Master Program in *Data Science and Business Informatics*

Statistics for Data Science

Lesson 09 - Expectation and variance. Computations with random variables

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Expectation of a discrete random variable

- Buy lottery ticket every week, $p = 1/10000$, what is probability of winning at k^{th} week?

$$X \sim Geo(p) \quad P(X = k) = (1-p)^{k-1} \cdot p \text{ for } k = 1, 2, \ldots$$

- What is the average number of weeks to wait (expected) before winning?

$$E[X] = \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} \cdot p = \frac{1}{p}$$

because $\sum_{k=1}^{\infty} k \cdot x^{k-1} = 1/(1-x)^2$

DEFINITION. The *expectation* of a discrete random variable X taking the values $a_1, a_2, \ldots$ and with probability mass function p is the number

$$\mathrm{E}[X] = \sum_i a_i \mathrm{P}(X = a_i) = \sum_i a_i p(a_i).$$

- Expected value, mean value (weighted by probability of occurrence), center of gravity

See **seeing-theory.brown.edu**

Expected value may be infinite or may not exist!

- Fair coin: win 2^k euros if first H appears at k^{th} toss **[St. Petersburg paradox]**
  - X with p.m.f. $p(2^k) = 2^{-k}$ for $k = 1, 2, \ldots$
  - $p()$ is a p.m.f. since $\sum_{k=1}^{\infty} 2^{-k} = 1$ using $\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$ for $|a| < 1$
  - $E[X] = \sum_{k=1}^{\infty} 2^k \cdot 2^{-k} = \sum_{k=1}^{\infty} 1 = \infty$
- Expectation does not exist when $\sum_i a_i p(a_i)$ does not converge
  - X with p.m.f. $p(2^k) = p(-2^k) = 2^{-k}$ for $k = 2, 3, \ldots$
  - $E[X] = \sum_{k=2}^{\infty}(2^k \cdot 2^{-k} - 2^k \cdot 2^{-k}) = \sum_{k=2}^{\infty}(1-1) = 0$ *wrong!*
  - $E[X] = \sum_{k=2}^{\infty} 2^k \cdot 2^{-k} - \sum_{k=2}^{\infty} 2^k \cdot 2^{-k} = \infty - \infty$ *undefined*
  - $E[X]$ is finite if $\sum_i |a_i| p(a_i) < \infty$
  - In the case above, $\sum_{k=2}^{\infty}(|2^k| \cdot 2^{-k} + |-2^k| \cdot 2^{-k}) = \infty$

Expectation of some other discrete distributions

- Expectation of some other discrete distributions
  - $X \sim U(m, M) \quad E[X] = {}^{(m+M)}/_{2}$
    - $\sum_{i=m}^{M} \frac{i}{M-m+1} = \frac{1}{M-m+1} \sum_{i=0}^{M-m} (m+i) = m + (M-m)/2 = \frac{m+M}{2}$
  - $X \sim Ber(p) \quad E[X] = p$
    - $0 \cdot (1-p) + 1 \cdot p = p$ *[Mean may not belong to the support]*
  - $X \sim Bin(n, p) \quad E[X] = n \cdot p$
    - Because . . . we'll see later
  - $X \sim NBin(n, p) \quad E[X] = \frac{n \cdot p}{1-p}$
    - Because . . . we'll see later
  - $X \sim Poi(\mu) \quad E[X] = \mu$
    - Because, when $n \to \infty$: $Bin(n, {}^{\mu}/_{n}) \to Poi(\mu)$

Expectation of a continuous random variable

DEFINITION. The *expectation* of a continuous random variable X with probability density function f is the number

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} x f(x)\, \mathrm{d}x.$$

- Expectation of some continuous distributions
  - $X \sim U(\alpha, \beta) \quad E[X] = (\alpha + \beta)/2$
  - $X \sim Exp(\lambda) \quad E[X] = {}^1/_\lambda$
    - Because $\int_0^\infty x\lambda e^{-\lambda x} dx = \left[-e^{-\lambda x}(x + {}^1/_\lambda)\right]_0^\infty = e^0(0 + {}^1/_\lambda)$ *[See Lesson 06]*
  - $X \sim N(\mu, \sigma^2) \quad E[X] = \mu$
    - Because: $\int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx = \mu + \int_{-\infty}^{\infty} (x-\mu) \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx =_{z=\frac{x-\mu}{\sigma}}$
      $= \mu + \sigma \int_{-\infty}^{\infty} z \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = \mu$
  - $X \sim Erl(n, \lambda) \quad E[X] = {}^n/_\lambda$
    - Because . . . we'll see later

Expected value may not exists!

