

Master Program in *Data Science and Business Informatics*

# Statistics for Data Science

Lesson 34 - Fitting distributions. Testing independence/association

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# Distribution fitting and quality of fitting

- Dataset  $x_1, \dots, x_n$  realization of  $X_1, \dots, X_n \sim F$
- **Distribution fitting:** What is a plausible  $F$ ?
  - ▶ Useful in Data Science for understanding the data generation process, for checking assumptions (e.g., normality of noise in LR), for checking data distribution changes, etc.
  - ▶ Parametric approaches:

- Assume  $F = F(\lambda)$  for some family  $F$ , and estimate  $\lambda$  as  $\hat{\lambda}$

- Maximum Likelihood Estimation (point estimate):

[See Lesson 19]

$$\hat{\lambda} = \operatorname{argmax}_{\lambda} L(\lambda)$$

- Parametric bootstrap ( $p$ -value):

[See Lesson 28]

$$T_{ks} = \sup_{a \in \mathbb{R}} |F_n^*(a) - F_{\hat{\lambda}^*}(a)|$$

- ▶ Non-parametric approaches:

- Empirical distribution  $F_n$

[Glivenko-Cantelli Thm]

- Kernel Density Estimation

[See Lesson 15]

- **Quality of fitting:** Among several fits  $F_1, \dots, F_k$ , which one is the best?
  - ▶ Goodness of fit: measure of how good/bad is  $F_i$  in fitting the data?
  - ▶ Comparison: which one between two  $F_1$  and  $F_2$  is better?

# Quality of fitting

- Loss functions (to be minimized)
  - ▶ Akaike information criterion (AIC), balances model fit against model simplicity

$$AIC(F(\lambda)) = 2|\lambda| - 2\ell(\lambda)$$

- ▶ Bayesian information criterion (BIC), stronger balances over model simplicity

$$BIC(F(\lambda)) = |\lambda| \log n - 2\ell(\lambda)$$

- Statistics (continuous data):

- ▶ **KS test**  $H_0 : X \sim F$     $H_1 : X \not\sim F$  with Kolmogorov-Smirnov (KS) statistic:

$$D = \sup_{a \in \mathbb{R}} |F_n(a) - F(a)| \sim K$$

- ▶ **LR test**  $H_0 : X \sim F_1$     $H_1 : X \sim F_2$  with the likelihood-ratio test:

$$\lambda_{LR} = \log \frac{L(F_1(\lambda_1))}{L(F_2(\lambda_2))} = \ell(F_1(\lambda_1)) - \ell(F_2(\lambda_2)) \quad \text{with} \quad -2\lambda_{LR} \sim \chi^2(1)$$

**See R script**

# Quality of fitting

- Statistics (discrete data):

- ▶ **Pearson's Chi-Square test**

$H_0 : X \sim F$     $H_1 : X \not\sim F$  with  $\chi^2$  statistic:

$$\chi^2 = \sum_{N_i > 0} \frac{(N_i - n_i)^2}{n_i} = n \cdot \sum_{N_i > 0} \frac{(N_i/n - p(i))^2}{p(i)} \sim \chi^2(df)$$

where  $N_i$  number of observations of value  $i$ ,  $n_i = n \cdot p(i)$  expected number of observations (rescaled), and  $df = |\{i \mid N_i > 0\}| - 1$  is the number of observed values minus 1.

$\chi^2 = \infty$  if for some  $i$ :  $n_i = 0$

- ▶ **Yates's correction for continuity**

It corrects for approximating the discrete probability of observed frequencies by the continuous chi-squared distribution

$$\chi^2 = \sum_{N_i > 0} \frac{(|N_i - n_i| - 0.5)^2}{n_i}$$

It increases Type II error, so do not use it!

**See R script**

# Comparing two datasets

- Dataset  $x_1, \dots, x_n$  realization of  $X_1, \dots, X_n \sim F_1$
- Dataset  $y_1, \dots, y_m$  realization of  $Y_1, \dots, Y_m \sim F_2$
- $H_0 : F_1 = F_2$      $H_1 : F_1 \neq F_2$ 
  - ▶ Useful to detect **covariate drift** (data stability) from source to target datasets
- Univariate data:
  - ▶ Continuous data: KS statistics  $D = \sup_{a \in \mathbb{R}} |F_1(a) - F_2(a)| \sim K$
  - ▶ Discrete data:  $\chi^2$  statistics

$$\chi^2 = \sum_{R_i > 0 \vee S_i > 0} \frac{(\sqrt{\frac{m}{n}} R_i - \sqrt{\frac{n}{m}} S_i)^2}{R_i + S_i} \sim \chi^2(df)$$

where  $R_i$  (resp.,  $S_i$ ) is the number of variables in  $X_1, \dots, X_n$  (resp.,  $Y_1, \dots, Y_m$ ) which are equal to  $i$ ,  $df = |\{i \mid R_i > 0 \vee S_i > 0\}| - 1$

