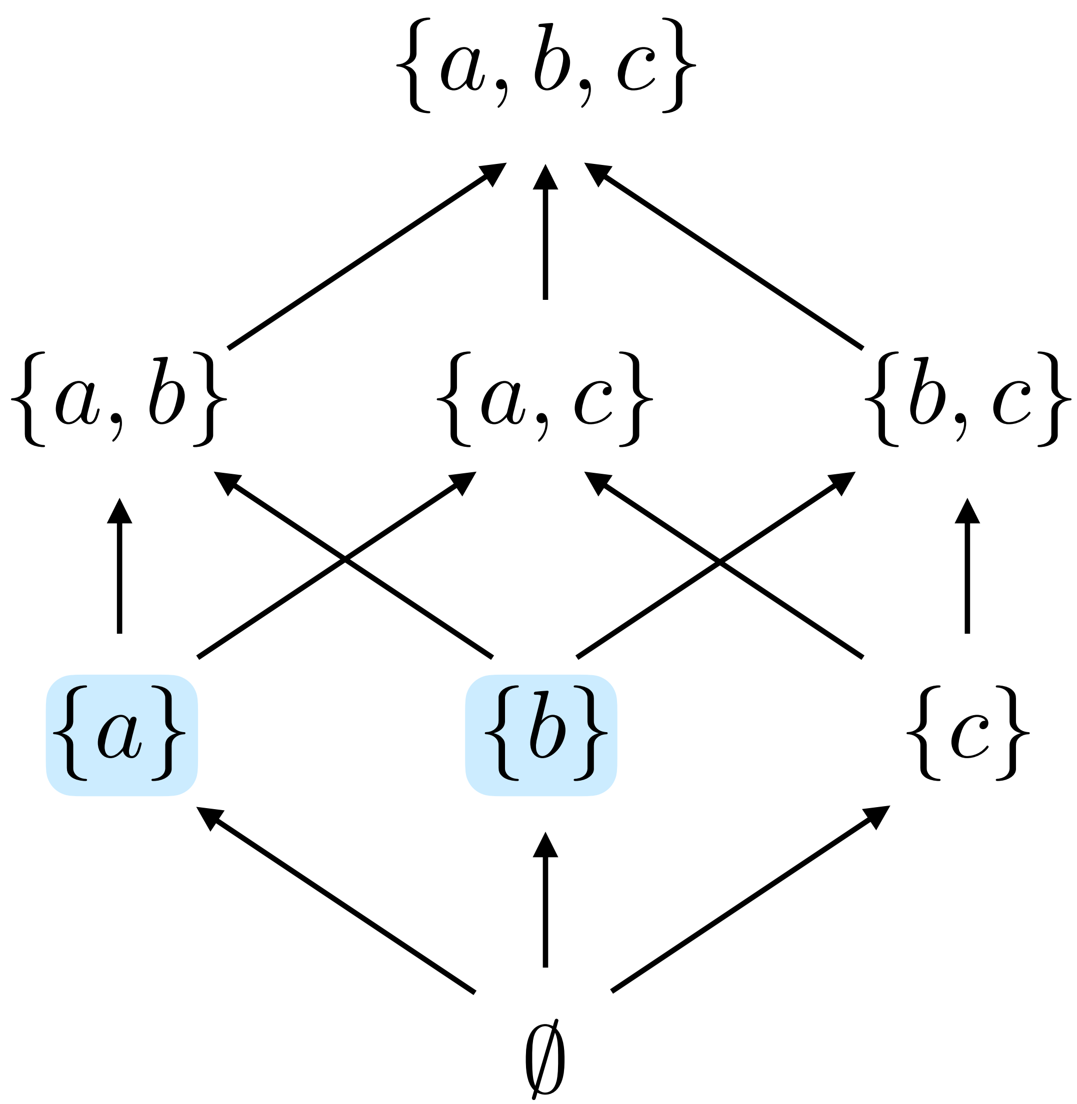
otherwise no lub



Exercise

$(\wp(S), \subseteq)$ $\quad Q \subseteq \wp(S)$ $\quad$ lub?

$$lub\ Q = \bigcup_{T \in Q} T$$



$$lub\ \{\{a\}, \{b\}\} = \{a, b\}$$

Complete partial orders (CPO)

Completeness: the idea

D a domain

$\sqsubseteq$ a way to compare element

$x \sqsubseteq y$ x is a (less precise) approximation of y

x and y are consistent, but y is more accurate than x

(PO)

$$x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq \cdots$$

- x_0: first approximation
- x_1: second approximation
- x_2: third approximation
- x_n: (n+1)th approximation

does any sequence of approximations tend to some limit?

