Master Program in *Data Science and Business Informatics*

Statistics for Data Science

Lesson 06 - Recalls on calculus

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J. Ward, J. Abdey. **Mathematics and Statistics**. *University of London, 2013. Chapters 1-8 of Part 1.*

- Errata-corrige at page 30: $\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + c \cdot b}{b \cdot d}$ and $\frac{a}{b} - \frac{c}{d} = \frac{a \cdot d - c \cdot b}{b \cdot d}$

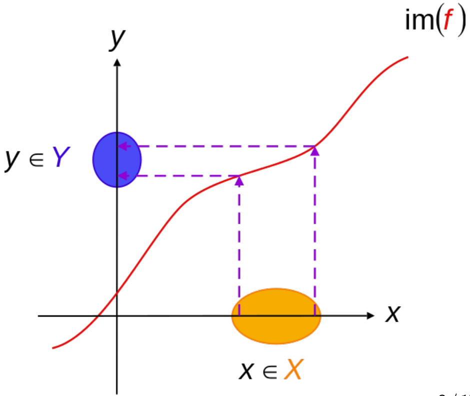
Sets and functions

- Numerical sets
  - $\mathbb{N} = \{0, 1, 2, \ldots\}$ *[Natural numbers]*
  - $\mathbb{Z} = \mathbb{N} \cup \{-1, -2, \ldots\}$ *[Integers]*
  - $\mathbb{Q} = \{^m/_n \mid m, n \in \mathbb{Z}, n \neq 0\}$ *[Rationals]*
  - $\mathbb{R} = \{$ fractional numbers with possibly infinitely many digits $\} \supseteq \mathbb{Q}$ *[Reals]*
  - $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ *[Irrationals]*
    - y such that $y \cdot y = 2$ belongs to $\mathbb{I}$
- Functions
  - $\mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}$ *[Cartesian product]*
  - $f : \mathbb{R} \to \mathbb{R}$ is a subset $f \subseteq \mathbb{R} \times \mathbb{R}$ such that $(x, y_0), (x, y_1) \in f$ implies $y_0 = y_1$ *[Functions]*
    - usually written $f(x) = y$ for $(x, y) \in f$
    - $f(x) = v$ for all x *[Constant functions]*
    - $f(x) = a \cdot x + b$ for fixed a, b *[Linear functions]*
    - $f(x) = a \cdot x^2 + b \cdot x + c$ for fixed a, b, c *[Quadratic functions]*
    - $f(x) = \sum_{i=0}^{n} a_i \cdot x^i$ for fixed $a_0, \ldots, a_n$ *[Polinomials]*

See R script

Functions

- $dom(f) = \{x \in \mathbb{R} \mid \exists\, y \in \mathbb{R}.(x, y) \in f\}$ *[Domain or Support]*
- $im(f) = \{y \in \mathbb{R} \mid \exists\, x \in \mathbb{R}.(x, y) \in f\}$ *[Co-domain or Image]*
- $f^{-1} = \{(y, x) \mid (x, y) \in f\}$ *[Inverse function, also f^{inv}]*
  - f^{-1} is a function iff f is injective
  - $f^{-1}(y) = x$ iff $f(x) = y$
  - $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$
- Examples
  - $\sqrt{y} = x$ iff $x^2 = y$ over $x \geq 0$
  - $\sqrt[n]{y} = x$ iff $x^n = y$ over $x \geq 0$ *[positive root]*



Powers and logarithms

Power laws

The power laws state that

$$a^n \cdot a^m = a^{n+m} \qquad \frac{a^n}{a^m} = a^{n-m} \qquad (a^n)^m = a^{nm}$$

provided that both sides of these expressions exist. In particular, we have

$$a^0 = 1 \qquad \text{and} \qquad a^{-n} = \frac{1}{a^n}.$$

If it exists, we also define the *positive* nth root of a, written $\sqrt[n]{a}$, to be $a^{\frac{1}{n}}$.

- $log_a(y) = x$ iff $a^x = y$ for $a \neq 1, x > 0$ *[Logarithms]*
- for $n/m \in \mathbb{Q}$: $\quad a^{n/m} \overset{\text{def}}{=} \sqrt[m]{a^n}$
- what is a^x for $x \in \mathbb{I}$?