- Cauchy distribution

$$f(x) = \frac{1}{\pi(1+x^2)}$$

- $X_1, X_2 \sim N(0,1)$ i.i.d., $X = X_1/X_2 \sim Cau(0,1)$

$$E[X] = \int_{-\infty}^{0} xf(x)dx + \int_{0}^{\infty} xf(x)dx$$

- $\int_{-\infty}^{0} xf(x)dx = \left[\frac{1}{2\pi}\log(1+x^2)\right]_{-\infty}^{0} = -\infty$
- $\int_{0}^{\infty} xf(x)dx = \left[\frac{1}{2\pi}\log(1+x^2)\right]_{0}^{\infty} = \infty$

$$E[X] = -\infty + \infty$$

- $E[X]$ is finite if $\int_{-\infty}^{\infty} |x|f(x)dx < \infty$

Mean value does not always make sense in your data analytics project!

$E[g(X)] \neq g(E[X])$

- Recall that *velocity* $=$ *space*/*time*, and then *time* $=$ *space*/*velocity*!
- Vector `v` of speed (Km/h) to reach school and their probabilities `p` using feet, bike, bus, train:

```
v = c(5, 10, 20, 30)    p = c(0.1, 0.4, 0.25, 0.25)
```

- Distance house-schools is 2 Km
- What is the average time to reach school?
  - `2/sum(v*p)` i.e., *space/E[velocity]*
  - `sum(2/v*p)` i.e., *E[space/velocity]*
- $X =$ velocity, $g(X) = 2/X$ time to reach school
  - $E[g(X)] \neq g(E[X])$

The change of variable formula (or rule of the lazy statistician)

- $X \sim U(0, 10)$, width of a square field, $E[X] = 5$
- $g(X) = X^2$ is the area of the field, $E[g(X)] = ?$ $\quad [E[g(X)] \neq g(E[X])]$
- $F_g(a) = P(g(X) \leq a) = P(X \leq \sqrt{a}) = \sqrt{a}/10$ for $0 \leq a \leq 100$
- Hence, $f_g(a) = {dF_g(a)}/{da} = 1/20\sqrt{a}$ *[later on, a general theorem]*
- $E[g(X)] = \frac{1}{20}\int_0^{100} \frac{x}{\sqrt{x}} dx = \frac{1}{20}\frac{2}{3}\left[x^{3/2}\right]_0^{100} = 100/3$
- Alternatively, $E[g(X)] = \int_0^{10} x^2 \frac{1}{10} dx = \frac{1}{10}\frac{1}{3}\left[x^3\right]_0^{10} = 100/3$

THE CHANGE-OF-VARIABLE FORMULA. Let X be a random variable, and let $g : \mathbb{R} \to \mathbb{R}$ be a function.
If X is discrete, taking the values $a_1, a_2, \ldots$, then

$$\mathrm{E}\left[g(X)\right] = \sum_i g(a_i)\mathrm{P}(X = a_i)\,.$$

If X is continuous, with probability density function f, then

$$\mathrm{E}\left[g(X)\right] = \int_{-\infty}^{\infty} g(x)f(x)\,\mathrm{d}x.$$

See R script

Change of units

Theorem (Change of units)

$$E[rX + s] = rE[X] + s$$

- Example: for $Y = 1.8X + 32$, we have $E[Y] = 1.8E[X] + 32$ *[Celsius to Fahrenheit]*
- **Corollary.**

$$E[X - E[X]] = E[X] - E[X] = 0$$

- **Theorem.** Expectation minimizes the square error, i.e., for $a \in \mathbb{R}$:

$$E[(X - E[X])^2] \leq E[(X - a)^2]$$

  - Proof. (sketch) set $\frac{d}{da} \int_{-\infty}^{\infty} (x - a)^2 f(x) dx = 0$

