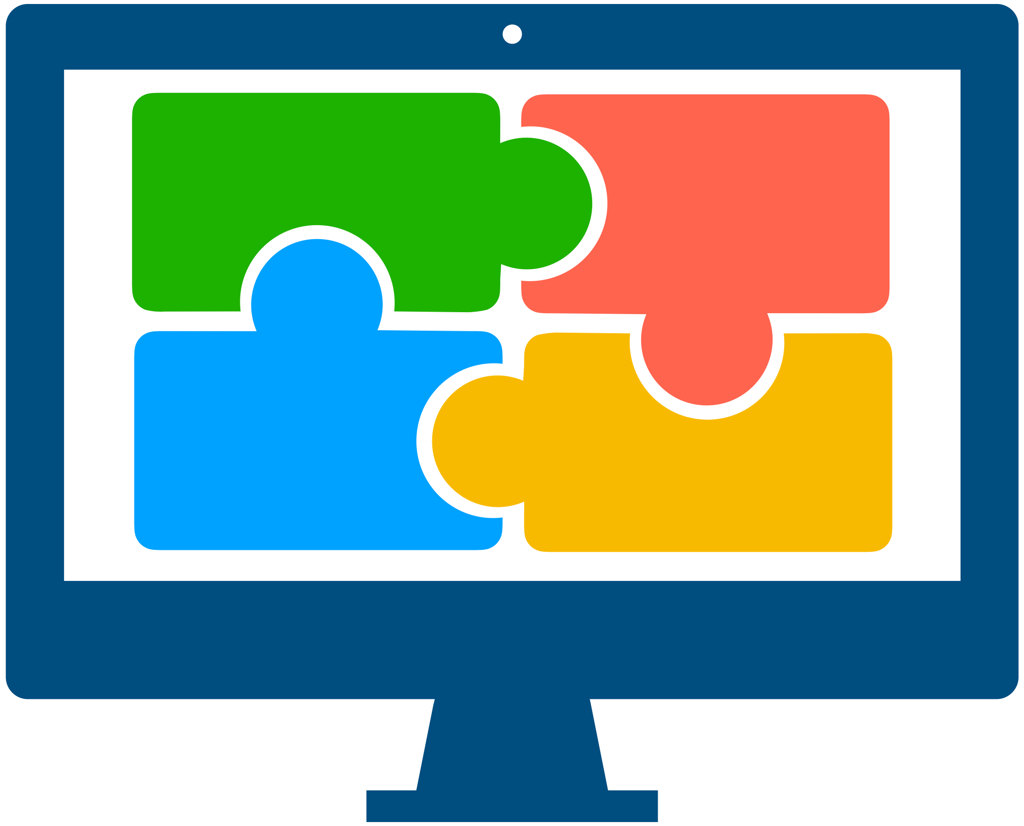

Linguaggi di Programmazione

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Esercitazione #5

HOFL, inferenza di tipi
e semantica operazionale

[Ex. 1] Determinare il tipo del termine HOFL

$$t \stackrel{\text{def}}{=} \mathbf{rec}\ x.\ ((\lambda y.\ \mathbf{if}\ y\ \mathbf{then}\ 0\ \mathbf{else}\ 0)\ x).$$

Poi calcolare la sua forma canonica (lazy).

Ex. 1, inferenza di tipi

$$t \triangleq \underbrace{\mathbf{rec}\ \underbrace{x}_{int}.\ \underbrace{(\ \underbrace{(\ \lambda \underbrace{y}_{int}.\ \underbrace{\mathbf{if}\ \underbrace{y}_{int}\ \mathbf{then}\ \underbrace{0}_{int}\ \mathbf{else}\ \underbrace{0}_{int}}_{int}\ )}_{int \to int}\ \underbrace{x}_{int}\ )}_{int}}_{int} \quad : int$$

Ex. 1, forma canonica?