- ▶ Other tests in the R package **twosamples**
- Multivariate data: see classifier 2-sample test and others in the R package **Ecume**

**See R script**

# Testing independence:

- **Pearson's Chi-Square test** of independence
- $X$  and  $Y$  discrete (finite) distributions
- $(x_1, y_1) \dots, (x_n, y_n)$  bivariate observed dataset
- $H_0 : X \perp\!\!\!\perp Y$      $H_1 : X \not\perp\!\!\!\perp Y$
- Test statistic:

$$\chi^2 = \sum_{i,j} \frac{(O_{i,j} - E_{i,j})^2}{E_{i,j}} = n \sum_{i,j} \frac{(O_{i,j}/n - p_{i,\cdot} p_{\cdot,j})^2}{p_{i,\cdot} p_{\cdot,j}} \sim \chi^2(df)$$

where  $O_{i,j}$  is the number of observations of value  $X = i$  and  $Y = j$ ,  $E_{i,j} = np_{i,\cdot} p_{\cdot,j}$  where  $p_{i,\cdot} = \sum_j O_{i,j}/n$  and  $p_{\cdot,j} = \sum_i O_{i,j}/n$ .  $df = (n_x - 1)(n_y - 1)$  where  $n_x$  (resp.,  $n_y$ ) is the size of the support of  $X$  (resp.,  $Y$ )

- Exact test when  $n$  is small: **Fisher's exact test**
- Paired data (e.g., before and after taking a drug): **McNemar's test**

**See R script**

# The G-test and Mutual Information

- **G-test** of independence
- $X$  and  $Y$  discrete (finite) distributions
- $(x_1, y_1) \dots, (x_n, y_n)$  bivariate observed dataset
- $H_0 : X \perp\!\!\!\perp Y$      $H_1 : X \not\perp\!\!\!\perp Y$
- Test statistics:

$$G = 2 \sum_{i,j} O_{i,j} \log \frac{O_{i,j}}{E_{i,j}} = 2 \sum_{i,j} O_{i,j} \log \frac{O_{i,j}}{np_{i,\cdot}p_{\cdot,j}} \sim \chi^2(df)$$

where  $O_{i,j}$  is the number of observations of value  $X = i$  and  $Y = j$ ,  $E_{i,j} = np_{i,\cdot}p_{\cdot,j}$  where  $p_{i,\cdot} = \sum_j O_{i,j}/n$  and  $p_{\cdot,j} = \sum_i O_{i,j}/n$ .  $df = (n_x - 1)(n_y - 1)$  where  $n_x$  (resp.,  $n_y$ ) is the size of the support of  $X$  (resp.,  $Y$ )

- Preferable to Chi-Squared when numbers ( $O_{ij}$  or  $E_{ij}$ ) are small, asymptotically equivalent
- $G = 2 \cdot n \cdot I(O, E)$  where  $I(O, E)$  is the mutual information between  $O$  and  $E$     [See Lesson 16]

**See R script**

# Other tests of independence

- $(x_1, y_1) \dots, (x_n, y_n)$  bivariate observed dataset
- Permutation tests:
  - ▶ reduces to comparing two datasets:  $(x_1, y_1) \dots, (x_n, y_n)$  and  $(x_1, y_{\pi_1}) \dots, (x_n, y_{\pi_n})$ , where  $\pi_1, \dots, \pi_n$  is a permutation of  $1, \dots, n$  *[see slide on comparing two datasets]*
- Continuous  $X$  and  $Y$ :
  - ▶ discretize both  $X$  and  $Y$  and then apply independence tests for discrete r.v.'s, or
  - ▶ test correlation (see later), or
  - ▶ **Hoeffding's test**, see R package **independence**
- Continuous  $X$  and discrete  $Y$ :
  - ▶ discretize  $X$  and then apply independence tests for discrete r.v.'s, or
  - ▶ a direct approach **Yang and Kim**, or
  - ▶ special case  $Y$  binary:  $X \perp\!\!\!\perp Y$  iff  $P(X|Y) = P(X)$  iff  $P(X|Y = 0) = P(X|Y = 1)$  *[see slide on comparing two datasets]*



# Measures of association (from Lesson 16)

- *Association*: one variable provides information on the other
  - ▶  $X \perp\!\!\!\perp Y$  independent, i.e.,  $P(X|Y) = P(X)$ : zero information
  - ▶  $Y = f(X)$  deterministic association with  $f$  invertible: maximum information
- *Correlation*: the two variables show an increasing/decreasing trend
  - ▶  $X \perp\!\!\!\perp Y$  implies  $Cov(X, Y) = 0$
  - ▶ the converse is not always true

Variable $Y$	Variable $X$		
	Nominal	Ordinal	Continuous
Nominal	$\phi$ or $\lambda$	Rank biserial	Point biserial
Ordinal	Rank biserial	$\tau_b$ or Spearman	$\tau_b$ or Spearman
Continuous	Point biserial	$\tau_b$ or Spearman	Pearson or Spearman

$\phi$  = phi coefficient,  $\lambda$  = Goodman and Kruskal's lambda,  
 $\tau_b$  = Kendall's  $\tau_b$ .