Entropy of a random variable

- The **Shannon's information entropy** is the average level of "information", "surprise", or "uncertainty" inherent to the variable's possible outcomes
  - Information is inversely proportional to probability $\frac{1}{p(a_i)}$
    - Highly likely events carry very little new information
    - Highly unlikely events carry more information
  - Information content $ic()$ of two independent events should sum up $\log \frac{1}{p(a_i)}$
    - $ic(p(A \cap B)) = ic(p(A)) + ic(p(B)) = ic(p(A)p(B))$
    - $ic(p(\Omega)) = ic(1) = 0$
    - $ic(p(A)) \geq 0$
- $H(X) = E[-\log p(X)]$ $(discrete)$ $\qquad$ $H(X) = E[-\log f(X)]$ $(continuous)$

$$H(X) = -\sum_i p(a_i) \log p(a_i) \qquad\qquad H(X) = -\int_{-\infty}^{\infty} f(x) \log f(x) dx$$

  - For X discrete, $H(X) \geq 0$ since $-\log p(X) = \log {}^1/_{p(X)} \geq 0$
    - reached when $p(a_1) = 1$ and $p(a_i) = 0$ for $i \neq 1$
  - For $X \sim Ber(p)$, $H(X) = -p \log p - (1-p) \log (1-p)$

See R script

Computation with discrete random variables

Theorem

For a discrete random variable X, the p.m.f. of $Y = g(X)$ is:

$$P_Y(Y = y) = \sum_{g(x)=y} P_X(X = x) = \sum_{x \in g^{-1}(y)} P_X(X = x)$$

- **Proof.** $\{Y = y\} = \{g(X) = y\} = \{x \in g^{-1}(y)\}$
- **Corollary** (the change-of-variable formula):

$$E[g(X)] = \sum_y y P_Y(Y = y) = \sum_y y \sum_{g(x)=y} P_X(X = x) = \sum_x g(x) P_X(X = x)$$

Example

- $X \sim U(1, 200)$ number of tickets sold
- Capacity is 150
- $Y = max\{X - 150, 0\}$ overbooked tickets

$$P_Y(Y = y) = \begin{cases} 150/200 & \text{if } y = 0 \\ 1/200 & \text{if } 1 \leq y \leq 50 \end{cases} \quad \begin{matrix} g^{-1}(0) = \{1, \ldots, 150\} \\ g^{-1}(y) = \{y + 150\} \end{matrix}$$

- Hence:

$$E[Y] = 0 \cdot \frac{150}{200} + \frac{1}{200} \cdot \sum_{y=1}^{50} y = 6.375$$

- or using the change-of-variable formula:

$$E[Y] = \frac{1}{200} \cdot \sum_{x=1}^{200} max\{X - 150, 0\} = \frac{1}{200} \cdot \sum_{x=151}^{200} (X - 150) = 6.375$$

Computation with continuous random variables

Theorem

For a continuous random variable X, the density functions of $Y = g(X)$ when $g()$ is increasing/decreasing are:

$$F_Y(y) = F_X(g^{-1}(y)) \qquad f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right|$$

- **Proof.** (for $g()$ increasing) Since $g()$ is invertible and $g(x) \leq y$ iff $x \leq g^{-1}(y)$:

$$F_Y(y) = P_Y(g(X) \leq y) = P_X(X \leq g^{-1}(y)) = F_X(g^{-1}(y))$$

and then:

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{dF_X(g^{-1}(y))}{dy} = \frac{dF_X(g^{-1}(y))}{dg^{-1}} \frac{dg^{-1}(y)}{dy} = f_X(g^{-1}(y)) \frac{dg^{-1}(y)}{dy}$$

Exercise: show the case $g()$ decreasing!

Change of units

CHANGE-OF-UNITS TRANSFORMATION. Let X be a continuous random variable with distribution function F_X and probability density function f_X. If we change units to $Y = rX + s$ for real numbers $r > 0$ and s, then

$$F_Y(y) = F_X\left(\frac{y-s}{r}\right) \quad \text{and} \quad f_Y(y) = \frac{1}{r} f_X\left(\frac{y-s}{r}\right).$$

- For $X \sim N(\mu, \sigma^2)$, how is $Z = \frac{X}{\sigma} + \frac{-\mu}{\sigma} = \frac{X-\mu}{\sigma}$ distributed?
- $f_Z(z) = \sigma f_X(\sigma y + \mu) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$
- Hence, $Z \sim N(0, 1)$
- In particular, for $X \sim N(\mu, \sigma^2)$, we have:

$$P(X \leq a) = P(Z \leq \frac{a-\mu}{\sigma}) = \Phi(\frac{a-\mu}{\sigma})$$

Example: $\Lambda(\mu, \sigma^2)$

Log-normal distribution $Y = e^X$ for $X \sim N(\mu, \sigma^2)$, i.e., $log(Y) \sim N(\mu, \sigma^2)$