$$t \triangleq \textbf{rec}\ x.\ (\ (\ \lambda y.\ \textbf{if}\ y\ \textbf{then}\ 0\ \textbf{else}\ 0\ )\ x\ ) \quad : int$$

$$t \to c \nwarrow\ (\ (\ \lambda y.\ \textbf{if}\ y\ \textbf{then}\ 0\ \textbf{else}\ 0\ )\ x\ )[^t/_x] \to c$$

$$= (\ (\ \lambda y.\ \textbf{if}\ y\ \textbf{then}\ 0\ \textbf{else}\ 0\ )\ t\ ) \to c$$

$$\nwarrow\ \lambda y.\ \textbf{if}\ y\ \textbf{then}\ 0\ \textbf{else}\ 0 \to \lambda x'.\ t'\ ,\ \ t'[^t/_{x'}] \to c$$

$$\nwarrow_{x'=y,t'=\textbf{if}\ldots}\ (\ \textbf{if}\ y\ \textbf{then}\ 0\ \textbf{else}\ 0\ )[^t/_y] \to c$$

$$= (\ \textbf{if}\ t\ \textbf{then}\ 0\ \textbf{else}\ 0\ ) \to c$$

$$\nwarrow\ t \to n\ ,\ \ 0 \to c$$ (it doesn't matter if $n = 0$)

same goal from which we started:
no canonical form

$[\mathbf{Ex.\ 2}]$ Determinare il tipo del termine

$$map \stackrel{\text{def}}{=} \lambda f.\ \lambda x.\ ((f\ \mathbf{fst}(x)), (f\ \mathbf{snd}(x)))$$

Poi calcolare le forme canoniche (lazy) dei termini seguenti.

$$t_1 \stackrel{\text{def}}{=} map\ (\lambda z.\ 2 \times z)\ (1,2) \qquad\qquad t_2 \stackrel{\text{def}}{=} \mathbf{fst}\ (\ map\ (\lambda z.\ 2 \times z)\ (1,2)\ )$$

Ex. 2, inferenza di tipi

$$map \triangleq \lambda f \,.\, \lambda x \,.\, (\,(\,f\ \mathbf{fst}(x)\,)\,,\,(\,f\ \mathbf{snd}(x)\,)\,)$$

- $f : \tau_1 \to \tau$
- $x : \tau_1 * \tau_1$
- $f : \tau_1 \to \tau$, $x : \tau_1 * \tau_2$
- $\mathbf{fst}(x) : \tau_1$
- $f : \tau_1 \to \tau$, $x : \tau_1 * \tau_2$
- $\mathbf{snd}(x) : \tau_2 = \tau_1$
- $(\,f\ \mathbf{fst}(x)\,) : \tau$
- $(\,f\ \mathbf{snd}(x)\,) : \tau$
- $(\,(\,f\ \mathbf{fst}(x)\,)\,,\,(\,f\ \mathbf{snd}(x)\,)\,) : \tau * \tau$
- $\lambda x \,.\, (\ldots) : \tau_1 * \tau_1 \to \tau * \tau$
- $\lambda f \,.\, \lambda x \,.\, (\ldots) : (\tau_1 \to \tau) \to \tau_1 * \tau_1 \to \tau * \tau$

$$map \ :\ (\tau_1 \to \tau) \to \tau_1 * \tau_1 \to \tau * \tau$$

Ex. 2a, forma canonica

$$map \triangleq \lambda f\ .\ \lambda x\ .\ (\ (\ f\ \mathbf{fst}(x)\ )\ ,\ (\ f\ \mathbf{snd}(x)\ )\ )$$

$$t_1 \triangleq map\ (\lambda z.\ 2 \times z)\ (1,2)$$

$$t_1 \to c\ \nwarrow\ \ (map\ (\lambda z.\ 2 \times z)) \to \lambda x'.\ t'\ ,\ t'[^{(1,2)}/_{x'}] \to c$$

$$\nwarrow\ map \to \lambda f'.\ t''\ ,\ t''[^{\lambda z.\ 2\times z}/_{f'}] \to \lambda x'.\ t'\ ,\ t'[^{(1,2)}/_{x'}] \to c$$

