

Master Program in *Data Science and Business Informatics*

Statistics for Data Science

Lesson 01 - Probabilities and independence

Salvatore Ruggieri

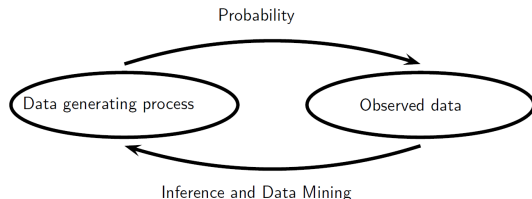
Department of Computer Science

University of Pisa, Italy

salvatore.ruggieri@unipi.it

Why Statistics in Data Science

We need grounded means for reasoning about data generated from real world with some degree of randomness.



What will you learn?

- Probability: properties of data generated by a known/assumed randomness model
- Statistics: properties of a randomness model that could have generated given data
- Simulation and R

Sample spaces and events

- An **experiment** is a measurement of a random process
- The **outcome** of a measurement takes values in some set Ω , called the **sample space**.

Examples:

- ▶ Tossing a coin: $\Omega = \{H, T\}$ *[Finite]*
- ▶ Month of birthdays $\Omega = \{\text{Jan}, \dots, \text{Dec}\}$ *[Finite]*
- ▶ Population of a city $\Omega = \mathbb{N} = \{0, 1, 2, \dots, \}$ *[Countably infinite]*
- ▶ Length of a street $\Omega = \mathbb{R}^+ = (0, \infty)$ *[Uncountably infinite]*
- ▶ Tossing a coin twice: $\Omega = \{H, T\} \times \{H, T\} = \{(H, H), (H, T), (T, H), (T, T)\}$
- ▶ Testing for Covid-19 (univariate): $\Omega = \{+, -\}$
- ▶ Testing for Covid-19 (multivariate): $\Omega = \{f, m\} \times \mathbb{N} \times \{+, -\}$, e.g., $(f, 25, -) \in \Omega$

Look at seeing-theory.brown.edu

- An **event** is some subset of $A \subseteq \Omega$ of possible outcomes of an experiment.
 - ▶ $L = \{ \text{Jan, March, May, July, August, October, December} \}$ *a long month with 31 days*
- We say that an event A **occurs** if the outcome of the experiment lies in the set A .
 - ▶ If the outcome is Jan then L occurs

Probability functions on finite sample space

A **probability function** is a mapping from events to **real numbers** that satisfies certain axioms. *Intuition: how likely is an event to occur.*

DEFINITION. A *probability function* P on a finite sample space Ω assigns to each event A in Ω a number $P(A)$ in $[0,1]$ such that

- (i) $P(\Omega) = 1$, and
- (ii) $P(A \cup B) = P(A) + P(B)$ if A and B are disjoint.

The number $P(A)$ is called the probability that A occurs.

- Fact: $P(\{a_1, \dots, a_n\}) = P(\{a_1\}) + \dots + P(\{a_n\})$ *[Generalized additivity]*
- Examples:
 - ▶ $P(\{H\}) = P(\{T\}) = 1/2$
 - ▶ $P(\{\text{Jan}\}) = 31/365, P(\{\text{Feb}\}) = 28/365, \dots P(\{\text{Dec}\}) = 31/365$
 - ▶ $P(L) = 7/12$ or $31 \cdot 7/365$?
- $P(\{a\})$ often abbreviated as $P(a)$, e.g., $P(H)$ instead of $P(\{H\})$

Properties of probability functions

- $P(A^c) = 1 - P(A)$
- $P(\emptyset) = 0$
- $A \subseteq B \Rightarrow P(A) \leq P(B)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A \cup B) = P(A) + P(B \setminus A)$
- probability that at least one coin toss over two lands head?