- $Y = g(X) = e^X$ Support is $]0, \infty[$
- $g(x) = e^x$ is increasing, and $g^{-1}(y) = \log y$, and $\frac{dg^{-1}(y)}{dy} = \frac{1}{y}$

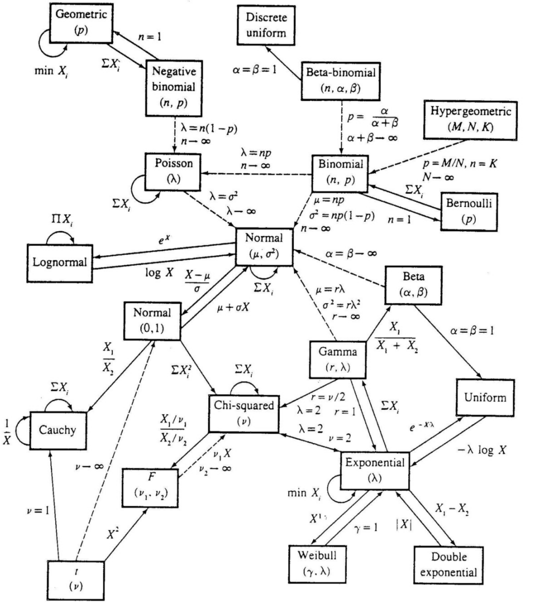
$$F_Y(y) = F_X(g^{-1}(y)) = \Phi(\frac{\log y - \mu}{\sigma}) \qquad f_Y(y) = f_X(g^{-1}(y))\frac{dg^{-1}(y)}{dy} = \frac{1}{y\sigma\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{\log y - \mu}{\sigma})^2}$$

- $E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx = \int_{-\infty}^{\infty} y f_Y(y) dy = e^{\mu + \sigma^2/2}$
- Plausible and empirically adequate model for:
  - length of comments in posts, dwell time reading online articles, length of chess games, ...
  - size of living tissue, number of hospitalized cases in epidemics, blood pressure, ...
  - income of 97%–99% of the population, the number of citations, log of city size, ...
  - times to repair a maintainable system, size of audio-video files, amount of internet traffic per unit time, ...

See R script

Common distributions

- **Probability distributions at Wikipedia**
- **Probability distributions in R**
- C. Forbes, M. Evans,
  N. Hastings, B. Peacock (2010)
  Statistical Distributions, 4th Edition
  Wiley



Relationships among common distributions. Solid lines represent transformations and special cases, dashed lines represent limits. Adapted from Leemis (1986).

Example

- $X \sim U(0,1)$ radius $\quad f_X(x) = 1 \quad F_X(x) = x$ for $x \in [0,1]$
- $Y = g(X) = \pi \cdot X^2$ Support is $[0, \pi]$
- $g(x) = \pi x^2$ is increasing, and $g^{-1}(y) = \sqrt{\frac{y}{\pi}}$, and $\frac{dg^{-1}(y)}{dy} = \frac{1}{2\sqrt{\pi y}}$

$$F_Y(y) = F_X(g^{-1}(y)) = \sqrt{\frac{y}{\pi}} \qquad f_Y(y) = f_X(g^{-1}(y))\frac{dg^{-1}(y)}{dy} = \frac{1}{2\sqrt{\pi y}}$$

Do not lift distributions from a data column
to a derived column in your data analytics project!
See R script

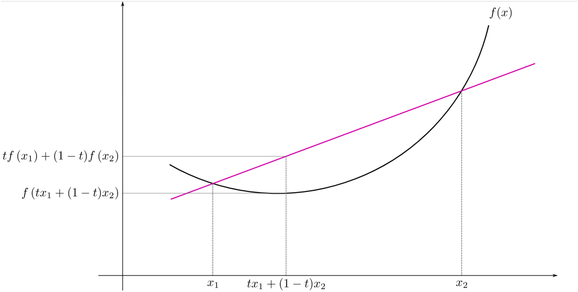
- Notice that: $g(E[X]) = \pi/4 \leq E[g(X)] = \int_0^1 g(x) f_X(x) dx = \int_0^\pi y f_Y(y) dy = \frac{\pi}{3}$

Jensen's inequality

JENSEN'S INEQUALITY. Let g be a convex function, and let X be a random variable. Then

$$g(\mathrm{E}[X]) \leq \mathrm{E}[g(X)].$$

- $f()$ is convex if $f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$ for $t \in [0, 1]$