$$\nwarrow_{f'=f, t''=\lambda x \ldots}\ (\lambda x.\ ((f\ \mathbf{fst}(x)), (f\ \mathbf{snd}(x))))[^{\lambda z.\ 2\times z}/_{f}] \to \lambda x'.\ t'\ ,\ t'[^{(1,2)}/_{x'}] \to c$$

$$= (\lambda x.\ (((\lambda z.\ 2 \times z)\ \mathbf{fst}(x)), ((\lambda z.\ 2 \times z)\ \mathbf{snd}(x)))) \to \lambda x'.\ t'\ ,\ t'[^{(1,2)}/_{x'}] \to c$$

$$\nwarrow_{x'=x, t'=(\ldots,\ldots)}\ (((\lambda z.\ 2 \times z)\ \mathbf{fst}(x)), ((\lambda z.\ 2 \times z)\ \mathbf{snd}(x)))[^{(1,2)}/_{x}] \to c$$

$$= (((\lambda z.\ 2 \times z)\ \mathbf{fst}(1,2)), ((\lambda z.\ 2 \times z)\ \mathbf{snd}(1,2))) \to c$$

$$\nwarrow_{c=(((\lambda z.\ 2\times z)\ \mathbf{fst}(1,2)),((\lambda z.\ 2\times z)\ \mathbf{snd}(1,2)))}\ \square$$

Ex. 2b, forma canonica

$$t_1 \to (\ ((\lambda z.\ 2 \times z)\ \mathbf{fst}(1,2))\ ,\ ((\lambda z.\ 2 \times z)\ \mathbf{snd}(1,2))\ )$$

$$\mathbf{fst}(t_1) \to c \ \nwarrow \ t_1 \to (t_1', t_2')\ ,\ t_1' \to c$$

$$\nwarrow^{*}_{t_1'=(\lambda z.\ 2\times z)\ \mathbf{fst}(1,2)\ ,\ t_2'=(\lambda z.\ 2\times z)\ \mathbf{snd}(1,2)} (\lambda z.\ 2 \times z)\ \mathbf{fst}(1,2) \to c$$

$$\nwarrow \lambda z.\ 2 \times z \to \lambda z'.\ t'\ ,\ t'[^{\mathbf{fst}(1,2)}/_{z'}] \to c$$

$$\nwarrow_{z'=z, t'=2\times z} (2 \times z)[^{\mathbf{fst}(1,2)}/_{z}] \to c$$

$$= (2 \times \mathbf{fst}(1,2)) \to c$$

$$\nwarrow_{c=n_1 \underline{\times} n_2} 2 \to n_1\ ,\ \mathbf{fst}(1,2) \to n_2$$

$$\nwarrow^{*}_{n_1=2} (1,2) \to (t_1'', t_2'')\ ,\ t_1'' \to n_2$$

$$\nwarrow_{t_1''=1, t_2''=2} 1 \to n_2$$

$$\nwarrow_{n_2=1} \square$$

$$c = n_1 \underline{\times} n_2 = 2 \underline{\times} 1 = 2$$

Teoria dei domini

[**Ex. 3**] Let $(D, \sqsubseteq_D)$ be a CPO and $f : D \to D$ be a continuous function. Prove that the set of fixpoints of f is itself a CPO (ordered by $\sqsubseteq_D$).

Ex. 3, CPO dei puntifissi

$(D, \sqsubseteq_D)$ CPO $\qquad f : D \to D$ continua

$\mathsf{FP}_f \triangleq \{ \ d \ | \ d = f(d) \ \}$ l'insieme di tutti i punti fissi di f

$(\mathsf{FP}_f, \sqsubseteq) \qquad \sqsubseteq \triangleq \sqsubseteq_D \cap (\mathsf{FP}_f \times \mathsf{FP}_f)$ CPO?

