Linguaggi di Programmazione

Roberta Gori

HOFL Sintassi e Tipi - 7.1

HOFL pre-termini
(linguaggio funzionale di ordine superiore)

Sintassi HOFL

Attenzione : t=0 e' true
t!=0 e' false !!!

$n \in \mathbb{Z}$

$\mathsf{op} \in \{+, -, \times\}$

$\textbf{if } t = 0 \textbf{ then } t_0 \textbf{ else } t_1$

$x \in Ide$

$$
\begin{aligned}
t \quad ::= \quad & x \mid n \mid t_0 \ \mathsf{op} \ t_1 \mid \textbf{if } t \textbf{ then } t_0 \textbf{ else } t_1 && \text{standard} \\
\mid \quad & (t_0, t_1) \mid \textbf{fst}(t) \mid \textbf{snd}(t) && \text{coppie proiezioni} \\
\mid \quad & \lambda x.\ t \mid t_0\ t_1 && \text{abstrazione applicazione} \\
\mid \quad & \textbf{rec } x.\ t && \text{ricorsione}
\end{aligned}
$$

$\mathsf{let}\ x = t\ \mathsf{in}\ t$

can contain x

Esercizio

$$\textbf{rec}\ f.\ \lambda x.\ \textbf{if}\ x\ \textbf{then}\ 1\ \textbf{else}\ x \times (f\ (x-1))$$

quale e' il significato del precedente pre-termine?

Definisce la funzione fattoriale

Esercizio

$$\textbf{rec}\ rep.\ \lambda n.\ \lambda f.\ \lambda x.\ \textbf{if}\ n\ \textbf{then}\ x$$
$$\textbf{else}\ f\ (rep\ (n-1)\ f\ x)$$

quale e'significato del precedente pre-termine?

$$rep\ n\ f\ x = f^n\ x$$

Esercizio

$$\lambda x. \left(\left(\begin{array}{ll} \mathbf{rec}\ f.\ \lambda y. & \mathbf{if}\ (x - y)\ \mathbf{then}\ 0 \\ & \mathbf{else\ if}\ (x + y)\ \mathbf{then}\ 1 \\ & \mathbf{else}\ f\ (y + 1) \end{array} \right)\ 0 \right)$$

quale e' significato del precedente pre-termine?

maggiore o uguale a 0

Esercizio (da consegnare)

assumiamo $true = 0$

$false =$ qls $n \neq 0$

riempire al posto dei puntini (in HOFL)

$$
\begin{aligned}
or &\triangleq \lambda n.\ \lambda m.\ \cdots \\
and &\triangleq \lambda n.\ \lambda m.\ \cdots \\
not &\triangleq \lambda m.\ \cdots \\
implies &\triangleq \lambda n.\ \lambda m.\ \cdots \\
iff &\triangleq \lambda n.\ \lambda m.\ \cdots
\end{aligned}
$$

Pre-termini

$$
\begin{array}{lcl}
t & ::= & x \mid n \mid t_0 \ \mathsf{op} \ t_1 \mid \textbf{if } t \textbf{ then } t_0 \textbf{ else } t_1 \\
 & \mid & (t_0, t_1) \mid \textbf{fst}(t) \mid \textbf{snd}(t) \\
 & \mid & \lambda x.\ t \mid t_0\ t_1 \\
 & \mid & \textbf{rec } x.\ t
\end{array}
$$

Perche' sono chiamati pre-termini?