Chain

$(P, \sqsubseteq)$ PO $\{d_i\}_{i \in \mathbb{N}}$ is a chain if $\forall i \in \mathbb{N}.\ d_i \sqsubseteq d_{i+1}$

$$d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots$$

any chain is an infinite list

finite chain: there are only finitely many distinct elements

$$\exists k \in \mathbb{N}.\ \forall i \geq k.\ d_i = d_{i+1}$$

or equivalently

$$\exists k \in \mathbb{N}.\ \forall i \geq k.\ d_i = d_k$$

Example

$(\mathbb{N}, \leq)$

$0 \leq 2 \leq 4 \leq \cdots \leq 2n \leq \cdots$ is an infinite chain

$0 \leq 1 \leq 3 \leq 3 \leq 5 \leq \cdots \leq 5 \leq \cdots$ is a finite chain

any chain has infinite length

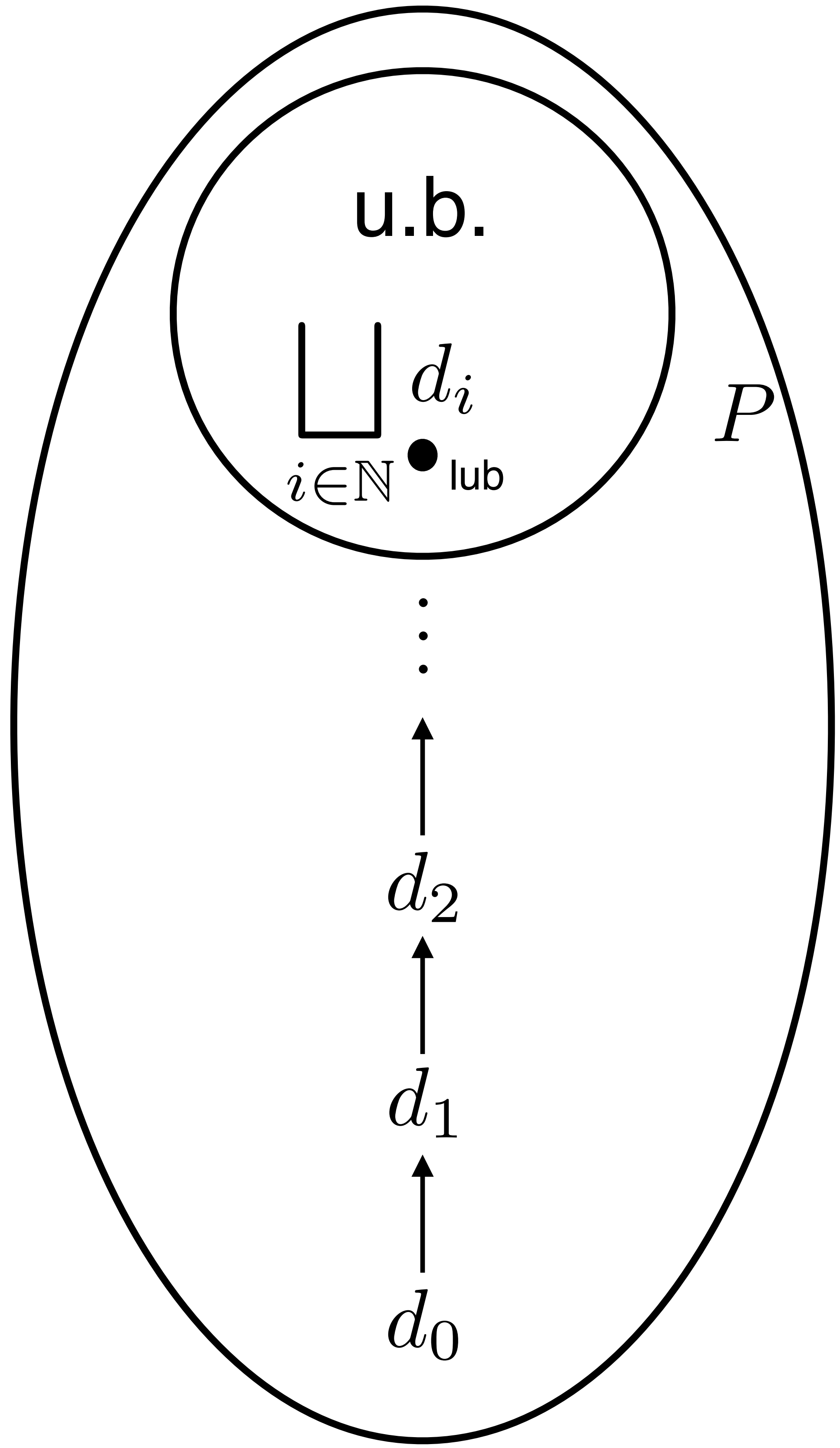
Limit of a chain

$(P, \sqsubseteq)$ PO $\{d_i\}_{i \in \mathbb{N}}$ a chain

we denote by $\bigsqcup_{i \in \mathbb{N}} d_i$ the lub of $\{d_i\}_{i \in \mathbb{N}}$ if it exists

and call it the limit of the chain

Limit illustrated



Example

$(\mathbb{N}, \leq)$

$0 \leq 2 \leq 4 \leq \cdots \leq 2n \leq \cdots$

has no lub
(empty set of upper bounds)

$0 \leq 1 \leq 3 \leq 3 \leq 5 \leq \cdots \leq 5 \leq \cdots$

has lub 5
(which upper bounds?)

Lemma on finite chains

Lemma (any finite chain has a limit)

$(P, \sqsubseteq)$ PO $\{d_i\}_{i \in \mathbb{N}}$ a finite chain $\Rightarrow$ $\bigsqcup_{i \in \mathbb{N}} d_i$ exists

proof.

$\{d_i\}_{i \in \mathbb{N}}$ finite $\Rightarrow$ $\exists k.\ \forall i.\ d_{i+k} = d_k$

the elements of the chain are totally ordered

d_k is the greatest element of the chain

d_k is an upper bound $\forall i.\ d_i \sqsubseteq d_k$

d_k is the least upper bound