[Impossible event]

[Inclusion-exclusion principle]

Defining probability functions

Assigning probability is **NOT** an easy task: a prob. function can be an approximation of reality

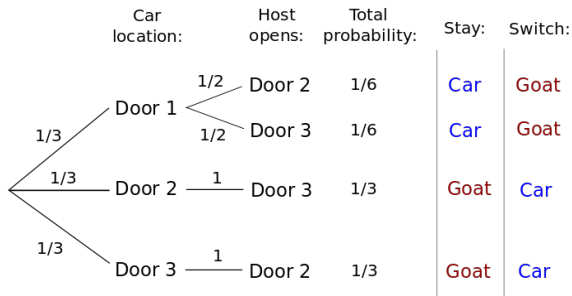
- **Frequentist** interpretation: probability measures a “*proportion of outcomes*” .
 - ▶ A fair coin lands on heads 50% of times
 - ▶ $P(A) = |A|/\Omega$ [Counting]
 - ▶ $P(\{\text{at least one H in two coin tosses}\}) = |\{(H, H), (H, T), (T, H)\}|/4 = 3/4$
- **Bayesian** (or epistemological) interpretation: probability measures a “*degree of belief*” .
 - ▶ Iliad and Odissey were composed by the same person at 90%

The Monty Hall problem

<https://math.andyou.com/tools/montyhallsimulator/montysim.htm>

(See also Exercise 2.14 of textbook [T])

Tree-based sequential description of probability function



Probability functions on countably infinite sample space

DEFINITION. A *probability function* on an infinite (or finite) sample space Ω assigns to each event A in Ω a number $P(A)$ in $[0, 1]$ such that

- (i) $P(\Omega) = 1$, and
- (ii) $P(A_1 \cup A_2 \cup A_3 \cup \dots) = P(A_1) + P(A_2) + P(A_3) + \dots$
if A_1, A_2, A_3, \dots are disjoint events.

- (ii) is equivalent to countable additivity (or σ -additivity): for $A_1 \subseteq A_2 \subseteq \dots$

$$P\left(\lim_{n \rightarrow \infty} A_i\right) = \lim_{n \rightarrow \infty} P(A_i)$$

- Example

- ▶ Experiment: we toss a coin repeatedly until H turns up.
- ▶ Outcome: the number of tosses needed.
- ▶ $\Omega = \{1, 2, \dots\} = \mathbb{N}^+$
- ▶ Suppose: $P(H) = p$. Then: $P(n) = (1 - p)^{n-1}p$
- ▶ Is it a probability function? $P(\Omega) = \dots$

Conditional probability

- Long months and months with 'r'
 - ▶ $L = \{ \text{Jan, Mar, May, July, Aug, Oct, Dec} \}$ *a long month with 31 days*
 - ▶ $R = \{ \text{Jan, Feb, Mar, Apr, Sep, Oct, Nov, Dec} \}$ *a month with 'r'*
 - ▶ $P(L) = 7/12$ $P(R) = 8/12$
- Anna is born in a long month. What is the probability she is born in a month with 'r'?

$$P(R|L) = \frac{P(L \cap R)}{P(L)} = \frac{P(\{ \text{Jan, Mar, Oct, Dec} \})}{P(L)} = \frac{4/12}{7/12} = \frac{4}{7}$$

- **Intuition:** probability of an event in the restricted sample space $\Omega \cap L$
 - ▶ *a-priori* probability $P(R) = 8/12$
 - ▶ *a-posteriori* probability $P(R|L) = 4/7 < 8/12$
- **Example (classification):** probab. of Covid given gender=f and age ≥ 60 : $P(C|G \cap A)$
 - ▶ $\Omega = \{f, m\} \times \mathbb{N} \times \{+, -\}$
 - ▶ $C = \{(-, -, +) \in \Omega\}$ $G = \{(f, -, -) \in \Omega\}$ $A = \{(-, a, -) \in \Omega \mid a \geq 60\}$
 - ▶ naming triples with features (gender, age, covid): $P(\text{covid} = + | \text{gender} = f, \text{age} \geq 60)$

Another example at seeing-theory.brown.edu

Conditional probability

DEFINITION. The **conditional probability** of A given C is given by:

$$P(A|C) = \frac{P(A \cap C)}{P(C)},$$

provided $P(C) > 0$.