e' un PO (perche' $\mathsf{FP}_f \subseteq D$)

proviamo che e' completo prendiamo una catena $\{d_i\}_{i \in \mathbb{N}} \subseteq \mathsf{FP}_f$

mostriamo $\bigsqcup_{i \in \mathbb{N}} d_i$ come calcolato in D e' un puntofisso di f

$$f\left(\bigsqcup_{i \in \mathbb{N}} d_i\right) = \bigsqcup_{i \in \mathbb{N}} f(d_i)$$ by continuita'

$$= \bigsqcup_{i \in \mathbb{N}} d_i$$ ogni d_i e' un puntofisso

HOFL semantica denotazionale

[**Ex. 4**] (Test for convergence) We would like to modify the denotational semantics of HOFL assigning to the construct

$$\mathbf{if}\ t\ \mathbf{then}\ t_0\ \mathbf{else}\ t_1$$

- the semantics of t_1 if the semantics of t is $\perp_{\mathbb{Z}_\perp}$, and
- the semantics of t_0 otherwise.

Is it possible? If not, why?

Ex. 4, test di convergenza

$$[\![\textbf{if } t \textbf{ then } t_0 \textbf{ else } t_1]\!]\rho \triangleq \mathsf{Cond}^{\perp}_{\tau}(\ [\![t]\!]\rho\ ,\ [\![t_0]\!]\rho\ ,\ [\![t_1]\!]\rho\ )$$

$$\mathsf{Cond}^{\perp}_{\tau}(v, d_0, d_1) \triangleq \begin{cases} d_0 & \text{se } v = \lfloor n \rfloor \text{ per qualche } n \\ d_1 & \text{altrimenti} \end{cases}$$

nessun problema?

$\mathsf{Cond}^{\perp}_{\tau}$ non e' monotona in *v* !

Counterexample $\perp_{\mathbb{Z}_\perp} \sqsubseteq_{\mathbb{Z}_\perp} \lfloor 1 \rfloor$ Take $d_1 \not\sqsubseteq_{D_\tau} d_0$

$$(\perp_{\mathbb{Z}_\perp}, d_0, d_1) \sqsubseteq_{\mathbb{Z}_\perp \times D_\tau \times D_\tau} (\lfloor 1 \rfloor, d_0, d_1)$$

$$\mathsf{Cond}^{\perp}_{\tau}(\perp_{\mathbb{Z}_\perp}, d_0, d_1) = d_1 \not\sqsubseteq_{D_\tau} d_0 = \mathsf{Cond}^{\perp}_{\tau}(\lfloor 1 \rfloor, d_0, d_1)$$

Ex. 4, test di convergenza

Per esempio prendiamo $d_0 = \lfloor 0 \rfloor \qquad d_1 = \lfloor 1 \rfloor$

$$[\![\mathbf{if}\ \mathbf{rec}\ x.\ x\ \mathbf{then}\ 0\ \mathbf{else}\ 1]\!]\rho = \lfloor 1 \rfloor$$

$$\not\sqsubseteq_{\mathbb{Z}_\perp}$$

$$[\![\mathbf{if}\ 1\ \mathbf{then}\ 0\ \mathbf{else}\ 1]\!]\rho = \lfloor 0 \rfloor$$

come conseguenza

$$t \triangleq \lambda x.\ \mathbf{if}\ x\ \mathbf{then}\ 0\ \mathbf{else}\ 1\ :\ int \rightarrow int$$

non puo' avere una semantica in $D_{int \rightarrow int} = [\mathbb{Z}_\perp \rightarrow \mathbb{Z}_\perp]_\perp$

perche' $[\![t]\!]\rho$ non e' continua (non e' monotona)

[**Ex. 5**] (Strict conditional) Modify the operational semantics of HOFL by taking the following rules for conditionals:

$$\frac{t \to 0 \quad t_0 \to c_0 \quad t_1 \to c_1}{\textbf{if } t \textbf{ then } t_0 \textbf{ else } t_1 \ \to \ c_0} \qquad \frac{t \to n \quad n \neq 0 \quad t_0 \to c_0 \ \ t_1 \to c_1}{\textbf{if } t \textbf{ then } t_0 \textbf{ else } t_1 \ \to \ c_1}.$$

Without changing the denotational semantics, prove that:

1. for any term t and canonical form c, we have $t \to c \ \Rightarrow \ \forall \rho.\ [\![t]\!]\,\rho = [\![c]\!]\,\rho$;
2. in general $t{\Downarrow} \not\Rightarrow t{\downarrow}$ (exhibit a counterexample).