take u such that $\forall i.\ d_i \sqsubseteq u$ then $d_k \sqsubseteq u$

Prefix independence

Lemma (prefix independence) $(P, \sqsubseteq)$ PO $\{d_i\}_{i\in\mathbb{N}}$ a chain

if $\bigsqcup_{i\in\mathbb{N}} d_i$ exists $\Rightarrow \forall k.\ \bigsqcup_{i\in\mathbb{N}} d_{i+k} = \bigsqcup_{i\in\mathbb{N}} d_i$

$$d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots \sqsubseteq d_k \sqsubseteq d_{k+1} \sqsubseteq \cdots \qquad \bigsqcup_{i\in\mathbb{N}} d_i$$

=

$$d_k \sqsubseteq d_{k+1} \sqsubseteq \cdots \qquad \bigsqcup_{i\in\mathbb{N}} d_{i+k}$$

Prefix independence

Lemma (prefix independence) $(P, \sqsubseteq)$ PO $\{d_i\}_{i \in \mathbb{N}}$ a chain

$$\text{if } \bigsqcup_{i \in \mathbb{N}} d_i \text{ exists } \Rightarrow \forall k.\ \bigsqcup_{i \in \mathbb{N}} d_{i+k} = \bigsqcup_{i \in \mathbb{N}} d_i$$

proof.

take a generic k

we prove that $\{d_i\}_{i \in \mathbb{N}}$ and $\{d_{i+k}\}_{i \in \mathbb{N}}$ have the same u.b.

(and thus the same lub)

1. if u is an u.b. of $\{d_i\}_{i \in \mathbb{N}}$ then is an u.b. of $\{d_{i+k}\}_{i \in \mathbb{N}}$
   because $\{d_{i+k}\}_{i \in \mathbb{N}} \subseteq \{d_i\}_{i \in \mathbb{N}}$
2. if u is an u.b. of $\{d_{i+k}\}_{i \in \mathbb{N}}$ we need to show $\forall j.\ d_j \sqsubseteq u$
   for $j \geq k$ it is obvious
   if $j < k$ then $d_j \sqsubseteq d_k \sqsubseteq u$ because $d_k \in \{d_{i+k}\}_{i \in \mathbb{N}}$

Complete partial order

$(P, \sqsubseteq)$ PO $\quad P$ is complete if each chain has a limit (lub)

TH. Any finite chain has a limit
(the last element in the sequence)

