Master Program in *Data Science and Business Informatics*

Statistics for Data Science

Lesson 31 - Two-sample tests of the mean and applications to classifier comparison

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Two sample test of the mean

- Dataset $x_1, \ldots, x_n$ realization of $X_1, \ldots, X_n \sim F_1$ with $E[X_i] = \mu_1$ and $Var(X_i) = \sigma_X^2$
- Dataset $y_1, \ldots, y_m$ realization of $Y_1, \ldots, Y_m \sim F_2$ with $E[Y_i] = \mu_2$ and $Var(Y_i) = \sigma_Y^2$
  - measurements for control and (medical) treatment groups of patients
  - performances on benchmark datasets/folds of two different classifiers
- $H_0 : \mu_1 = \mu_2 \qquad H_1 : \mu_1 \neq \mu_2$
- Wald test statistics: $T = \frac{\bar{X}_n - \bar{Y}_m}{\sqrt{Var(\bar{X}_n - \bar{Y}_m)}} = \frac{\bar{X}_n - \bar{Y}_m}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}}$
- We distinguish a few cases:
  - F_1, F_2 are normal distributions
    - σ_X^2 and σ_Y^2 are known *[z-test]*
    - σ_X^2 and σ_Y^2 are unknown and $\sigma_X^2 = \sigma_Y^2$ *[t-test]*
    - σ_X^2 and σ_Y^2 are unknown and $\sigma_X^2 \neq \sigma_Y^2$ *[Welch test]*
  - F_1, F_2 are general distributions
    - Large sample *[t-test]*
    - $F_1(x - \Delta) = F_2(x)$ location-shift *[Wilcoxon test]*
    - Bootstrap two sample test
  - Paired data *[paired t-test]*

Normal data with known σ_X^2 and σ_Y^2: z-test

- $X_1, \ldots, X_n \sim \mathcal{N}(\mu_1, \sigma_X^2)$ and $Y_1, \ldots, Y_m \sim \mathcal{N}(\mu_2, \sigma_Y^2)$
- $H_0 : \mu_1 = \mu_2$
- $H_1 : \mu_1 \neq \mu_2$ *[Two-tailed test]*
- $100(1-\alpha)\%$, e.g., 95% or 99% or 99.9% *[Confidence level]*
  - i.e., $\alpha = 0.05$ or $\alpha = 0.01$ or $\alpha = 0.001$ *[Significance level]*
- $Z = \frac{\bar{X}_n - \bar{Y}_m}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}} \sim \mathcal{N}(0,1)$ test statistics when H_0 is true
- z value is $\frac{\bar{x}_n - \bar{y}_m}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}}$ and p-value $p = P(|Z| \geq |z|) = 2(1 - \Phi(|z|))$
- $P(Z \leq -z_{\alpha/2}) = \alpha/2$ and $P(Z \geq z_{\alpha/2}) = \alpha/2$ *[Critical values]*
- Output of the test at confidence level $100(1-\alpha)\%$ using critical values
  - $|z| \geq z_{\alpha/2}$: H_0 is rejected *[Critical region]*
  - otherwise: H_0 cannot be rejected

See R script

Unknown $\sigma_X^2 = \sigma_Y^2 = \sigma^2$ and pooled variance

- We need to estimate $Var(\bar{X}_n - \bar{Y}_m) = \sigma^2 \left(\frac{1}{n} + \frac{1}{m}\right)$
- Recall

$$S_X^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 \quad \text{and} \quad S_Y^2 = \frac{1}{m-1}\sum_{i=1}^{m}(Y_i - \bar{Y}_m)^2$$

  are unbiased estimators of σ_X^2 and σ_Y^2
- The *pooled variance*:

$$S_p^2 = \frac{(n-1)S_X^2 + (m-1)S_Y^2}{n+m-2}\left(\frac{1}{n} + \frac{1}{m}\right) = \frac{\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 + \sum_{i=1}^{m}(Y_i - \bar{Y}_m)^2}{n+m-2}\left(\frac{1}{n} + \frac{1}{m}\right)$$

  is an unbiased estimator of $\sigma^2 \left(\frac{1}{n} + \frac{1}{m}\right)$

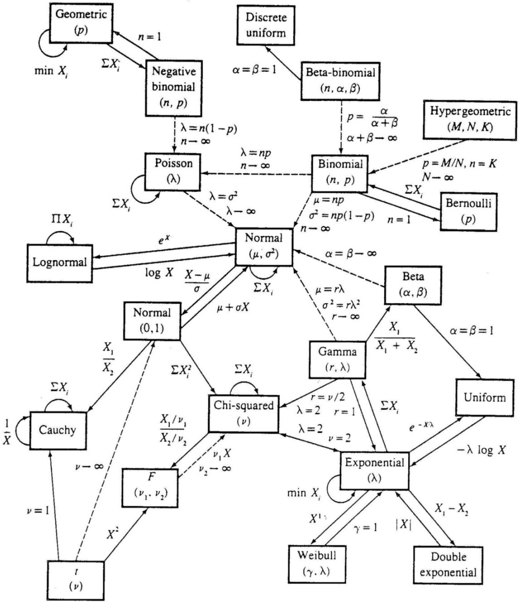
Testing equal variances for normal data: F-test