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots = \sum_{n \geq 0} \frac{x^n}{n!}$$

and $a^x = (e^{log_e(a)})^x = e^{x \cdot log_e(a)}$

- $X \sim Poi(\mu)$, $\quad \sum_{k=0}^{\infty} \frac{\mu^k}{k!} e^{-\mu} = e^{-\mu} \sum_{k=0}^{\infty} \frac{\mu^k}{k!} = e^{-\mu} \cdot e^{\mu} = 1$

See R script

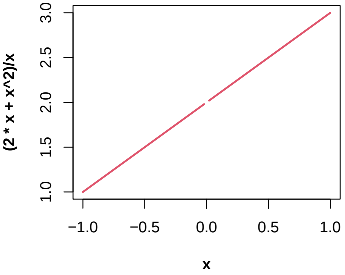
Limits

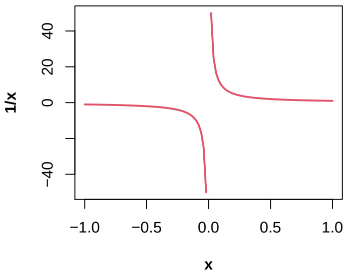
For a function $f()$, and $a \in \mathbb{R} \cup \{-\infty, \infty\}$

$$\lim_{x \to a} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to a$$

if $f(x)$ can be made as close to L as desired, by making x close enough, but not equal, to a.

- Example: $\lim_{x \to 0} \frac{2 \cdot x + x^2}{x} = 2$
- A function $f()$ is called *continuous* at c, if $\lim_{x \to c} f(x) = f(c)$



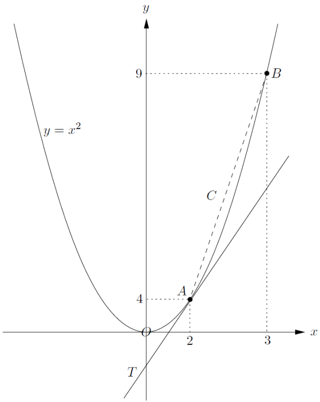


- The limit may not exist, e.g., $\lim_{x \to 0} 1/x$

Gradient and derivatives

- The **gradient** is a measure of how ‘steep’ a function is.
  - For $f(x) = m \cdot x + b$, m is the (constant!) gradient and b the intercept (i.e., $f(x)$ at $x = 0$)
- For $f(x) = x^2$?
  - Tangent at $x = a$ is $y = m \cdot x + b$ where:
    - $m = \frac{f(a+\delta)-f(a)}{\delta} = \frac{2 \cdot a \cdot \delta + \delta^2}{\delta} \to 2 \cdot a$ for $\delta \to 0$
    - since $f(a) = m \cdot a + b$, we have $b = f(a) - m \cdot a = -a^2$
- In general, for $f(x)$?
  - Since m depends on a, we write m as $f'(a)$
  - $f'(a) = \lim_{\delta \to 0} \frac{f(a+\delta)-f(a)}{\delta}$ is called the **derivative** of $f()$,
  - $f'(x)$ also written $\frac{\delta f}{\delta x}$ or $\frac{df}{dx}$
  - Not all functions are differentiable!

See R script or **this Colab Notebook**



Derivatives

Standard derivatives

- If k is a constant, then $f(x) = k$ gives $f'(x) = 0$.
- If $k \neq 0$ is a constant, then $f(x) = x^k$ gives $f'(x) = kx^{k-1}$.
- $f(x) = \mathrm{e}^x$ gives $f'(x) = \mathrm{e}^x$.
- $f(x) = \ln x$ gives $f'(x) = \dfrac{1}{x}$.