If P has only finite chains it is complete

If P is finite it is complete

Any discrete order is complete

Any flat order is complete

Example

$(\mathbb{N}, \leq)$ is not complete
(it is enough to exhibit a chain with no limit)

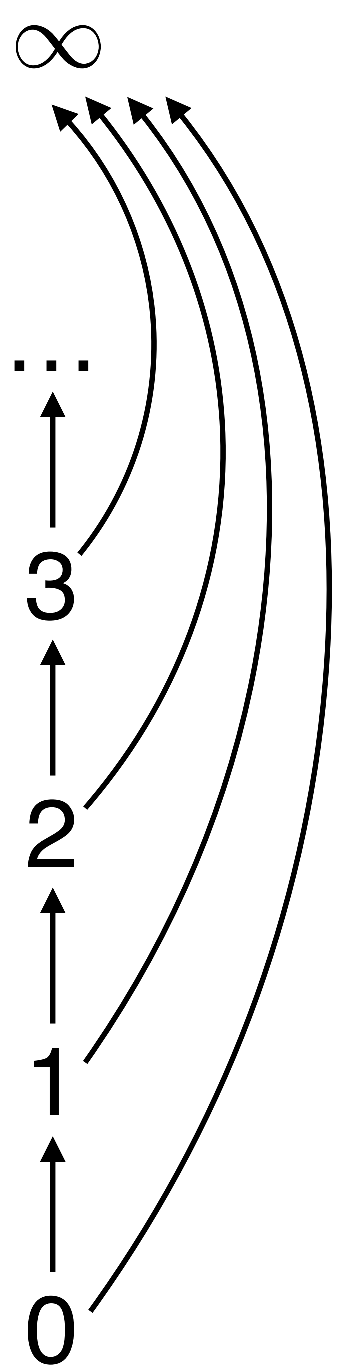
$$0 \leq 2 \leq 4 \leq \cdots \leq 2n \leq \cdots$$

has no lub
(empty set of u.b.)

Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

complete?



any infinite chain has limit ∞
(set of u.b. $\{\infty\}$)