Properties:

- $P(A|C) \neq P(C|A)$, in general
- $P(\Omega|C) = 1$
- if $A \cap B = \emptyset$ then $P(A \cup B|C) = P(A|C) + P(B|C)$ $P(\cdot|C)$ is a probability function

THE MULTIPLICATION RULE. For any events A and C :

$$P(A \cap C) = P(A|C) \cdot P(C).$$

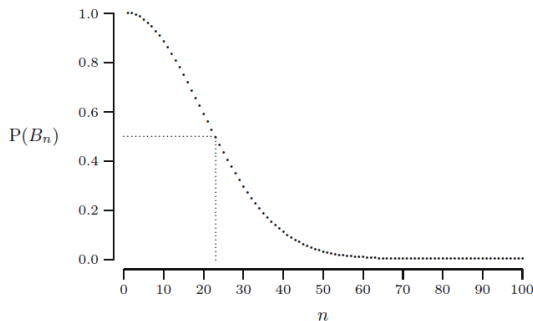
More generally, the **Chain Rule**:

$$P(A_1 \cap A_2 \cap A_3 \dots \cap A_n) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_1 \cap A_2) \cdot \dots \cdot P(A_n|\cap_{i=1}^{n-1} A_i)$$

Example: no coincident birthdays

- $B_n = \{n \text{ different birthdays}\}$
- For $n = 1$, $P(B_1) = 1$
- For $n > 1$,

$$\begin{aligned}P(B_n) &= P(B_{n-1}) \cdot P(\{\text{the } n\text{-th person's birthday differs from the other } n-1\} | B_{n-1}) \\ &= P(B_{n-1}) \cdot \left(1 - \frac{n-1}{365}\right) = \dots = \prod_{i=1}^{n-1} \left(1 - \frac{i}{365}\right)\end{aligned}$$



Example: case-based reasoning

Factory 1's light bulbs work for over 5000 hours in 99% of cases.

Factory 2's bulbs work for over 5000 hours in 95% of cases.

Factory 1 supplies 60% of the total bulbs on the market and Factory 2 supplies 40% of it.

What is the chance that a purchased bulb will work for longer than 5000 hours?

- $A = \{\text{bulbs working for longer than 5000 hours}\}$
- $C = \{\text{bulbs made by Factory 1}\}$, hence $C^c = \{\text{bulbs made by Factory 2}\}$
- Since $A = (A \cap C) \cup (A \cap C^c)$ with $(A \cap C)$ and $(A \cap C^c)$ disjoint:

$$P(A) = P(A \cap C) + P(A \cap C^c)$$

- and then by the multiplication rule:

$$P(A) = P(A|C) \cdot P(C) + P(A|C^c) \cdot P(C^c)$$

Answer: $P(A) = 0.99 \cdot 0.6 + 0.95 \cdot 0.4 = 0.974$

The law of total probability

THE **LAW OF TOTAL PROBABILITY**. Suppose C_1, C_2, \dots, C_m are disjoint events such that $C_1 \cup C_2 \cup \dots \cup C_m = \Omega$. The probability of an arbitrary event A can be expressed as:

$$P(A) = P(A | C_1)P(C_1) + P(A | C_2)P(C_2) + \dots + P(A | C_m)P(C_m).$$

- **Intuition:** case-based reasoning

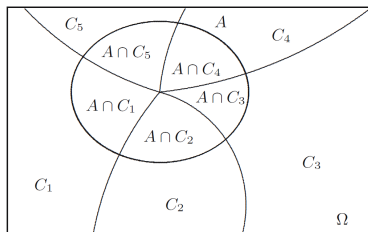


Fig. 3.2. The law of total probability (illustration for $m = 5$).

Exercise: Prisoners and guard dilemma

3 prisoners, 2 of which will be released.