✓	✗
$x + 1$	$1 + (0, 5)$
$\textbf{if } x \textbf{ then } x + 1 \textbf{ else } x - 1$	$2 \times \lambda x.\ x$
$(0, \lambda x.\ x)$	$3\ \lambda x.\ x + 1$
$\textbf{fst}(0, \lambda x.\ x)$	$\textbf{fst}(3)$
$(\lambda x.\ x + 1)\ 3$	$\textbf{if } x \textbf{ then } \lambda x.\ x \textbf{ else } (x, x)$
$\textbf{rec } f.\ \lambda x.\ x + (f\ 0)$	$\textbf{rec } f.\ \lambda x.\ f + x$

abbiamo bisogno di un sistema di tipi

tipi HOFL

Tipi

$$
\begin{array}{rcl}
t & ::= & x \mid n \mid t_0 \ \mathsf{op}\ t_1 \mid \textbf{if}\ t\ \textbf{then}\ t_0\ \textbf{else}\ t_1 \\
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& \mid & \lambda x.\ t \mid t_0\ t_1 \\
& \mid & \textbf{rec}\ x.\ t
\end{array}
$$

quali tipi?

infinite combinazioni!

coppie

funzioni

int

$int * int$

$int \to int$

$int * (int \to int)$

$(int * int) \to int$

$int * (int * int)$

$(int \to int) \to int$

$(int \to int) * (int \to int)$

$(int \to int) \to (int \to (int * int))$

Sintassi dei tipi

$$\tau \ ::= \ int \ \mid \ \tau_0 * \tau_1 \ \mid \ \tau_0 \to \tau_1$$

$\mathcal{T}$

insieme di tutti i tipi

perche' non le liste?

per la stessa ragione per cui evitiamo la divisione
testa e coda non sono funzioni totali

assumiamo variabili tipate

$$Ide = \{Ide_\tau\}_{\tau \in \mathcal{T}}$$

$$\widehat{\cdot} : Ide \to \mathcal{T}$$

$\widehat{x}$ denota il tipo di x

Type judgements

si legge "ha tipo"

formula: $t : \tau$

sono assegnati ai pre-termini
usando un insieme di regole di inferenza
(induzione strutturale sulla sintassi HOFL)

Sistema di tipi

$$\frac{}{x : \widehat{x}} \qquad \frac{}{n : int} \qquad \frac{t_0 : int \quad t_1 : int}{t_0 \text{ op } t_1 : int} \qquad \frac{t : int \quad t_0 : \tau \quad t_1 : \tau}{\textbf{if } t \textbf{ then } t_0 \textbf{ else } t_1 : \tau}$$

$$\frac{t_0 : \tau_0 \quad t_1 : \tau_1}{(t_0, t_1) : \tau_0 * \tau_1} \qquad \frac{t : \tau_0 * \tau_1}{\textbf{fst}(t) : \tau_0} \qquad \frac{t : \tau_0 * \tau_1}{\textbf{snd}(t) : \tau_1}$$

$$\frac{x : \tau_0 \quad t : \tau_1}{\lambda x.\, t : \tau_0 \rightarrow \tau_1} \qquad \frac{t_1 : \tau_0 \rightarrow \tau_1 \quad t_0 : \tau_0}{t_1\ t_0 : \tau_1}$$

$$\frac{x : \tau \quad t : \tau}{\textbf{rec } x.\, t : \tau}$$

Termini ben formati

$$
\begin{array}{rcl}
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 & \mid & \lambda x.\ t \mid t_0\ t_1 \\
 & \mid & \mathbf{rec}\ x.\ t
\end{array}
$$

$$\tau \ ::= \ int \mid \tau_0 * \tau_1 \mid \tau_0 \rightarrow \tau_1$$

$\mathcal{T}$ insieme di tutti i tipi

un pre-termine t e' ben formato se $\exists \tau \in \mathcal{T}.\ t : \tau$

cioe' se possiamo assegnargli un tipo
anche detto ben tipato *o tipabile*

T_τ insieme di tutti i termini ben formati di tipo τ

Controllo dei tipi
Church Type Theory

Church Type Theory

- Le variabili sono etichettate con tipi (dichiarati)
- Deduciamo il tipo dei termini per induzione strutturale cioe' usando le regole di inferenza in modo bottom-up

Esempio

$$\frac{}{x : \widehat{x}} \qquad \frac{}{n : int} \qquad \frac{t_0 : int \quad t_1 : int}{t_0 \ \mathsf{op}\ t_1 : int} \qquad \frac{t : int \quad t_0 : \tau \quad t_1 : \tau}{\mathbf{if}\ t\ \mathbf{then}\ t_0\ \mathbf{else}\ t_1 : \tau}$$

