Master Program in *Data Science and Business Informatics*

Statistics for Data Science

Lesson 22 - Issues with linear regression. Logistic regression

Salvatore Ruggieri

Department of Computer Science
University of Pisa, Italy
salvatore.ruggieri@unipi.it

Issues: Omitted variable bias

- Suppose we omit a variable z_i that belongs to the true model
$$Y_i = \alpha + \beta_1 x_i + \beta_2 z_i + U_i$$
with $\beta_2 \neq 0$ (i.e., Y is determined by Z)
  - Under-specification of the model, e.g., due to lack of data
- Fitted model $Y_i = \alpha + \beta_1 x_i + U_i'$
  - We have: $E[U_i'] = E[\beta_2 z_i + U_i] = \beta_2 z_i + E[U_i] = \beta_2 z_i \neq 0$
  - The assumption $E[U_i'] = 0$ is not met! Hence, estimators will be biased!
- Let $\hat{\alpha}$ and $\hat{\beta}_1$ be the LSE estimators of the fitted model. It turns out (proof not included):
$$E[\hat{\beta}_1] = \beta_1 + \beta_2 \delta \qquad Bias(\hat{\beta}_1) = \beta_2 \delta$$
where δ is the slope of the regression of $Z_i = \gamma + \delta x_i + U_i''$, i.e.:
$$\delta = r_{xz} \frac{s_z}{s_x}$$
- $Bias(\hat{\beta}_1) \neq 0$ if X and Z correlated

See R script

Issues: Multi-collinearity and variance inflation factors

- *Multicollinearity*: two or more independent variables (regressors) are strongly correlated.
- $Y_i = \alpha + \beta_1 x_i^1 + \beta_2 x_i^2 + U_i$
- It can be shown that for $j \in \{1, 2\}$:

$$Var(\hat{\beta}_j) = \frac{1}{(1 - r^2)} \cdot \frac{\sigma^2}{SXX_j}$$

  where $r = cor(x^1, x^2)$, $\sigma^2 = Var(U_i)$ and $SXX_j = \sum_1^n (x_i^j - \bar{x}_n^j)^2$
- Correlation between regressors increases the variance of the estimators
- In general, for more than 2 variables:

$$Var(\hat{\beta}_j) = \frac{1}{(1 - R_j^2)} \cdot \frac{\sigma^2}{SXX_j}$$

  where R_j^2 is the coefficient of determination (R^2) in the regression of x_j from all other x_i's.
- The term $1/(1-R_j^2)$ is called *variance inflation factor*

See R script

Variable selection

- Recall: when $U_i \sim N(0, \sigma^2)$, we have $Y_i \sim N(\boldsymbol{x}_i \cdot \boldsymbol{\beta}, \sigma^2)$, hence we can apply MLE
- Log-likelihood is $\ell(\boldsymbol{\beta}) = \sum_{i=1}^{n} \log\left(\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y_i - \boldsymbol{x}_i \cdot \boldsymbol{\beta}}{\sigma^2}\right)^2}\right)$
- Akaike information criterion (AIC), balances model fit against model simplicity

$$AIC(\boldsymbol{\beta}) = 2|\boldsymbol{\beta}| - 2\ell(\boldsymbol{\beta})$$

- stepAIC(model, direction="backward") algorithm

1. $S = \{x^1, \dots, x^k\}$
2. $b = AIC(S)$
3. repeat
    - 3.1 $x = arg\,\min_{x \in S} AIC(S \setminus \{x\})$
    - 3.2 $v = AIC(S \setminus \{x\})$
    - 3.3 if $v < b$ then $S, b = S \setminus \{x\}, v$
4. until no change in S
5. return S

See R script

Regularization methods: Ridge/Tikhonov

$$\hat{\boldsymbol{\beta}} = arg \min_{\boldsymbol{\beta}} S(\boldsymbol{\beta})$$

- Ordinary Least Square Estimation (OLS):

$$S(\boldsymbol{\beta}) = \|\boldsymbol{y} - \boldsymbol{X} \cdot \boldsymbol{\beta}\|^2$$

  where $\|(v_1, \ldots, v_n)\| = \sqrt{\sum_{i=1}^{n} v_i^2}$ is the Euclidian norm

  - Performs poorly as for prediction (overfitting) and interpretability (number of variables)

- Ridge regression:

$$S(\boldsymbol{\beta}) = \|\boldsymbol{y} - \boldsymbol{X} \cdot \boldsymbol{\beta}\|^2 + \lambda_2 \|\boldsymbol{\beta}\|^2$$

  where $\|\boldsymbol{\beta}\| = \sqrt{\alpha^2 + \sum_{i=1}^{k} \beta_i^2}$.

  - Notice that λ_2 is not in the parameters of the minimization problem!
  - Variables with minor contribution have their coefficients **close** to zero
  - It improves prediction error by reducing overfitting through a bias-variance trade-off
  - It is **not** a parsimonious method, i.e., does not reduce features

Regularization methods: Lasso and Penalized

- Lasso (Least Absolute Shrinkage and Selection Operator) regression:

$$S(\boldsymbol{\beta}) = \|\boldsymbol{y} - \boldsymbol{X} \cdot \boldsymbol{\beta}\|^2 + \lambda_1 \|\boldsymbol{\beta}\|_1$$

  where $\|\boldsymbol{\beta}\|_1 = |\alpha| + \sum_{i=1}^{k} |\beta_i|$.
  - Notice that λ_1 is not in the parameters of the minimization problem!
  - Variable with minor contribution have their coefficients **equal** to zero
  - It improves prediction error by reducing overfitting through a bias-variance trade-off
  - It **is** a parsimonious method, i.e., it reduces the number of features
- Penalized linear regression:

$$S(\boldsymbol{\beta}) = \|\boldsymbol{y} - \boldsymbol{X} \cdot \boldsymbol{\beta}\|^2 + \lambda_2 \|\boldsymbol{\beta}\|^2 + \lambda_1 \|\boldsymbol{\beta}\|_1$$

  - Both Ridge and Lasso regularization parameters
- How to solve the minimization problems? **Lagrange multiplier method** and the methods studied at the *Optimization for Data Science* course
- How to find the best λ_1 and/or λ_2? Cross-validation!