- if $f''(x) \geq 0$ then $f()$ is convex, e.g., $g(x) = \pi x^2$ or $g(x) = 1/x$ for $x \geq 0$

Corollary and Example

Corollary (see [T, Ex. 8.11]. For a concave function g, namely $g''(x) \leq 0$: $g(E[X]) \geq E[g(X)]$

- $\log(x)$ is concave since $\log''(x) = -1/x^2 \leq 0$
- Let X be discrete with finite domain of n elements
  - By corollary above:
$$H(X) = E[\log \frac{1}{p(X)}] \leq \log E[\frac{1}{p(X)}]$$
  - By change of variable:
$$E[\frac{1}{p(X)}] = \sum_i \frac{p(a_i)}{p(a_i)} = n$$
    and then maximum entropy is:
$$H(X) \leq \log n$$
  - E.g., $X \sim Ber(p)$, maximum entropy (uncertainty) for equiprobable events $p = 1/2$

Variance

- **Investment A.** $P(X = 450) = 0.5 \quad P(X = 550) = 0.5 \quad E[X] = 500$
- **Investment B.** $P(X = 0) = 0.5 \quad P(X = 1000) = 0.5 \quad E[X] = 500$
- Spread around the mean is important!

Variance and standard deviations

The *variance* $Var(X)$ of a random variable X is the number:

$$Var(X) = E[(X - E[X])^2]$$

$\sigma_X = \sqrt{Var(X)}$ is called the *standard deviation* of X.

- The standard deviation has the same dimension as $E[X]$ (and as X)
- For X discrete, $Var(X) = \sum_i (a_i - E[X])^2 p(a_i)$
- **Investment A.** $Var(X) = 50^2$ and $\sigma_X = 50$
- **Investment B.** $Var(X) = 500^2$ and $\sigma_X = 500$

Examples

- For $a \in \mathbb{R}$:

$$E[|X - a|] \leq \sqrt{E[(X - a)^2]}$$

  - Apply Jensen's ineq. for $g(y) = y^2$ convex on the r.v. $Y = |X - a|$

- Median minimizes absolute deviation, i.e., for $a \in \mathbb{R}$:

$$E[|X - m_X|] \leq E[|X - a|]$$

  - **Prove it!** (for continuous functions) Hint: $\frac{d}{dx}|x| = x/|x|$

- Maximum distance between expectation and median:

$$|E[X] - m_X| \leq E[|X - m_X|] \leq E[|X - E[X]|] \leq \sqrt{E[(X - E[X])^2]} = \sigma_X$$

  - Apply Jensen's ineq. for $g(y) = |y|$ convex on the r.v. $Y = X - m_X$ plus two results above

Mode

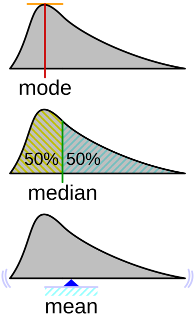
- For discrete r.v. X with p.m..f. $p()$: the values a such that $p(a)$ is maximum, i.e.:

$$\arg\max_a p(a)$$

  - Can be more than one, e.g., in $Ber(0.5)$
- For continuous r.v. X with d.f. $f()$: the values x such that $f(x)$ is a local maximum, e.g.:

$$f'(x) = 0 \quad \text{and} \quad f''(x) < 0$$

  - Notice: **local** maximum!
- Unimodal distribution = that have only one mode



Variance

Theorem. $Var(X) = E[X^2] - E[X]^2$
Proof.

$$\begin{aligned} Var(X) &= E[(X - E[X])(X - E[X])] \\ &= E[X^2 + E[X]^2 - 2XE[X]] \\ &= E[X^2] + E[X]^2 - E[2XE[X]] \\ &= E[X^2] + E[X]^2 - 2E[X]E[X] = E[X^2] - E[X]^2 \end{aligned}$$

- $E[X^2]$ is called the *second moment* of X $\int_{-\infty}^{\infty} x^2 f(x)dx$

  Corollary.

  $$Var(rX + s) = r^2 Var(X)$$

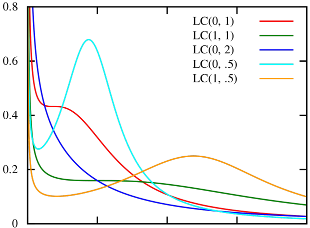
  Prove it!

- Variance insensitive to shift s!