- Constant multiple rule:

$$\frac{d}{dx}[k \cdot f(x)] = k \cdot \frac{df}{dx}(x)$$

- Sum rule:

$$\frac{d}{dx}[f(x) + g(x)] = \frac{df}{dx}(x) + \frac{dg}{dx}(x)$$

Derivatives

- Product rule:

$$\frac{d}{dx}[f(x) \cdot g(x)] = \frac{df}{dx}(x) \cdot g(x) + f(x) \cdot \frac{dg}{dx}(x)$$

- Quotient rule:

$$\frac{d}{dx}[\frac{f(x)}{g(x)}] = \left[\frac{df}{dx}(x) \cdot g(x) - f(x) \cdot \frac{dg}{dx}(x)\right] \cdot \frac{1}{g(x)^2}$$

- Chain rule:

$$\frac{d}{dx}[f(g(x))] = \frac{df}{dg}(g(x)) \cdot \frac{dg}{dx}(x)$$

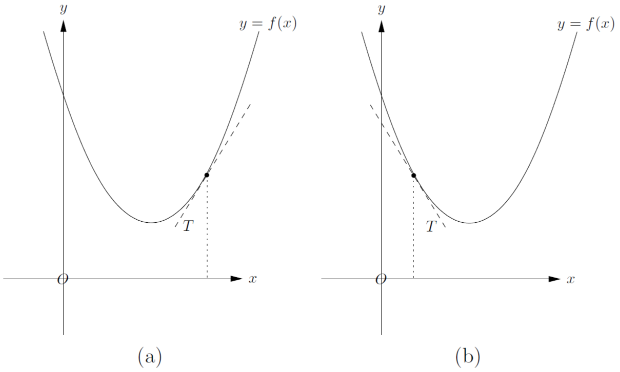
- $\frac{d}{dx}e^{-x} = \dots$
- Inverse rule:

$$\frac{d}{dx}[f^{-1}(x)] = \frac{1}{\frac{df}{dx}(f^{-1}(x))}$$

- $\frac{d}{dx}log\, x = \dots$

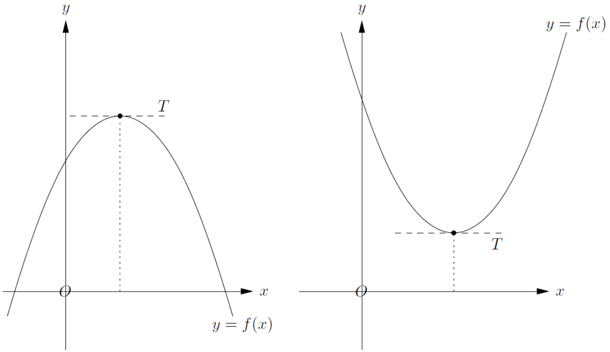
See R script or this Colab Notebook

Optimization



- $f'(x) > 0$ implies $f()$ is increasing at x
- $f'(x) < 0$ implies $f()$ is decreasing at x
- $f'(x) = 0$ we cannot say *[Stationary point]*

Optimization - second derivatives



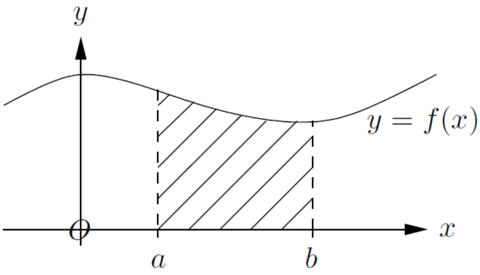
(a) (b)

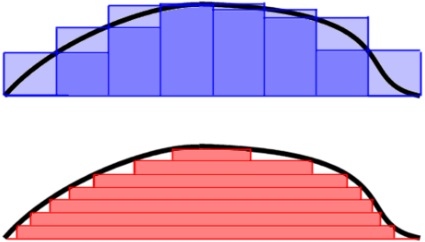
- $f''(x) < 0$ implies $f(x)$ is a maximum
- $f''(x) > 0$ implies $f(x)$ is a minimum
- $f''(x) = 0$ we cannot say *[Maximum, minimum, or point of inflection]*

See this Colab Notebook

Integration

- Given $f(x)$, what is $F(x)$ such that $f(x) = \frac{d}{dx}F(x)$? i.e, such that $F'(x) = f(x)$
- Quick answer: $F(x) = \int_{-\infty}^{x} f(t)dt$
  - Integration is the inverse of differentiation *[Fundamental theorem of calculus]*
- Geometrical definition of integrals:
  - $\int_a^b f(x)dx$ is the area below $f(x)$
  - defined as approximation of domain partitioning (**Riemann–Darboux integrals**) or image partitioning (**Lebesgue integrals**). For $f(x)$ continuous, the two integrals do coincide.