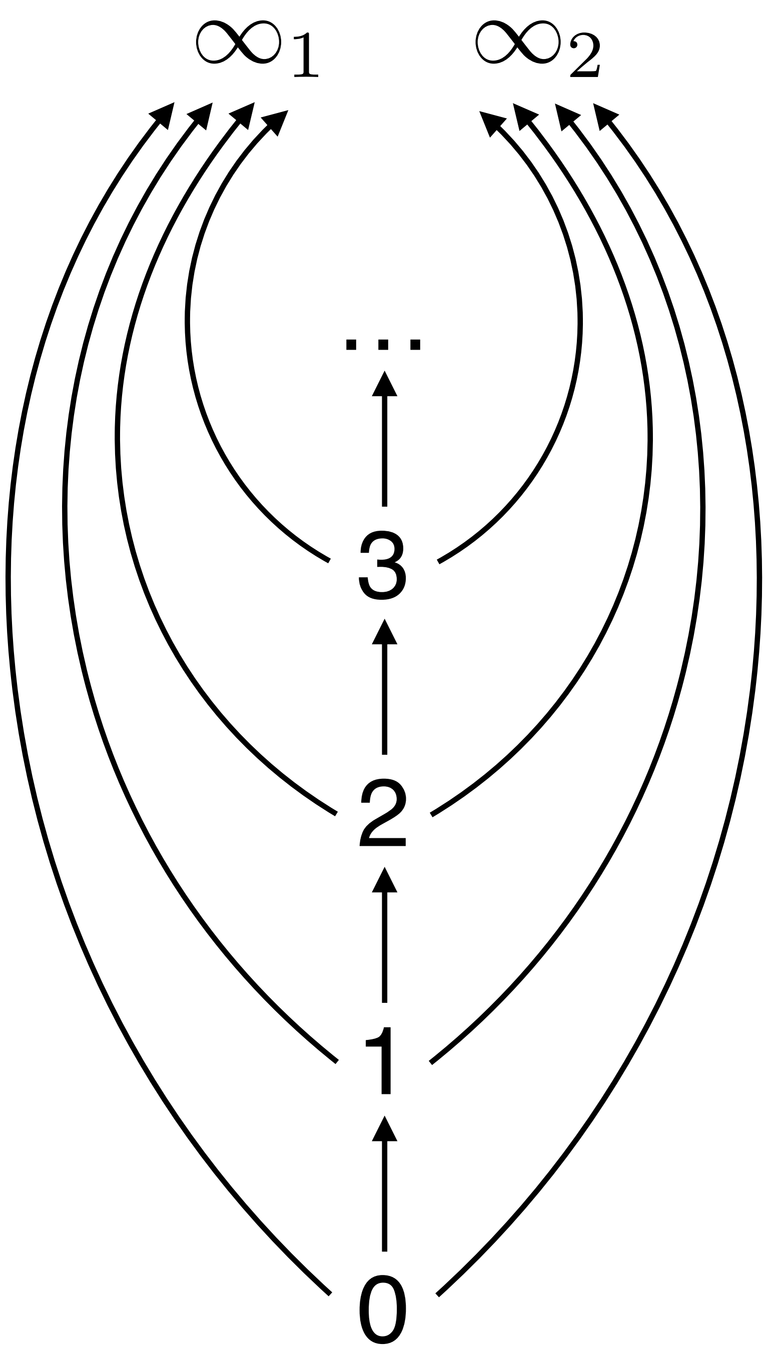
Exercise

$(\wp(S), \subseteq)$ complete? ✓

$\{S_i\}_{i\in\mathbb{N}}$ $\qquad \bigsqcup_{i\in\mathbb{N}} S_i = \bigcup_{i\in\mathbb{N}} S_i = \{x \mid \exists k \in \mathbb{N}. x \in S_k\}$

Exercise

$(\mathbb{N} \cup \{\infty_1, \infty_2\}, \leq)$ complete? ✖



any infinite chain has no limit
(set of u.b. $\{\infty_1, \infty_2\}$)

Badge exercise

Let D be a CPO

let $\{d_i\}_{i\in\mathbb{N}}$ be a chain in D

let $\{k_j\}_{j\in\mathbb{N}}$ be an infinite chain in $(\mathbb{N}, \leq)$

1. Prove that $\{d_{k_j}\}_{j\in\mathbb{N}}$ is a chain in D
2. Prove or disprove that $\bigsqcup_{j\in\mathbb{N}} d_{k_j} = \bigsqcup_{i\in\mathbb{N}} d_i$

Partial functions

Comparing functions

given two functions $f, g : A \to B$, when can we say $f = g$?

$$\forall a \in A \,.\, f(a) = g(a)$$

if we see functions as relations

$$\{\ (a, f(a)) \mid a \in A\ \}\ \subseteq\ A \times B$$

we can use set equality

Example

$$f(n) = n!$$

$$f(n) = n(n+1)/2$$

$n \quad f(n)$

$$f = \{ \quad (0,1), (1,1), (2,2), (3,6), \ldots (k, k!), \ldots \}$$

$$f = \{ \quad (0,0), (1,1), (2,3), (3,6), \ldots (k, T_k), \ldots \}$$

Partial functions

let $f\colon A \rightharpoonup B$, or equivalently $f\colon A \to B \cup \{ \perp \}$

the function f can be undefined on some inputs

we can still see partial functions as relations

$$\{ (a, f(a)) \mid a \in A, f(a) \neq \perp \} \subseteq A \times B$$

omit pairs where $f(a)$ is undefined

Partial functions

$D = (A \rightharpoonup B) = \mathbf{Pf}(A, B) = \{f : A \rightharpoonup B\}$ partial functions

$f \sqsubseteq g$ if $f(a)$ is defined, $g(a)$ is defined and $g(a) = f(a)$

but $g(a)$ can be defined when $f(a)$ is not

if we see partial functions as relations

$$\{(x, f(x)) \mid f(x) \neq \bot\} \subseteq A \times B$$

$f \sqsubseteq g$ means essentially $f \subseteq g$

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ \bot & \text{otherwise} \end{cases}$$

$$\begin{aligned} & \quad\ {\color{red} n \quad f(n)} \\ f = \{ \quad & (0, 0), \\ & (2, 1), \\ & (4, 2), \\ & (6, 3), \\ & \ldots \\ & (2k, k), \\ & \ldots \qquad\qquad \} \end{aligned}$$

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$g(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ 2 \cdot n & \text{otherwise} \end{cases}$$

$$\begin{aligned} g = \{ \quad & (0,0), (1,2), \\ & (2,1), (3,6), \\ & (4,2), (5,10), \\ & (6,3), (7,14), \\ & \ldots \\ & (2k,k), (1+2k, 2+4k), \\ & \ldots \qquad \} \end{aligned}$$