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$$fact \triangleq \mathbf{rec}\ f : int \to int.\ \lambda x : int.\ \mathbf{if}\ x\ \mathbf{then}\ 1\ \mathbf{else}\ x \times (f\ (x-1))$$

$$\frac{}{fact : int \to int}$$

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scritto in maniera più semplice

$$fact \stackrel{\text{def}}{=} \mathbf{rec}\ \underset{int \to int}{\underset{\sqcup}{f}}\ .\lambda\ \underset{int}{\underset{\sqcup}{x}}\ .\mathbf{if}\ x\ \mathbf{then}\ 1\ \mathbf{else}\ x \times (\ \ f\ \ (\ x - 1\ ))$$

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=

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Esempio

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scritto in maniera più semplice

$$fact \stackrel{\text{def}}{=} \mathbf{rec}\ \underbrace{f}_{int\rightarrow int} .\lambda \underbrace{x}_{int} .\mathbf{if}\ \underbrace{x}_{int}\ \mathbf{then}\ \underbrace{1}_{int}\ \mathbf{else}\ \underbrace{\underbrace{x}_{int} \times (\underbrace{\underbrace{f}_{int\rightarrow int}\ (\underbrace{\underbrace{x}_{int} - \underbrace{1}_{int}}_{int})}_{int})}_{int}$$

Esempio

$$fact \triangleq \textbf{rec}\ f : int \to int.\ \lambda x : int.\ \textbf{if}\ x\ \textbf{then}\ 1\ \textbf{else}\ x \times (f\ (x-1))$$