You are one of the prisoners:

- at your's present state of knowledge, the probability of being released is $2/3$
 - ▶ $P(A1) = P(A1 \cap A2^c \cap A3) + P(A1 \cap A2 \cap A3^c) = 2/3$
- if you ask a friendly guard to tell you who is the prisoner other than yourself that will be released, your probability of being released will become $1/2$
 - ▶ $P(A1|A2) = P(A1 \cap A2)/P(A2) = (1/3)/(2/3) = 1/2$
 - ▶ $P(A1) = P(A1|A2)P(A2) + P(A1|A3)P(A3) - P(A1|A2 \cap A3)P(A2 \cap A3) = 1/2 \cdot 2/3 + 1/2 \cdot 2/3 - 0 \cdot 1/3 = 2/3$

What is wrong with this line of reasoning?

- $A1 = \{ \text{Prisoner 1 is released} \}$, $A2 = \{ \text{Prisoner 2 is released} \}$, $A3 = \{ \text{Prisoner 3 is released} \}$

$$P(A1^c \cap A2 \cap A3) = P(A1 \cap A2^c \cap A3) = P(A1 \cap A2 \cap A3^c) = 1/3$$

Independence of events

Intuition: whether one event provides any information about another.

Independence

An event A is independent of B , if $P(B) > 0$ or

$$P(A|B) = P(A)$$

- For $P(R|L) = 4/7 \neq 8/12 = P(R)$ - knowing Anna was born in a long month change the probability she was born in a month with 'r'!
- Tossing 2 coins:
 - ▶ A_1 is "H on toss 1" and A_2 is "H on toss 2"
 - ▶ $P(A_1) = P(A_2) = 1/2$
 - ▶ $P(A_2|A_1) = P(A_2 \cap A_1)/P(A_1) = 1/4/1/2 = 1/2 = P(A_2)$
- Physical and stochastic independence
- Properties:
 - ▶ A independent of B iff $P(A \cap B) = P(A) \cdot P(B)$
 - ▶ A independent of B iff B independent of A
 - ▶ A independent of B iff A^c independent of B

[Symmetry]

Conditional independence of events

Intuition: whether one event provides any information about another given a third event occurred. Technically, consider $P(\cdot|C)$ in independence.

Conditional independence

An event A is conditionally independent of B given C such that $P(C) > 0$, if $P(B|C) = 0$ or

$$P(A|B \cap C) = P(A|C)$$

- Properties:
 - ▶ A conditionally independent of B iff $P(A \cap B|C) = P(A|C) \cdot P(B|C)$
 - ▶ A conditionally independent of B iff B conditionally independent of A
- **Exercise at home.** Prove or disprove:
 - ▶ If A is independent of B then A is conditionally independent of B given C

[Symmetry]

Independence of two or more events

INDEPENDENCE OF TWO OR MORE EVENTS. Events A_1, A_2, \dots, A_m are called independent if

$$P(A_1 \cap A_2 \cap \dots \cap A_m) = P(A_1)P(A_2) \dots P(A_m)$$

and this statement *also* holds when any number of the events A_1, \dots, A_m are replaced by their complements throughout the formula.

Alternative definition

Events A_1, A_2, \dots, A_m are called independent if for every $J \subseteq \{1, \dots, m\}$:

$$P\left(\bigcap_{i \in J} A_i\right) = \prod_{i \in J} P(A_i)$$

- **Exercise at home:** show the two definitions are equivalent

Independence of two or more events

Alternative definition

Events A_1, A_2, \dots, A_m are called independent if for every $J \subseteq \{1, \dots, m\}$:

$$P\left(\bigcap_{i \in J} A_i\right) = \prod_{i \in J} P(A_i)$$

- It is **stronger** than **pairwise independence**

$$P(A_i \cap A_j) = P(A_i) \cdot P(A_j) \text{ for } i \neq j \in \{1, \dots, m\}$$

- Example: what is the probability of at least one head in the first 10 tosses of a coin?
 $A_i = \{\text{head in } i\text{-th toss}\}$

$$P\left(\bigcup_{i=1}^{10} A_i\right) = 1 - P\left(\bigcap_{i=1}^{10} A_i^c\right) = 1 - \prod_{i=1}^{10} P(A_i^c) = 1 - \prod_{i=1}^{10} (1 - P(A_i))$$