See R script

Towards logistic regression

- Consider a bivariate dataset
$$(x_1, y_1), \ldots, (x_n, y_n)$$
where $y_i \in \{0, 1\}$, i.e., Y_i is a binary variable
- Using directly linear regression:
$$Y_i = \alpha + \beta x_i + U_i$$
results in poor performances (R^2)

See R script

Towards logistic regression

- Consider a bivariate dataset

$$(x_1, y_1), \ldots, (x_n, y_n)$$

where $y_i \in \{0, 1\}$, i.e., Y_i i binary variable

- Group by x values:

$$(d_1, f_1), \ldots, (d_m, f_m)$$

where $d_1, \ldots, d_m$ are the distinct values of $x_1, \ldots, x_n$ and f_i is the fraction of 1's:

$$f_i = \frac{|\{j \in [1, n] \mid x_j = d_i \wedge y_j = 1\}|}{|\{j \in [1, n] \mid x_j = d_i\}|}$$

and the linear model (we continue using x_i but it should be d_i):

$$F_i = \alpha + \beta x_i + U_i$$

where $F_i = P(Y_i = 1)$

See R script

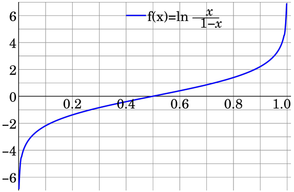
Towards logistic regression

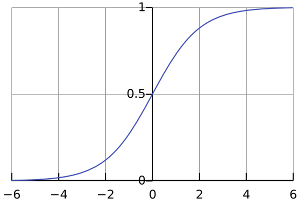
- Rather than F_i, we model the log odds of F_i (called the *logit*)

$$logit(F_i) = \alpha + \beta x_i + U_i$$

  where logit and its inverse (**logistic function**) are:

$$logit(p) = \log \frac{p}{1-p} \qquad inv.logit(x) = \frac{e^x}{1+e^x} = \frac{1}{1+e^{-x}}$$





- Why?
  - $F_i \in [0,1]$ while the $\alpha + \beta x_i + U_i$ is in $\mathbb{R}$, hence inadequate to model probabilities
  - Relation between x_i's and F_i is sigmoidal, not linear, hence the use of logistic function
  - **Other sigmoid functions** beyond the logistic one (see also FisherZ in Lesson 18)

See R script

Logistic regression

- Since $F_i = P(Y_i = 1)$, we actually estimate p_i for $Y_i \sim Ber(p_i)$, and U_i is not necessary

$$p_i = inv.logit(\alpha + \beta x_i) = \frac{e^{\alpha+\beta x_i}}{1 + e^{\alpha+\beta x_i}} \tag{1}$$

- Since distribution is known, MLE can be adopted for estimating α and β in logistic regression:

$$\ell(\alpha, \beta) = \sum_{i=1}^{n} \left[y_i \log\left(inv.logit(\alpha + \beta x_i)\right) + (1 - y_i) \log\left(1 - inv.logit(\alpha + \beta x_i)\right)\right]$$

  recalling the p.m.f. of $Ber(p_i)$: $p_i^a \cdot (1 - p_i)^{(1-a)}$

- Since $p_i/(1 - p_i) = e^{\alpha+\beta x_i}$, then e^β can be interpreted as:
  - the expected change in odds after a unit change in x_i,
  - e.g., $\beta = 0.38$ in predicting heart disease from smoking: the smoking group $(x = 1)$ has $e^\beta = 1.46$ times the odds of the non-smoking group $(x = 0)$ of having heart disease.
- By (1) for $x_i = 0$, then $e^\alpha/(1 + e^\alpha)$ can be interpreted as the base probability:
  - e.g., $\alpha = -1.93$ means the probability a non-smoker $(x = 0)$ has heart disease is $e^\alpha/(1 + e^\alpha) = 0.13$.

Generalized linear models

- **Generalized linear models**: family = distribution + link function
  - E.g., Binomial + logit for logistic regression
  - Actually Bernoulli + logit *[Binary logistic regression]*

See R script

Elastic net logistic regression

- Penalized linear regression minimizes:

$$\|\boldsymbol{y} - \boldsymbol{X} \cdot \boldsymbol{\beta}\|^2 + \lambda_2 \|\boldsymbol{\beta}\|^2 + \lambda_1 \|\boldsymbol{\beta}\|_1$$

  - $\lambda_1 = 0$ is the Ridge penalty
  - $\lambda_2 = 0$ is the Lasso penalty

- Elastic net regularization for logistic regression minimizes:

$$-\ell(\boldsymbol{\beta}) + \lambda \left(\frac{(1-\alpha)}{2} \|\boldsymbol{\beta}\|^2 + \alpha \|\boldsymbol{\beta}\|_1 \right)$$

  - $\alpha = 0$ is the Ridge penalty
  - $\alpha = 1$ is the Lasso penalty
  - λ is to be found, e.g., by cross-validation

See R script

Optional references

Michael David W. Hosmer, Stanley Lemeshow, and Rodney X. Sturdivant (2013)
Applied Logistic Regression.
3rd edition*John Wiley & Sons, Inc.*