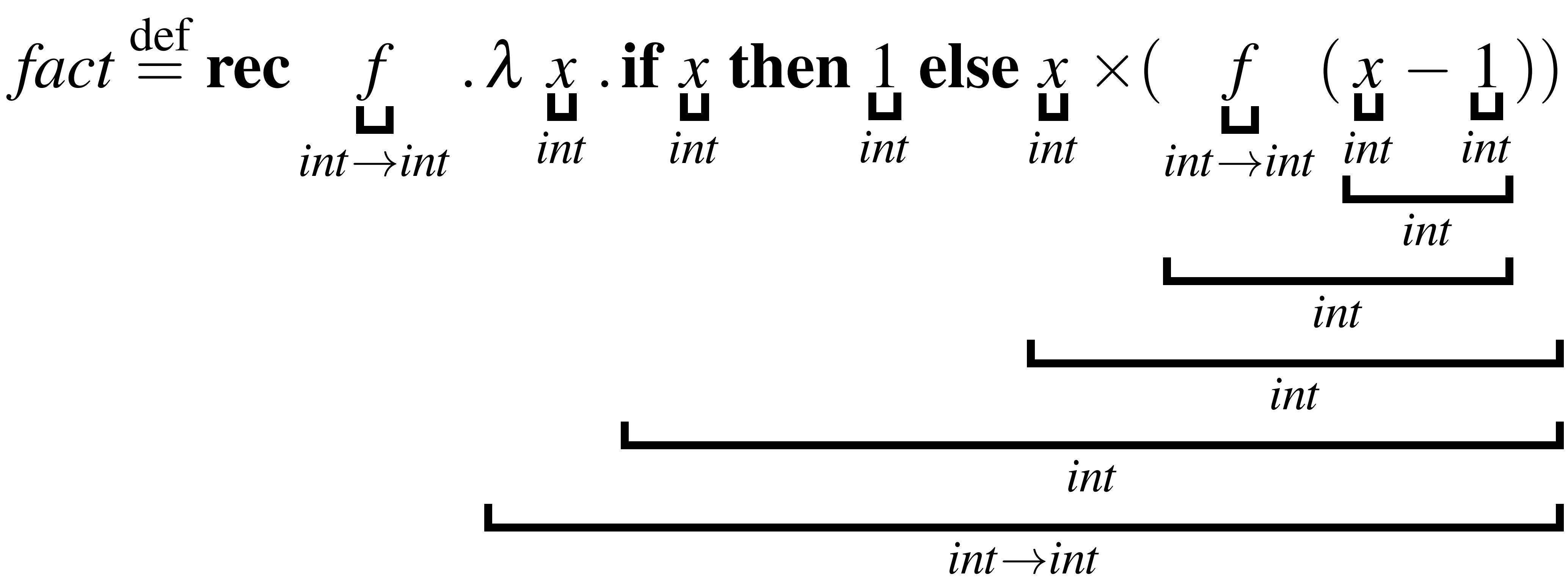
scritto in maniera più semplice

$$fact \stackrel{\text{def}}{=} \textbf{rec}\ \underbrace{f}_{int\to int}.\lambda\ \underbrace{x}_{int}.\underbrace{\textbf{if}\ \underbrace{x}_{int}\ \textbf{then}\ \underbrace{1}_{int}\ \textbf{else}\ \underbrace{\underbrace{x}_{int} \times (\underbrace{\underbrace{f}_{int\to int}\ (\underbrace{\underbrace{x}_{int} - \underbrace{1}_{int}}_{int}}_{int}))}_{int}}_{int}$$

Esempio

$$fact \triangleq \mathbf{rec}\ f : int \to int.\ \lambda x : int.\ \mathbf{if}\ x\ \mathbf{then}\ 1\ \mathbf{else}\ x \times (f\ (x-1))$$

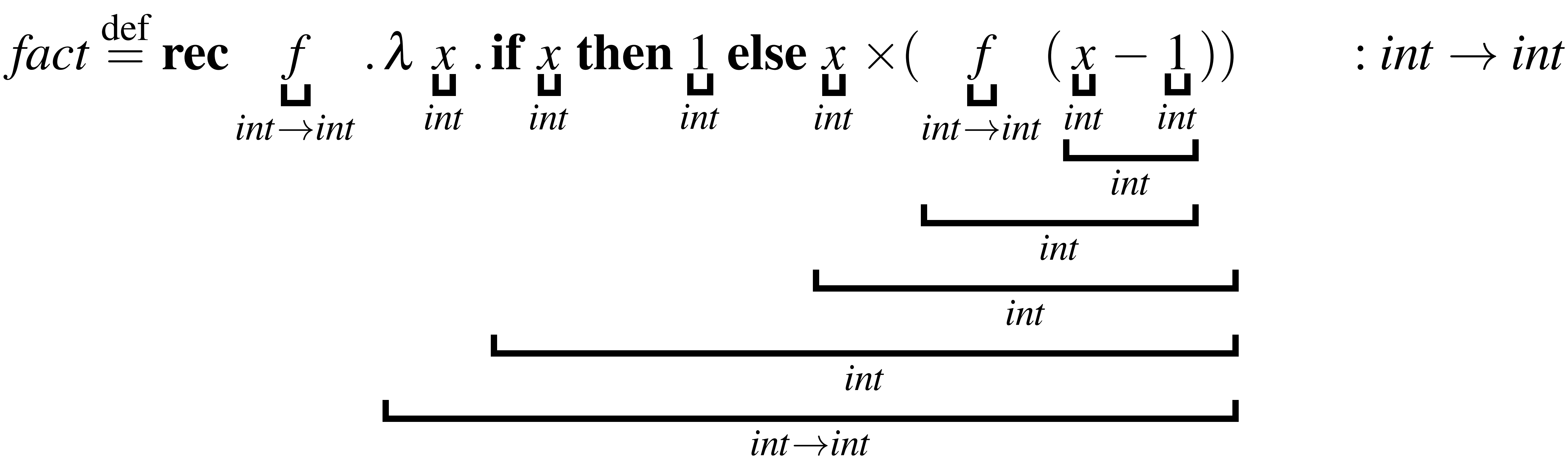
scritto in maniera più semplice



Esempio

$$fact \triangleq \mathbf{rec}\ f : int \to int.\ \lambda x : int.\ \mathbf{if}\ x\ \mathbf{then}\ 1\ \mathbf{else}\ x \times (f\ (x - 1))$$

scritto in maniera più semplice



Inferenza di tipi
Curry Type theory

Curry Type Theory

- I tipi delle variabili non devono necessariamente essere (dichiarati)