Variance may be infinite or may not exist!

Standard deviation σ_X is a measure of the margin of error around a predicted value (e.g., temperature "20 ± 1.5").
An infinite or non-existent margin of error is no prediction at all.

- Variance may not exists!
  - If expectation does not exist!
  - Also in cases when expectation exists
    - We'll see later *Power laws*.
- Variance can be infinite
  - Distributions have fat upper tails that decrease at an extremely slow rate.
  - The slow decay of probability increases the odds of very extreme values (*outliers*)
  - E.g., e^X for $X \sim Cau(0,1)$ *[log-Cauchy distribution]*



Variance

- Variance of some discrete distributions
  - $X \sim U(m, M) \quad E[X] = \frac{(m+M)}{2} \quad Var(X) = \frac{(M-m+1)^2-1}{12}$
    - use $Var(X) = Var(X-m)$, call $n = M - m + 1$ and $\sum_{i=1}^{n-1} i^2 = \frac{(n-1)n(2n-1)}{6}$
  - $X \sim Ber(p) \quad E[X] = p \quad Var(X) = p^2(1-p) + (1-p)^2 p = p(1-p)$
  - $X \sim Bin(n, p) \quad E[X] = n \cdot p \quad Var(X) = np(1-p)$
    - Because . . . we'll see later
  - $X \sim Geo(p) \quad E[X] = \frac{1}{p} \quad Var(X) = \frac{1-p}{p^2}$
    - Hint: use $Var(X) = E[X^2] - E[X]^2$ and $\sum_{k=1}^{\infty} k^2 \cdot x^{k-1} = \frac{1+x}{(1-x)^3}$
  - $X \sim NBin(n, p) \quad E[X] = \frac{n \cdot p}{1-p} \quad Var(X) = n\frac{1-p}{p^2}$
    - Because . . . we'll see later
  - $X \sim Poi(\mu) \quad E[X] = \mu \quad Var(X) = \mu$
    - Because, when $n \to \infty$: $Bin(n, \mu/n) \to Poi(\mu)$

See **seeing-theory.brown.edu**

Variance

- Variance of some continuous distributions
  - $X \sim U(\alpha, \beta) \quad E[X] = (\alpha + \beta)/2 \quad Var(X) = (\beta - \alpha)^2/12$
    - **Prove it!** Recall that $f(x) = 1/(\beta - \alpha)$
  - $X \sim Exp(\lambda) \quad E[X] = 1/\lambda \quad Var(X) = 1/\lambda^2$
    - **Prove it!** Recall that $f(x) = \lambda e^{-\lambda x}$
  - $X \sim N(\mu, \sigma^2) \quad E[X] = \mu \quad Var(X) = \sigma^2$
    - **Prove it!** Hint: use $z = \frac{x - \mu}{\sigma}$ and integration by parts.
  - $X \sim Erl(n, \lambda) \quad E[X] = n/\lambda \quad Var(X) = n/\lambda^2$
    - Because . . . we'll see later

$E[]$ and $Var()$ of random variables with bounded support

Assume $a \leq X \leq b$, or more generally $P(a \leq X \leq b) = 1$ *[***almost surely** *or a.s.]*

It turns out that expectation and variance are finite!

- $a \leq E[X] \leq b$
  - E.g., for X continuous, $E[X] = \int_a^b xf(x)dx \leq \int_a^b bf(x)dx = b$
- $0 \leq Var(X) \leq {(b-a)^2}/4$

  Proof.

  - Since $0 \leq (X - E[X])^2$, we have $0 \leq E[(X - E[X])^2] = Var(X)$
  - For any $\gamma \in \mathbb{R}$, consider $E[(X - \gamma)^2] = \gamma^2 - 2\gamma E[X] + E[X^2]$
    - It is mimimum for $\gamma = E[X]$ (consider $\frac{d}{d\gamma}(\gamma^2 - 2\gamma a + b)$)
    - Thus, $E[(X - E[X])^2] = Var(X) \leq E[(X - \gamma)^2]$
  - For $\gamma = {(a+b)}/2$, we have $(X - \gamma)^2 \leq (b - \gamma)^2$, and then:

$$Var(X) \leq E[(X - \gamma)^2] \leq (b - \gamma)^2 = (b - \frac{(a+b)}{2})^2 = \frac{(b-a)^2}{4}$$

- **Exercise at home:** show that the bound ${(b-a)^2}/4$ can be reached.