Ex. 5.1, correttezza

$$\frac{t \to 0 \quad t_0 \to c_0 \quad t_1 \to c_1}{\textbf{if } t \textbf{ then } t_0 \textbf{ else } t_1 \ \to \ c_0} \qquad \frac{t \to n \quad n \neq 0 \quad t_0 \to c_0 \quad t_1 \to c_1}{\textbf{if } t \textbf{ then } t_0 \textbf{ else } t_1 \ \to \ c_1}.$$

$$t \to c \ \Rightarrow \ \forall \rho.\ [\![t]\!]\,\rho = [\![c]\!]\,\rho$$

$$P(t \to c) \triangleq \forall \rho.\ [\![t]\!]\rho = [\![c]\!]\rho$$

estendiamo la prova di correttezza (per induzione sulle regole) per considerare le nuove regole

Ex. 5.1, correttezza

$$P(t \to c) \triangleq \forall \rho.\ [\![t]\!]\rho = [\![c]\!]\rho$$

$$\frac{t \to 0 \quad t_0 \to c_0 \quad t_1 \to c_1}{\textbf{if } t \textbf{ then } t_0 \textbf{ else } t_1 \ \to \ c_0}$$

assumiamo

$$P(t \to 0) \triangleq \forall \rho.\ [\![t]\!]\rho = [\![0]\!]\rho = \lfloor 0 \rfloor$$

$$P(t_0 \to c_0) \triangleq \forall \rho.\ [\![t_0]\!]\rho = [\![c_0]\!]\rho$$

$$P(t_1 \to c_1) \triangleq \forall \rho.\ [\![t_1]\!]\rho = [\![c_1]\!]\rho$$

vogliamo provare

$$P(\textbf{if } t \textbf{ then } t_0 \textbf{ else } t_1 \to c_0) \triangleq \forall \rho.\ [\![\textbf{if } t \textbf{ then } t_0 \textbf{ else } t_1]\!]\rho = [\![c_0]\!]\rho$$

$$[\![\textbf{if } t \textbf{ then } t_0 \textbf{ else } t_1]\!]\rho = \mathsf{Cond}_\tau([\![t]\!]\rho, [\![t_0]\!]\rho, [\![t_1]\!]\rho) \quad \text{per def}$$

$$= \mathsf{Cond}_\tau(\lfloor 0 \rfloor, [\![c_0]\!]\rho, [\![c_1]\!]\rho) \quad \text{per ip.ind.}$$

$$= [\![c_0]\!]\rho \quad \text{per Cond}$$

Ex. 5.1, correttezza

$$P(t \to c) \triangleq \forall \rho.\ [\![t]\!]\rho = [\![c]\!]\rho$$

$$\frac{t \to n \quad n \neq 0 \quad t_0 \to c_0 \quad t_1 \to c_1}{\textbf{if}\ t\ \textbf{then}\ t_0\ \textbf{else}\ t_1 \ \to\ c_1}$$

assumiamo $P(t \to n) \triangleq \forall \rho.\ [\![t]\!]\rho = [\![n]\!]\rho = \lfloor n \rfloor \quad n \neq 0$

$$P(t_0 \to c_0) \triangleq \forall \rho.\ [\![t_0]\!]\rho = [\![c_0]\!]\rho$$

$$P(t_1 \to c_1) \triangleq \forall \rho.\ [\![t_1]\!]\rho = [\![c_1]\!]\rho$$

vogliamo provare

$$P(\textbf{if}\ t\ \textbf{then}\ t_0\ \textbf{else}\ t_1 \to c_1) \triangleq \forall \rho.\ [\![\textbf{if}\ t\ \textbf{then}\ t_0\ \textbf{else}\ t_1]\!]\rho = [\![c_1]\!]\rho$$