- Inferiamo il tipo dei termini e' inferito usando le regole per derivare vincoli di tipo (equazioni di tipo) le cui soluzioni (tramite unificazione) definiscono il tipo principale

Esempio

$$t \stackrel{\text{def}}{=} \textbf{rec}\ p.\ \lambda x.\ (x, (p\ (x+2)))$$

intuivamente

$$t\ 0 \equiv (0, (t\ 2)) \equiv (0, (2, (t\ 4))) \equiv \cdots \equiv (0, (2, (4, \ldots)))$$

sequenza di tutti i numeri pari

possiamo digitare sequenze di interi di lunghezza fissa
non abbiamo un tipo per sequenze di lunghezza qualsiasi/infinita

Esempio

$$t \stackrel{\text{def}}{=} \mathbf{rec}\ p.\ \lambda x.\ (x, (p\ (x+2)))$$

Haskell

```
Prelude> let p x = (x, p (x+2))

<interactive>:…:5: error:
    • Occurs check: cannot construct the infinite type: b ~ (t, b)
      Expected type: t -> b
        Actual type: t -> (t, b)
    • Relevant bindings include
        p :: t -> b (bound at <interactive>:…:5)
```

Esempio

$$t \stackrel{\text{def}}{=} \textbf{rec}\ p.\ \lambda x.\ (x,(p\ (x+2)))$$

$$t = \textbf{rec}\ p.\ \lambda x.\ (x,(p\ (x+2))) : \tau$$

$\nwarrow_{\widehat{p}=\tau}$ $\lambda x.\ (x,(p\ (x+2))) : \tau$

$\nwarrow_{\tau=\tau_1\to\tau_2,\ \widehat{x}=\tau_1}$ $(x,(p\ (x+2))) : \tau_2$

$\nwarrow_{\tau_2=\tau_3*\tau_4}$ $x : \tau_3,\ (p\ (x+2)) : \tau_4$

$\nwarrow_{\widehat{x}=\tau_3}$ $(p\ (x+2)) : \tau_4$

$\nwarrow$ $p : \tau_5 \to \tau_4,\ (x+2) : \tau_5$

$\nwarrow_{\widehat{p}=\tau_5\to\tau_4}$ $(x+2) : \tau_5$

$\nwarrow_{\tau_5=int}$ $x : int,\ 2 : int$

$\nwarrow^{*}_{\widehat{x}=int}$ $\square$

$$\left.\begin{array}{l}\widehat{x} = \tau_1\\ \widehat{x} = \tau_3\\ \widehat{x} = int\end{array}\right\}\ \tau_1 = \tau_3 = int$$

$$\left.\begin{array}{l}\widehat{p} = \tau = \tau_1 \to \tau_2\\ \widehat{p} = \tau_5 \to \tau_4\end{array}\right\}\ \begin{array}{l}\tau_1 = \tau_5 = int\\ \tau_2 = \tau_4\end{array}$$

$$\left.\begin{array}{l}\tau_2 = \tau_4\\ \tau_2 = \tau_3 * \tau_4\end{array}\right\}\ \text{fail! (occur check)}$$

$$\frac{}{x : \widehat{x}} \qquad \frac{}{n : int} \qquad \frac{t_0 : int \quad t_1 : int}{t_0\ \textbf{op}\ t_1 : int} \qquad \frac{t : int \quad t_0 : \tau \quad t_1 : \tau}{\textbf{if}\ t\ \textbf{then}\ t_0\ \textbf{else}\ t_1 : \tau}$$