Integration

Key concepts in integration

If $F(x)$ is a function whose derivative is the function $f(x)$, then we have

$$\int f(x)\,\mathrm{d}x = F(x) + c,$$

where c is an arbitrary constant. In particular, we call the

- function, $f(x)$, the *integrand* as it is what we are integrating,
- function, $F(x)$, **an** *antiderivative* as its derivative is $f(x)$,
- constant, c, a *constant of integration* which is completely arbitrary,[†] and
- integral, $\int f(x)\,\mathrm{d}x$, an *indefinite integral* since, in the result, c is arbitrary.

- Definite integrals over an interval $[a, b]$:

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

Integration

Standard integrals

- If $k \neq -1$ is a constant, then $\int x^k \, \mathrm{d}x = \frac{x^{k+1}}{k+1} + c.$

In particular, if $k = 0$, we have $\int 1 \, \mathrm{d}x = \int x^0 \, \mathrm{d}x = x + c.$

- $\int x^{-1} \, \mathrm{d}x = \ln |x| + c.$
- $\int \mathrm{e}^x \, \mathrm{d}x = \mathrm{e}^x + c.$

- Constant multiple rule:

$$\int [k \cdot f(x)] dx = k \cdot \int f(x) dx$$

- Sum rule:

$$\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$$

See R script

Integration by parts

- From the product rule of derivatives:

$$\frac{d}{dx}[f(x) \cdot g(x)] = \frac{df}{dx}(x) \cdot g(x) + f(x) \cdot \frac{dg}{dx}(x)$$

- take the inverse of both sides:

$$f(x) \cdot g(x) = \int f'(x) \cdot g(x)dx + \int f(x) \cdot g'(x)dx$$

- and then:

$$\int f(x) \cdot g'(x)dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x)dx$$

- $\int \lambda x e^{-\lambda x} dx = \ldots = -e^{-\lambda x}(x + {}^{1}/_{\lambda})$
  - consider $f(x) = x$ and $g'(x) = \lambda e^{-\lambda x}$
  - $g(x) = -e^{-\lambda x}$ and $f'(x) = 1$

Integration by change of variable

- Change of variable rule:

$$\int f(y)dy =_{y=g(x)} \int f(g(x))g'(x)dx$$

- $\int \frac{log x}{x} dx = \int y dy = {}^{y^2}/_2$ for $y = log x$ hence, $\int \frac{log x}{x} dx = {}^{(log x)^2}/_2$
    - consider $f(y) = y$ and $g(x) = log x$

Functions of two or more variables

- Symmetry of second derivatives *[Schwarz's theorem]*

$$\frac{d}{dx}\frac{d}{dy}f(x,y)=\frac{d}{dy}\frac{d}{dx}f(x,y)$$

- Leibniz integral rule

$$\frac{d}{dx}\int_a^b f(x,y)dy=\int_a^b \frac{d}{dx}f(x,y)dy$$

- Gradient (pronounced "del") *[direction and rate of fastest increase]*

$$\nabla f(x,y)=\begin{pmatrix} \frac{d}{dx}f(x,y) \\ \frac{d}{dy}f(x,y) \end{pmatrix}$$

- Hessian matrix (2×2 case): *[Generalize the second derivative test for max/min]*

$$\mathbf{H}_2(x,y)=\begin{pmatrix} \frac{d}{dx}\frac{d}{dx}f(x,y) & \frac{d}{dx}\frac{d}{dy}f(x,y) \\ \frac{d}{dy}\frac{d}{dx}f(x,y) & \frac{d}{dy}\frac{d}{dy}f(x,y) \end{pmatrix}$$

Feyman's trick

$$F(t) = \int_0^\infty e^{-tx} dx = \left[-\frac{e^{-tx}}{t} \right]_0^\infty = \frac{1}{t}$$

- using Leibniz integral rule

$$\frac{d}{dt}F(t) = \frac{d}{dt}\int_0^\infty e^{-tx} dx = \int_0^\infty \frac{d}{dt}e^{-tx} dx = -\int_0^\infty xe^{-tx} dx = -\frac{1}{t^2}$$

- Taking further derivatives yields:

$$\int_0^\infty x^{n-1}e^{-tx} dx = \frac{(n-1)!}{t^n}$$

and for $t = 1$: *[**Euler's** $\Gamma(n)$]*

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x} dx = (n-1)!$$

See R script