Example

$\mathbf{Pf}(\mathbb{N},\mathbb{N})$

$$g(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ 2 \cdot n & \text{otherwise} \end{cases}$$

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ \bot & \text{otherwise} \end{cases}$$

$$g = \{ \begin{array}{l} (0,0),(1,2), \\ (2,1),(3,6), \\ (4,2),(5,10), \\ (6,3),(7,14), \\ \dots \\ (2k,k),(1+2k,2+4k), \\ \dots \end{array} \}$$

$f \sqsubseteq g$? ✓

$g \sqsubseteq f$? ✗

$$f = \{ \begin{array}{l} (0,0), \\ (2,1), \\ (4,2), \\ (6,3), \\ \dots \\ (2k,k), \\ \dots \end{array} \}$$

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$\emptyset \sqsubseteq$ { (0,0) } $\sqsubseteq$ { (0,0), (1,1) } $\sqsubseteq$ …

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$\emptyset \sqsubseteq$ { (0,0) } $\sqsubseteq$ { (0,0), (1,1) } $\sqsubseteq$ { (0,0), (1,1), (2,2) } $\sqsubseteq$ ...

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$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$\emptyset \sqsubseteq \{(0,0)\} \sqsubseteq \{(0,0), (1,1)\} \sqsubseteq \{(0,0), (1,1), (2,2)\} \sqsubseteq \{(0,0), (1,1), (2,2), (3,3)\} \sqsubseteq \ldots$$

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$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$\emptyset \sqsubseteq \{ (0,0) \} \sqsubseteq \{ (0,0), (1,1) \} \sqsubseteq \{ (0,0), (1,1), (2,2) \} \sqsubseteq \{ (0,0), (1,1), (2,2), (3,3) \} \sqsubseteq \{ (0,0), (1,1), (2,2), (3,3), (4,4) \} \sqsubseteq \cdots$$

which function(s) are we approximating?

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$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$\emptyset \sqsubseteq \{ (0,1) \} \sqsubseteq \{ (0,1), (1,1) \} \sqsubseteq \{ (0,1), (1,1), (2,2) \} \sqsubseteq \ldots$$

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$\emptyset \sqsubseteq \{ (0,1) \} \sqsubseteq \{ (0,1), (1,1) \} \sqsubseteq \{ (0,1), (1,1), (2,2) \} \sqsubseteq \{ (0,1), (1,1), (2,2), (3,6) \} \sqsubseteq \ldots$$

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$\emptyset \sqsubseteq \{ (0,1) \} \sqsubseteq \{ (0,1), (1,1) \} \sqsubseteq \{ (0,1), (1,1), (2,2) \} \sqsubseteq \{ (0,1), (1,1), (2,2), (3,6) \} \sqsubseteq \{ (0,1), (1,1), (2,2), (3,6), (4,24) \} \sqsubseteq \ldots$$

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N},\mathbb{N})$

$$\emptyset \sqsubseteq \{ (0,1) \} \sqsubseteq \{ (0,1), (4,24) \} \sqsubseteq \{ (0,1), (1,1), (4,24) \} \sqsubseteq \{ (0,1), (1,1), (3,6), (4,24) \} \sqsubseteq \{ (0,1), (1,1), (2,2), (3,6), (4,24) \} \sqsubseteq \ldots$$

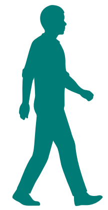
which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$\emptyset \sqsubseteq \{ (1,1) \} \sqsubseteq \{ (1,1), (2,4) \} \sqsubseteq \{ (1,1), (2,4), (3,81) \} \sqsubseteq \{ (1,1), (2,4), (3,81), (4,256) \} \sqsubseteq \ldots$$

which function(s) are we approximating?

 Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$\emptyset \sqsubseteq \{ (1,6) \} \sqsubseteq \left\{ \begin{array}{l} (1,6), \\ (2,28) \end{array} \right\} \sqsubseteq \left\{ \begin{array}{l} (1,6), \\ (2,28), \\ (3,496) \end{array} \right\} \sqsubseteq \left\{ \begin{array}{l} (1,6), \\ (2,28), \\ (3,496), \\ (4,8128) \end{array} \right\} \sqsubseteq \ldots$$

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$\emptyset \sqsubseteq \{ (4,2) \} \sqsubseteq \{ (4,2), (6,3) \} \sqsubseteq \{ (4,2), (6,3), (8,4) \} \sqsubseteq \{ (4,2), (6,3), (8,4), (9,3) \} \sqsubseteq \{ (4,2), (6,3), (8,4), (9,3), (10,5) \} \sqsubseteq \ldots$$

which function(s) are we approximating?