$$\frac{t_0 : \tau_0 \quad t_1 : \tau_1}{(t_0, t_1) : \tau_0 * \tau_1} \qquad \frac{t : \tau_0 * \tau_1}{\textbf{fst}(t) : \tau_0} \qquad \frac{t : \tau_0 * \tau_1}{\textbf{snd}(t) : \tau_1}$$

$$\frac{x : \tau_0 \quad t : \tau_1}{\lambda x.\ t : \tau_0 \to \tau_1} \qquad \frac{t_1 : \tau_0 \to \tau_1 \quad t_0 : \tau_0}{t_1\ t_0 : \tau_1}$$

$$\frac{x : \tau \quad t : \tau}{\textbf{rec}\ x.\ t : \tau}$$

Esempio

$$t \stackrel{\text{def}}{=} \mathbf{rec}\ p.\ \lambda x.\ (x, (p\ (x+2)))$$

scritto in maniera più semplice

$$t = \mathbf{rec}\ p.\quad \lambda\ x\ .\ (\ x\ ,(\quad p \quad (\ x + \underbrace{2}_{int})))$$

Esempio

$$t \stackrel{\text{def}}{=} \textbf{rec}\ p.\ \lambda x.\ (x, (p\ (x+2)))$$

scritto in maniera più semplice

$$t = \textbf{rec}\ p.\quad \lambda\ x\ .\ (\ x\ , (\quad p \quad (\underbrace{x}_{int} + \underbrace{2}_{int})))$$

Esempio

$$t \stackrel{\text{def}}{=} \mathbf{rec}\ p.\ \lambda x.\ (x, (p\ (x+2)))$$

scritto in maniera più semplice

$$t = \mathbf{rec}\ p.\quad \lambda \underbrace{x}_{int}.\ (\underbrace{x}_{int}, (\quad p \quad (\underbrace{x}_{int} + \underbrace{2}_{int})))$$

Esempio

$$t \stackrel{\text{def}}{=} \mathbf{rec}\ p.\ \lambda x.\ (x, (p\ (x+2)))$$

scritto in maniera più semplice

$$t = \mathbf{rec}\ p.\quad \lambda\ \underbrace{x}_{int}\ .\ (\underbrace{x}_{int}, (\quad p \quad \underbrace{(\underbrace{x}_{int} + \underbrace{2}_{int})}_{int})))$$

Esempio

$$t \stackrel{\text{def}}{=} \textbf{rec}\ p.\ \lambda x.\ (x, (p\ (x+2)))$$

scritto in maniera più semplice

$$t = \textbf{rec}\ p.\quad \lambda\, \underbrace{x}_{int}\,.\ (\underbrace{x}_{int}, (\ \underbrace{p}_{int \to \tau_4}\ \ \underbrace{(\underbrace{x}_{int} + \underbrace{2}_{int})}_{int})))$$

Esempio

$$t \stackrel{\text{def}}{=} \mathbf{rec}\ p.\ \lambda x.\ (x,(p\ (x+2)))$$

scritto in maniera più semplice

$$t = \mathbf{rec}\ \underbrace{p}_{int \to \tau_4}.\quad \lambda\ \underbrace{x}_{int}\ .\ (\underbrace{x}_{int},(\ \underbrace{p}_{int \to \tau_4}\ \ \underbrace{(\underbrace{x}_{int} + \underbrace{2}_{int}}_{int})))$$

