Master Program in *Data Science and Business Informatics*

Statistics for Data Science

Lesson 12 - Simulation

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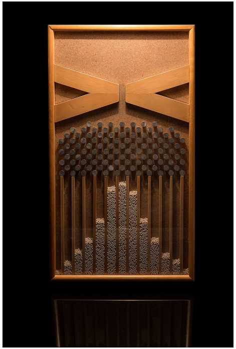
Simulation

- Not all problems can be solved with calculus!
- Complex interactions among random variables can be simulated
- Generated random values are called *realizations*
- Basic issue: *how to generate realizations?*
    - The **Galton Board**



Simulation

- Not all problems can be solved with calculus!
- Complex interactions among random variables can be simulated
- Generated random values are called *realizations*
- Basic issue: *how to generate realizations?*
  - in R: $rnorm(5), rexp(2), rbinom(\ldots), \ldots$
- Ok, but how do they work?
- **Assumption**: we are only given $runif()$!
- **Problem**: derive all the other random generators

Simulation: discrete distributions

Bernoulli random variables

Suppose U has a $U(0,1)$ distribution. To construct a $Ber(p)$ random variable for some $0 < p < 1$, we define

$$X = \begin{cases} 1 & \text{if } U < p, \\ 0 & \text{if } U \geq p \end{cases}$$

so that

$$\begin{aligned} \mathrm{P}(X = 1) &= \mathrm{P}(U < p) = p, \\ \mathrm{P}(X = 0) &= \mathrm{P}(U \geq p) = 1 - p. \end{aligned}$$

This random variable X has a Bernoulli distribution with parameter p.

- For $X_1, \ldots, X_n \sim Ber(p)$ i.i.d., we have: $\sum_{i=1}^{n} X_i \sim Binom(n, p)$

See R script

$X \sim Cat(\mathbf{p})$

DEFINITION. A discrete random variable X has a *Bernoulli distribution* with parameter p, where $0 \leq p \leq 1$, if its probability mass function is given by

$$p_X(1) = \mathrm{P}(X = 1) = p \quad \text{and} \quad p_X(0) = \mathrm{P}(X = 0) = 1 - p.$$

We denote this distribution by $Ber(p)$.

- Alternative definition: $p_X(a) = p^a \cdot (1-p)^{1-a}$ for $a \in \{0, 1\}$
- Categorical distribution generalizes to $n_C \geq 2$ possible values

$X \sim Cat(\mathbf{p})$

Categorical distribution

A discrete random variable X has a Categorical distribution with parameters $p_0, \ldots, p_{n_C - 1}$ where $\sum_i p_i = 1$ and $p_i \in [0, 1]$ if its p.m.f. is given by:

$$p_X(i) = P(X = i) = p_i \quad \text{for } i = 0, \ldots, n_C - 1$$

- Alternative definition: $p_X(a) = \prod_i p_i^{\mathbb{1}_{a=i}}$ for $a \in \{0, \ldots, n_C - 1\}$

Notation. Indicator variable: $\mathbb{1}_\varphi(x) = \begin{cases} 1 & \text{if } \varphi(x) \\ 0 & \text{otherwise} \end{cases}$

$X \sim Mult(n, \mathbf{p})$

- $X \sim Bin(n, p)$ models the number of successes in n Bernoulli trials
- **Intuition**: for $X_1, X_2, \dots, X_n$ i.i.d. $X_i \sim Ber(p)$: $X = \sum_{i=1}^{n} X_i \sim Bin(n, p)$
- $X \sim Mult(n, \mathbf{p})$ models the number of categories in n Categorical trials
- **Intuition**: for $X_1, X_2, \dots, X_n$ such that $X_i \sim Cat(\mathbf{p})$ and independent (**i.i.d.**), define:

$$Y_1 = \sum_{i=1}^{n} \mathbb{1}_{X_i=0} \sim Bin(n, p_0) \quad \dots \quad Y_{n_C} = \sum_{i=1}^{n} \mathbb{1}_{X_i=n_C-1} \sim Bin(n, p_{n_C-1})$$

$$X = (Y_1, \dots, Y_{n_C}) \sim Mult(n, \mathbf{p})$$

Multinomial distribution

A discrete random variable $X = (Y_1, \dots, Y_{n_C})$ has a Multinomial distribution with parameters $p_0, \dots, p_{n_C-1}$ where $\sum_i p_i = 1$ and $p_i \in [0, 1]$ if its p.m.f. is given by:

$$p_X(i_0, \dots, i_{n_C-1}) = P(X = (i_0, \dots, i_{n_C-1})) = \frac{n!}{i_0! i_1! \dots i_{n_C-1}!} p_0^{i_0} p_1^{i_1} \dots p_{(n_C-1)}^{i_{(n_C-1)}}$$

$X \sim Mult(n, \mathbf{p})$

- Example: student selection from a population with $n_C = 3$:
  - $p_0 = 60\%$ undergraduates
  - $p_1 = 30\%$ graduate
  - $p_2 = 10\%$ PhD students
- Assume $n = 20$ students are randomly selected
- $X \sim (Y_1, Y_2, Y_3)$ where:
  - Y_1 number of undergraduate students selected
  - Y_2 number of graduate students selected
  - Y_3 number of PhD students selected
- $P(X = (10, 6, 4)) = \frac{20!}{10!6!4!}(0.6)^{10}(0.3)^6(0.1)^4 = 9.6\%$

See R script

Simulation: continuous distributions

- $F(x) = P_X(X \leq x)$
- $F : \mathbb{R} \to [0, 1]$ invertible as $F^{-1} : [0, 1] \to \mathbb{R}$
  - E.g., F *strictly* increasing
  - N.B., the textbook notation for F^{-1} is F^{inv}
- For $Y \sim U(0, 1)$ and $0 \leq b \leq 1$

  $$P_Y(Y \leq b) = b$$

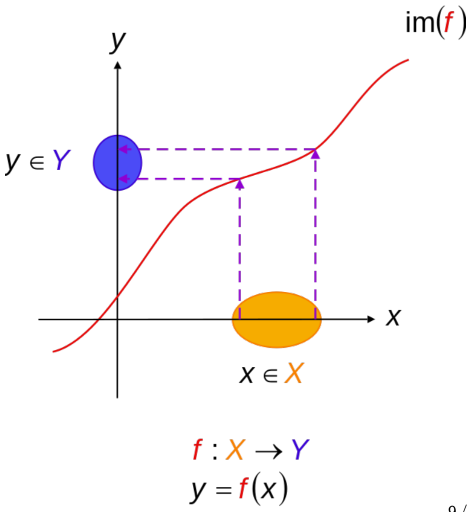
  then, for $b = F(x)$

  $$P_Y(Y \leq F(x)) = F(x)$$

  and then by inverting $X = F^{-1}(Y)$

  $$P_X(X \leq x) = P_Y(F^{-1}(Y) \leq x) = F(x)$$
- In summary:

  $$X = F^{-1}(Y) \sim F \text{ for } Y \sim U(0, 1)$$
- Example: $F(x) = 1 - e^{-\lambda x}$ for $Exp(\lambda)$
  - $F^{-1}(y) = -{}^{1}/_{\lambda} \log(1 - y)$
  - See also quantiles in Lesson 08



See R script

Optional reference

William H. Press, Saul A. Teukolsky, William T. Vetterling, Brian P. Flannery (2007)
Numerical Recipes - The Art of Scientific Computing
Chapter 7: Random Numbers
online book