# Association between nominal variables: Pearson $\chi^2$ -based

- $\phi$  **coefficient** (or MCC, Matthews correlation coefficient)

▶ For  $2 \times 2$  contingency tables:

*[Exercise. Show  $\phi = |r_{xy}|$ ]*

$$\phi = \sqrt{\frac{\chi^2}{n}} \in [0, 1]$$

- **Cramer's  $V$**

▶ For contingency tables larger than  $2 \times 2$ :

$$V = \sqrt{\frac{\chi^2}{n \cdot \min\{r-1, c-1\}}} \in [0, 1]$$

where  $r$  and  $c$  are the number of rows and columns

- **Tschuprov's  $T$**

*[same as  $V$  if  $r = c$ ]*

▶ For contingency tables larger than  $2 \times 2$ :

$$T = \sqrt{\frac{\chi^2}{n \cdot \sqrt{(r-1)(c-1)}}} \in [0, 1]$$

where  $r$  and  $c$  are the number of rows and columns

**See R script**

# Testing correlation: continuous data

- Population correlation:

$$\rho = \frac{E[(X - \mu_X) \cdot (Y - \mu_Y)]}{\sigma_X \cdot \sigma_Y}$$

- Pearson's correlation coefficient:

$$r = \frac{\sum_{i=1}^n (x_i - \bar{x}) \cdot (y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2 \cdot \sum_{i=1}^n (y_i - \bar{y})^2}}$$

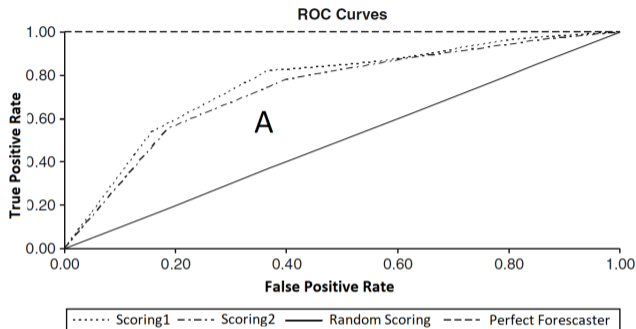
- Assumption: joint distribution of  $X, Y$  is bivariate normal (or large sample)
- $(x_1, y_1) \dots, (x_n, y_n)$  bivariate observed dataset
- $H_0 : \rho = 0 \quad H_1 : \rho \neq 0$
- Test statistics:

$$T = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}} \sim t(n-2)$$

- ▶ Recall that  $X \perp\!\!\!\perp Y$  implies  $\rho = 0$ : if  $H_0$  can be rejected, then  $X \perp\!\!\!\perp Y$  can be rejected

**See R script**

# Testing AUC-ROC



- Binary classifier score  $s_{\theta}(w) \in [0, 1]$  where  $s_{\theta}(w)$  estimate  $\eta(w) = P_{\theta_{TRUE}}(C = 1|W = w)$

- ROC Curve

- ▶  $TPR(p) = P(s_{\theta}(w) \geq p|C = 1)$  and  $FPR(p) = P(s_{\theta}(w)|C = 0)$
- ▶ ROC Curve is the scatter plot  $TPR(p)$  over  $FPR(p)$  for  $p$  ranging from 1 down to 0
- ▶ AUC-ROC is the area below the curve

**What does AUC-ROC estimate?**

- ▶ Linearly related to Somer's D correlation index (a.k.a. Gini coefficient) [See Lesson 16]

# Testing AUC-ROC

- AUC is the probability of correct identification of the order between two instances:

$$AUC = P_{\theta_{TRUE}}(s_{\theta}(W1) < s_{\theta}(W2) | C_{W1} = 0, C_{W2} = 1)$$

where  $(W1, C_{W1}) \sim f_{\theta_{TRUE}}$  and  $(W2, C_{W2}) \sim f_{\theta_{TRUE}}$

- $s_{\theta}(W_1), \dots, s_{\theta}(W_n) \sim F_{\theta_{TRUE}} |_{C=1}$  and  $s_{\theta}(V_1), \dots, s_{\theta}(V_m) \sim F_{\theta_{TRUE}} |_{C=0}$

$$U = \sum_{i=1}^n \sum_{j=1}^m S(s_{\theta}(W_i), s_{\theta}(V_j)) \quad S(X, Y) = \begin{cases} 1 & \text{if } X > Y \\ 1/2 & \text{if } X = Y \\ 0 & \text{if } X < Y \end{cases}$$

▶  $AUC\text{-}ROC = U/(n \cdot m)$  is an estimator of  $AUC$

- $U$  statistics of the **Wilcoxon rank-sum test** [See Lesson 31]
- Normal approximation, **DeLong's algorithm**, bootstrap, **Brunner-Munzel** tests and confidence intervals

See R script