Esempio

$$t \stackrel{\text{def}}{=} \mathbf{rec}\ p.\ \lambda x.\ (x, (p\ (x+2)))$$

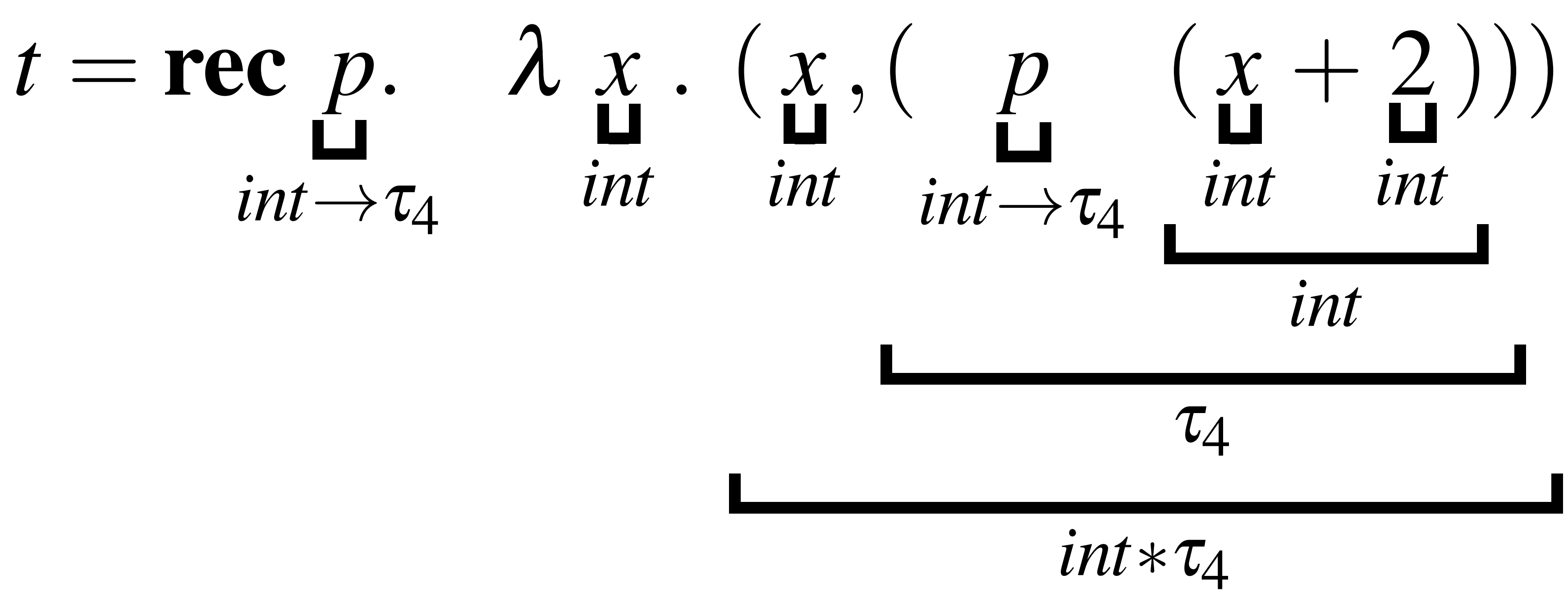
scritto in maniera più semplice

$$t = \mathbf{rec}\ \underbrace{p}_{int \to \tau_4}.\ \lambda\ \underbrace{x}_{int}\ .\ (\underbrace{x}_{int}, \underbrace{(\underbrace{p}_{int \to \tau_4}\ \underbrace{(\underbrace{x}_{int} + \underbrace{2}_{int})}_{int})}_{\tau_4})$$