- $X_1, \ldots, X_n \sim \mathcal{N}(\mu_1, \sigma_X^2)$ and $Y_1, \ldots, Y_m \sim \mathcal{N}(\mu_2, \sigma_Y^2)$
- $H_0 : \sigma_X^2 = \sigma_Y^2$
- $H_1 : \sigma_X^2 \neq \sigma_Y^2$ *[Two-tailed test]*
- $100(1-\alpha)\%$, e.g., 95% or 99% or 99.9% *[Confidence level]*
  - i.e., $\alpha = 0.05$ or $\alpha = 0.01$ or $\alpha = 0.001$ *[Significance level]*
- $F = \frac{S_X^2}{S_Y^2} \sim F(n-1, m-1)$ test statistics when H_0 is true *[**Fisher-Snedecor distribution**]*
- f value is $\frac{s_X^2}{s_Y^2}$ and p-value is $p = 2\min\{P(F \leq f), 1 - P(F \leq f)\}$ *[Asymmetric]*
- $P(F \leq l) = \alpha/2$ and $P(F \geq u) = \alpha/2$ *[Critical values]*
- Output of the test at confidence level $100(1-\alpha)\%$ using critical values
  - $f \leq l$ or $f \geq u$: H_0 is rejected *[Critical region]*
  - otherwise: H_0 cannot be rejected

See R script

Common distributions

- **Probability distributions at Wikipedia**
- **Probability distributions in R**
- C. Forbes, M. Evans,
  N. Hastings, B. Peacock (2010)
  Statistical Distributions, 4th Edition
  Wiley



Relationships among common distributions. Solid lines represent transformations and special cases, dashed lines represent limits. Adapted from Leemis (1986).

Normal data with unknown $\sigma_X^2 = \sigma_Y^2 = \sigma^2$: t-test

- $X_1, \ldots, X_n \sim \mathcal{N}(\mu_1, \sigma^2)$ and $Y_1, \ldots, Y_m \sim \mathcal{N}(\mu_2, \sigma^2)$
- $H_0 : \mu_1 = \mu_2$
- $H_1 : \mu_1 \neq \mu_2$ *[Two-tailed test]*
- $100(1-\alpha)\%$, e.g., 95% or 99% or 99.9% *[Confidence level]*
  - i.e., $\alpha = 0.05$ or $\alpha = 0.01$ or $\alpha = 0.001$ *[Significance level]*
- $T_p = \frac{\bar{X}_n - \bar{Y}_m}{S_p} \sim t(n+m-2)$ test statistics when H_0 is true
- t value is $\frac{\bar{x}_n - \bar{y}_m}{\sqrt{\frac{(n-1)s_X^2 + (m-1)s_Y^2}{n+m-2}\left(\frac{1}{n}+\frac{1}{m}\right)}}$ and p-value $p = P(|T_p| \geq |t|)$
- $P(T_p \leq -t_{n+m-2,\alpha/2}) = \alpha/2$ and $P(T_p \geq t_{n+m-2,\alpha/2}) = \alpha/2$ *[Critical values]*
- Output of the test at confidence level $100(1-\alpha)\%$ using critical values
  - $|t| \geq t_{n+m-2,\alpha/2}$: H_0 is rejected *[Critical region]*
  - otherwise: H_0 cannot be rejected

See R script

Normal data with unknown $\sigma_X^2 \neq \sigma_Y^2$

- The *nonpooled variance*:
$$S_d^2 = \frac{S_X^2}{n} + \frac{S_Y^2}{m}$$
is an unbiased estimator of $Var(\bar{X}_n - \bar{Y}_m) = \frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}$
- The test statistics $T_d = \frac{\bar{X}_n - \bar{Y}_m}{S_d} \approx t(\upsilon)$ when H_0 is true, with
$$\upsilon = \frac{\left(\frac{1}{n} + \frac{u}{m}\right)^2}{\frac{1}{n^2(n-1)} + \frac{u^2}{m^2(m-1)}} \quad \text{and} \quad u = \frac{s_Y^2}{s_X^2}$$

Normal data with unknown $\sigma_X^2 \neq \sigma_Y^2$: Welch t-test

- $X_1, \ldots, X_n \sim \mathcal{N}(\mu_1, \sigma_X^2)$ and $Y_1, \ldots, Y_m \sim \mathcal{N}(\mu_2, \sigma_Y^2)$
- $H_0 : \mu_1 = \mu_2$
- $H_1 : \mu_1 \neq \mu_2$ *[Two-tailed test]*
- $100(1-\alpha)\%$, e.g., 95% or 99% or 99.9% *[Confidence level]*
  - i.e., $\alpha = 0.05$ or $\alpha = 0.01$ or $\alpha = 0.001$ *[Significance level]*
- $T_d = \frac{\bar{X}_n - \bar{Y}_m}{S_d} \approx t(\upsilon)$ test statistics when H_0 is true, with $\upsilon = \frac{\left(\frac{1}{n} + \frac{u}{m}\right)^2}{\frac{1}{n^2(n-1)} + \frac{u^2}{m^2(m-1)}}$ and $u = \frac{s_Y^2}{s_X^2}$
- t value is $\frac{\bar{x}_n - \bar{y}_m}{\sqrt{\frac{s_X^2}{n} + \frac{s_Y^2}{m}}}$ and p-value $p = P(|T_d| \geq |t|)$
- $P(T_d \leq -t_{\upsilon,\alpha/2}) = \alpha/2$ and $P(T_d \geq t_{\upsilon,\alpha/2}) = \alpha/2$ *[Critical values]*
- Output of the test at confidence level $100(1-\alpha)\%$ using critical values
  - $|t| \geq t_{\upsilon,\alpha/2}$: H_0 is rejected *[Critical region]*
  - otherwise: H_0 cannot be rejected