$$[\![\textbf{if}\ t\ \textbf{then}\ t_0\ \textbf{else}\ t_1]\!]\rho = \mathsf{Cond}_\tau([\![t]\!]\rho, [\![t_0]\!]\rho, [\![t_1]\!]\rho) \quad \text{per def}$$

$$= \mathsf{Cond}_\tau(\lfloor n \rfloor, [\![c_0]\!]\rho, [\![c_1]\!]\rho) \quad \text{per ip.ind.}$$

$$= [\![c_1]\!]\rho \quad \text{per Cond}$$

Ex. 5.2, inconsistenza

vogliamo trovare un termine t tale che $t \Downarrow$ $t \uparrow$

consideriamo $t \triangleq \textbf{if } 0 \textbf{ then } 1 \textbf{ else rec } x.\ x \ : \ int$

$$[\![t]\!]\rho = \mathsf{Cond}_{int}([\![0]\!]\rho, [\![1]\!]\rho, [\![\textbf{rec } x.\ x]\!]\rho)$$

$$= \mathsf{Cond}_{int}(\lfloor 0 \rfloor, \lfloor 1 \rfloor, \bot_{\mathbb{Z}_\bot}) \quad = \lfloor 1 \rfloor \qquad t \Downarrow$$

$$t \to c \ \nwarrow \ 0 \to 0 \ , \ 1 \to c \ , \ \textbf{rec } x.\ x \to c_1$$

$$\nwarrow^*_{c=1} \ \textbf{rec } x.\ x \to c_1$$

$$\nwarrow \ x[^{\textbf{rec } x.\ x}/_x] \to c_1$$

$$= \textbf{rec } x.\ x \to c_1 \qquad t \uparrow$$

[**Ex. 6**] Determine the type of the HOFL term

$$t \stackrel{\text{def}}{=} \mathbf{rec}\ f.\ (\ \lambda x.1\ ,\ \mathbf{fst}(f)\ 0\ )$$

Then, compute the (lazy) denotational semantics of t.

Ex. 6, inferenza di tipi

$$t \triangleq \underbrace{\mathbf{rec}\ \underbrace{f}_{(int \to \tau_1) * \tau_2}.\ \underbrace{(\ \underbrace{\lambda \underbrace{x}_{\tau}.\ \underbrace{1}_{int}}_{\tau \to int}\ ,\ \underbrace{(\underbrace{\mathbf{fst}(\ \underbrace{f}_{(int \to \tau_1) * \tau_2}\ )}_{int \to \tau_1}\ \underbrace{0}_{int}\ )}_{\tau_1}\ )}_{(\tau \to int) * \tau_1}}_{(int \to \tau_1) * \tau_2 = (\tau \to int) * \tau_1} \quad : (int \to int) * int$$

$$\begin{cases} int = \tau \\ \tau_1 = int \\ \tau_2 = \tau_1 \end{cases}$$

$$\tau = \tau_1 = \tau_2 = int$$

Ex. 6, semantica den

$$t \triangleq \textbf{rec}\ f.\ (\ \lambda x.\ 1\ ,\ (\textbf{fst}(\ f\ )\ 0\ )\ )\quad : (int \to int) * int$$

$$[\![t]\!]\rho = \mathit{fix}\ \lambda d_f.\ [\![(\lambda x.\ 1, \textbf{fst}(f)\ 0)]\!]\rho[^{d_f}/_f]$$

$$= \mathit{fix}\ \lambda d_f.\ \lfloor\ (\ [\![\lambda x.\ 1]\!]\rho[^{d_f}/_f]\ ,\ [\![\textbf{fst}(f)\ 0]\!]\rho[^{d_f}/_f]\ )\ \rfloor$$

$$\rho' = \rho[^{d_f}/_f]$$

$$= \mathit{fix}\ \lambda d_f.\ \lfloor\ (\ \lfloor\ \lambda d_x.\ [\![1]\!]\rho'[^{d_x}/_x]\ \rfloor\ ,\ (\textbf{let}\ \varphi \Leftarrow [\![\textbf{fst}(f)]\!]\rho'.\ \varphi([\![0]\!]\rho'))\ )\ \rfloor$$