Functional property

$\mathbf{Pf}(A,B) = \{f : A \rightharpoonup B\}$ partial functions

$$\mathbf{Pf}(A,B) = \{f \subseteq A \times B \mid \boxed{\forall a \in A. \forall b_1, b_2 \in B. (a, b_1) \in f \wedge (a, b_2) \in f \Rightarrow b_1 = b_2}\}$$

functional property

$f(a)\downarrow \triangleq \exists b \in B.\ (a,b) \in f$ function f is defined on a

$$f \sqsubseteq g \Leftrightarrow (\forall a \in A. \quad f(a)\downarrow \ \Rightarrow \ (g(a)\downarrow \wedge f(a) = g(a))\,)$$
$$\Leftrightarrow f \subseteq g$$

$(\mathbf{Pf}(A,B), \sqsubseteq)$ is a PO with bottom

what is bottom?

is it complete?

the empty relation
(the function always undefined)

Is Pf complete?

$(\mathbf{Pf}(A,B), \sqsubseteq)$

complete?

Given a chain $\{f_i\}_{i\in\mathbb{N}}$ let us consider $\bigcup_{i\in\mathbb{N}} f_i \subseteq A \times B$

we want to prove that $\bigcup_{i\in\mathbb{N}} f_i \in \mathbf{Pf}(A,B)$

i.e. that $f = \bigcup_{i\in\mathbb{N}} f_i$ satisfies the functional property

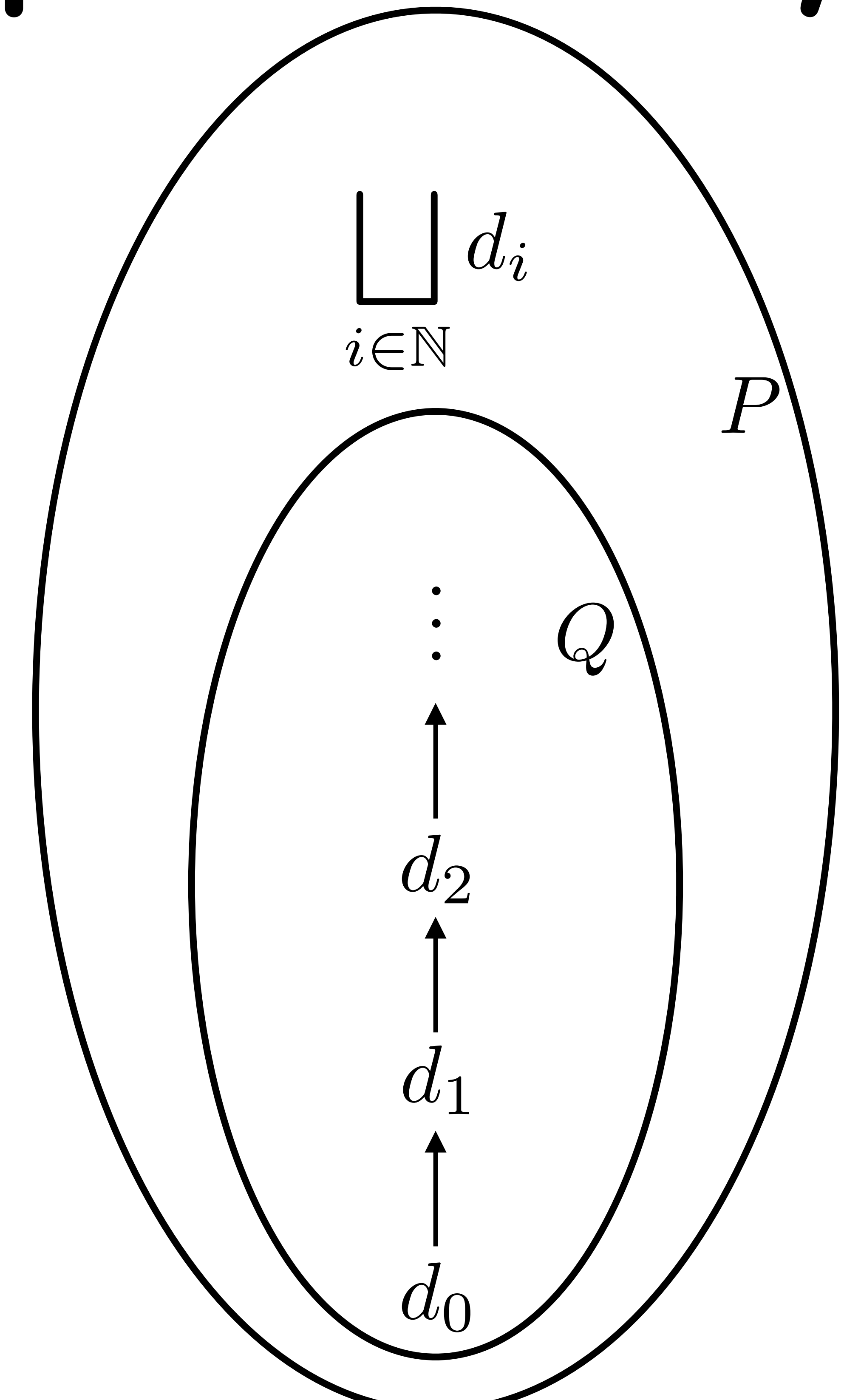
we know that each f_i is functional

$$\forall i \in \mathbb{N}.\forall a \in A.\forall b_1, b_2 \in B.(a, b_1) \in f_i \wedge (a, b_2) \in f_i \Rightarrow b_1 = b_2$$

we need to prove f is functional

$$\forall a \in A.\forall b_1, b_2 \in B.(a, b_1) \in f \wedge (a, b_2) \in f \Rightarrow b_1 = b_2$$

pictorially



is the limit in Q ?

Pf is complete

we need to prove f is functional

$$\forall a \in A. \forall b_1, b_2 \in B. (a, b_1) \in f \wedge (a, b_2) \in f \Rightarrow b_1 = b_2$$