See R script

General data, large sample: t-test

- $X_1, \ldots, X_n \sim F_1$ and $Y_1, \ldots, Y_m \sim F_2$
- $H_0 : \mu_1 = \mu_2$
- $H_1 : \mu_1 \neq \mu_2$ *[Two-tailed test]*
- $100(1-\alpha)\%$, e.g., 95% or 99% or 99.9% *[Confidence level]*
  - i.e., $\alpha = 0.05$ or $\alpha = 0.01$ or $\alpha = 0.001$ *[Significance level]*
- $T_d = \frac{\bar{X}_n - \bar{Y}_m}{S_d} \approx \mathcal{N}(0,1)$
- t value is $\frac{\bar{x}_n - \bar{y}_m}{\sqrt{\frac{s_X^2}{n} + \frac{s_Y^2}{m}}}$ and p-value $p = P(|T_d| \geq |t|)$
- $P(T_d \leq -z_{\alpha/2}) = \alpha/2$ and $P(T_d \geq z_{\alpha/2}) = \alpha/2$ *[Critical values]*
- Output of the test at confidence level $100(1-\alpha)\%$ using critical values
  - $|t| \geq z_{\alpha/2}$: H_0 is rejected *[Critical region]*
  - otherwise: H_0 cannot be rejected

See R script

General data, location-shift: Wilcoxon rank-sum test

- Also called as: **Mann–Whitney U test** or Mann–Whitney–Wilcoxon (MWW)
- $X_1, \ldots, X_n \sim F_1$ and $Y_1, \ldots, Y_m \sim F_2$
- $H_0 : \mu_1 = \mu_2$ and $H_1 : \mu_1 \neq \mu_2$ *[Two-tailed test]*
  - actually, $H_0 : F_1(x - \Delta) = F_2(x)$ where $\Delta = \mu_2 - \mu_1$ *[Location-shift model]*
  - we should test that empirical distributions have **the same shape**
- $W = \sum_{i=1}^{n} S_i \sim W(n, m)$ when H_0 is true *[or $U = W - m \cdot (m+1)/2$]*
  - where S_i is the rank of X_i in sorted$(X_1, \ldots, X_n, Y_1, \ldots, Y_m)$
  - `pwilcox` in R, or large sample Normal approx
- w value is $\sum_{i=1}^{n} s_i$ and p-value $p = P(|W| \geq |w|)$
- $P(W \leq -w_{\alpha/2}) = \alpha/2$ and $P(T_p \geq w_{\alpha/2}) = \alpha/2$ *[Critical values]*
- Output of the test at confidence level $100(1-\alpha)\%$ using critical values
  - $|w| \geq w_{\alpha/2}$: H_0 is rejected *[Critical region]*
  - otherwise: H_0 cannot be rejected
- Generalized test (without location-shift assumption): **Brunner-Munzel** test

See R script

General data: bootstrap test

- Equal variance ($\sigma_X^2 = \sigma_Y^2$)
  - bootstrap of pooled studentized mean difference

$$t_p^\star = \frac{(\bar{x}_n^\star - \bar{y}_m^\star) - (\bar{x}_n - \bar{y}_m)}{s_p^\star}$$

- Non-equal variance ($\sigma_X^2 \neq \sigma_Y^2$)
  - bootstrap of nonpooled studentized mean difference

$$t_d^\star = \frac{(\bar{x}_n^\star - \bar{y}_m^\star) - (\bar{x}_n - \bar{y}_m)}{s_d^\star}$$

See R script

Paired data

- Datasets $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are measurement for the same experimental unit
  - unit: a person before and after a (medical) treatment
  - unit: a dataset/fold used to train two different classifiers
- The theory is essentially based on taking differences $x_1 - y_1, \ldots, x_n - y_n$ and thus reducing the problem to that of a one-sample test.
- $H_0 : \mu_1 = \mu_2 \Rightarrow H_0 : \mu_1 - \mu_2 = 0$
- Advantage: better power / lower Type II risk of the test w.r.t. unpaired version
  - $P_{paired}(p \leq \alpha | H_1) \geq P_{unpaired}(p \leq \alpha | H_1)$

See R script

Optional reference

- On confidence intervals and statistical tests (with R code)

Myles Hollander, Douglas A. Wolfe, and Eric Chicken (2014)
Nonparametric Statistical Methods.
3rd edition, *John Wiley & Sons, Inc.*