$$= \mathit{fix}\ \lambda d_f.\ \lfloor\ (\ \lfloor\ \lambda d_x.\ \lfloor 1 \rfloor\ \rfloor\ ,\ (\textbf{let}\ \varphi \Leftarrow \pi_1^*([\![f]\!]\rho').\ \varphi\ \lfloor 0 \rfloor)\ )\ \rfloor$$

$$= \mathit{fix}\ \lambda d_f.\ \lfloor\ (\ \lfloor\ \lambda d_x.\ \lfloor 1 \rfloor\ \rfloor\ ,\ (\textbf{let}\ \varphi \Leftarrow \pi_1^*\ d_f.\ \varphi\ \lfloor 0 \rfloor)\ )\ \rfloor$$

Ex. 6, semantica den

$$[\![t]\!]\rho = \mathit{fix}\ \lambda d_f.\ \lfloor\ (\ \lfloor\ \lambda d_x.\ \lfloor 1 \rfloor\ \rfloor\ ,\ (\mathbf{let}\ \varphi \Leftarrow \pi_1^*\ d_f.\ \varphi\ \lfloor 0 \rfloor)\ )\ \rfloor$$

$$f_0 = \bot_{D_{(int \to int) * int}}$$

$$f_1 = \lfloor\ (\ \lfloor\ \lambda d_x.\ \lfloor 1 \rfloor\ \rfloor\ ,\ (\mathbf{let}\ \varphi \Leftarrow \pi_1^*\ f_0.\ \varphi\ \lfloor 0 \rfloor)\ )\ \rfloor$$

$$= \lfloor\ (\ \lfloor\ \lambda d_x.\ \lfloor 1 \rfloor\ \rfloor\ ,\ \bot_{D_{int}}\ )\ \rfloor$$

$$f_2 = \lfloor\ (\ \lfloor\ \lambda d_x.\ \lfloor 1 \rfloor\ \rfloor\ ,\ (\mathbf{let}\ \varphi \Leftarrow \pi_1^*\ f_1.\ \varphi\ \lfloor 0 \rfloor)\ )\ \rfloor$$

$$= \lfloor\ (\ \lfloor\ \lambda d_x.\ \lfloor 1 \rfloor\ \rfloor\ ,\ (\mathbf{let}\ \varphi \Leftarrow \lfloor\ \lambda d_x.\ \lfloor 1 \rfloor\ \rfloor.\ \varphi\ \lfloor 0 \rfloor)\ )\ \rfloor$$

$$= \lfloor\ (\ \lfloor\ \lambda d_x.\ \lfloor 1 \rfloor\ \rfloor\ ,\ (\lambda d_x.\ \lfloor 1 \rfloor)\ \lfloor 0 \rfloor\ )\ \rfloor$$

$$= \lfloor\ (\ \lfloor\ \lambda d_x.\ \lfloor 1 \rfloor\ \rfloor\ ,\ \lfloor 1 \rfloor\ )\ \rfloor$$ maximal element!

Ex. 6, semantica den

$$t \triangleq \mathbf{rec}\ f.\ (\ \lambda x.\ 1\ ,\ (\mathbf{fst}(\ f\ )\ 0\ )\ )\quad : (int \to int) * int$$

$$[\![t]\!]\rho = \mathit{fix}\ \lambda d_f.\ \lfloor\ (\ \lfloor\ \lambda d_x.\ \lfloor 1 \rfloor\ \rfloor\ ,\ (\mathbf{let}\ \varphi \Leftarrow \pi_1^*\ d_f.\ \varphi\ \lfloor 0 \rfloor)\ )\ \rfloor$$

$$[\![t]\!]\rho = \lfloor\ (\ \lfloor\ \lambda d_x.\ \lfloor 1 \rfloor\ \rfloor\ ,\ \lfloor 1 \rfloor\ )\ \rfloor$$