Take $a \in A, b_1, b_2 \in B$ such that $(a, b_1) \in f \wedge (a, b_2) \in f$

we need to prove $b_1 = b_2$

$$(a, b_1) \in f = \bigcup_{i \in \mathbb{N}} f_i \Leftrightarrow \exists k \in \mathbb{N}. (a, b_1) \in f_k$$

$$m \triangleq \max\{k, h\}$$

$$(a, b_2) \in f = \bigcup_{i \in \mathbb{N}} f_i \Leftrightarrow \exists h \in \mathbb{N}. (a, b_2) \in f_h$$

Clearly $f_k \subseteq f_m$ $f_h \subseteq f_m$ f_m is functional

$$(a, b_1) \in f_m \quad (a, b_2) \in f_m \quad \Rightarrow \quad b_1 = b_2$$

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$
\begin{array}{ll}
f_0 & \emptyset \\
\subseteq & \{(0,1)\} \quad f_1 \\
\subseteq & \{(0,1),(1,1)\} \quad f_2 \\
\subseteq & \{(0,1),(1,1),(2,2)\} \quad f_3 \\
\subseteq & \{(0,1),(1,1),(2,2),(3,6)\} \quad f_4 \\
\subseteq & \{(0,1),(1,1),(2,2),(3,6),(4,24)\} \quad f_5 \\
\subseteq & \cdots
\end{array}
$$

$\bigcup_{i \in \mathbb{N}} f_i$ is (maybe) the factorial function

note: the limit of partial functions can be a total function

Total functions

$\mathbf{Tf}(A, B) = (A \to B)$ total functions

$\mathbf{Pf}(A, B) \equiv \mathbf{Tf}(A, B_\perp)$ $B_\perp \triangleq B \uplus \{\perp\}$

$\sqsubseteq_{B_\perp} \triangleq$ flat order

$$f \sqsubseteq g \Leftrightarrow \forall a \in A.\ f(a) \sqsubseteq_{B_\perp} g(a)$$

PO? immediate to check

bottom? $f_\perp(a) = \perp$ for any $a \in A$

complete? we will prove it later

(as an instance of a more general result)

$$(\bigsqcup_{i \in \mathbb{N}} f_i)(a) \triangleq \bigsqcup_{i \in \mathbb{N}} f_i(a)$$ (flat order, limit exists)