Esempio

$$t \stackrel{\text{def}}{=} \mathbf{rec}\ p.\ \lambda x.\ (x, (p\ (x+2)))$$

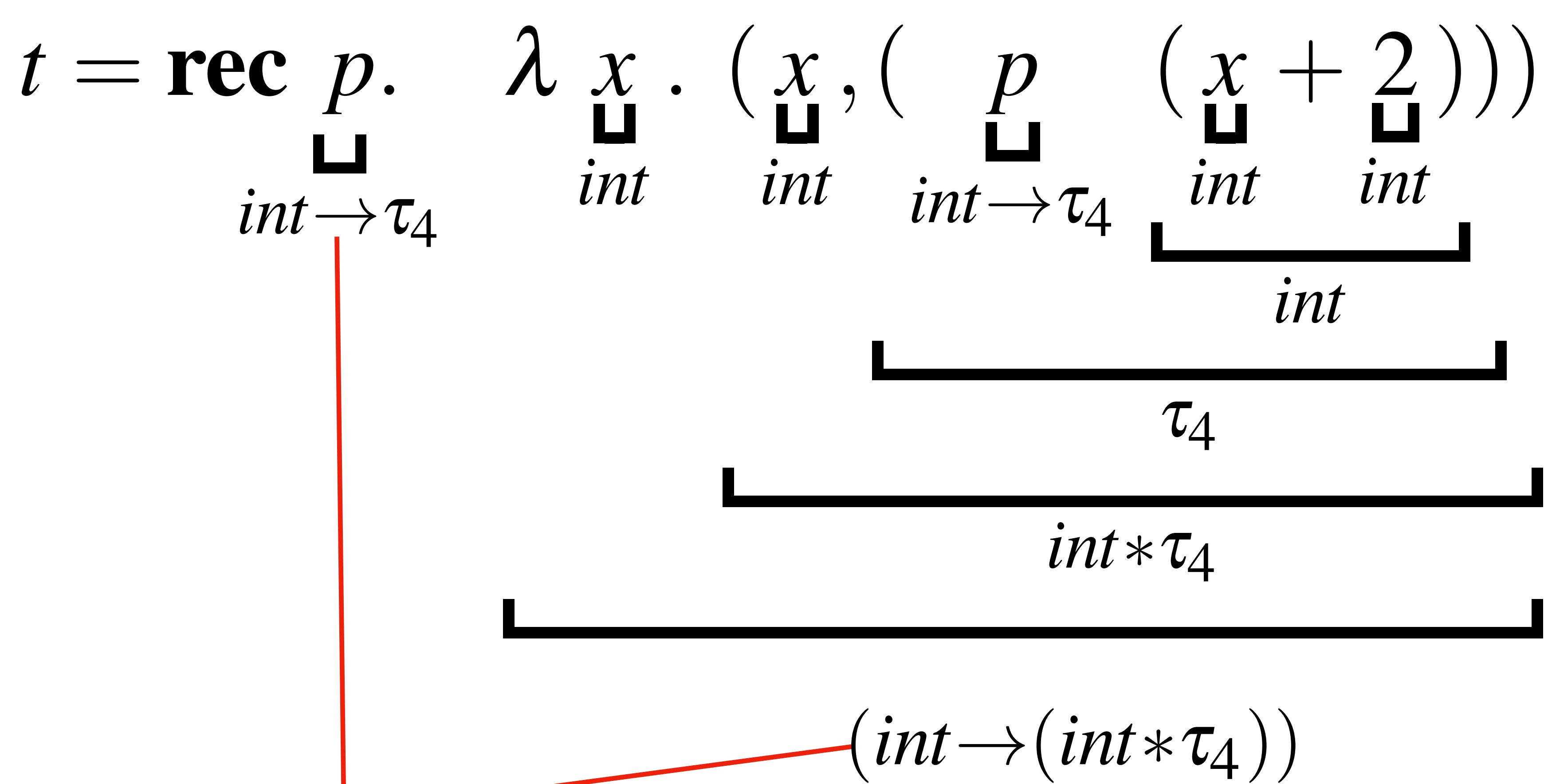
scritto in maniera più semplice



Esempio

$$t \stackrel{\text{def}}{=} \textbf{rec}\ p.\ \lambda x.\ (x, (p\ (x+2)))$$

scritto in maniera più semplice



$$(int \to (int * \tau_4)) = (int \to \tau_4) \Rightarrow \tau_4 = (int * \tau_4)$$

fallisce (occur check)

Esercizio

inferire il tipo del termine sotto

$$\frac{}{x : \widehat{x}} \qquad \frac{}{n : int} \qquad \frac{t_0 : int \quad t_1 : int}{t_0 \ \mathbf{op}\ t_1 : int} \qquad \frac{t : int \quad t_0 : \tau \quad t_1 : \tau}{\mathbf{if}\ t\ \mathbf{then}\ t_0\ \mathbf{else}\ t_1 : \tau}$$

$$\frac{t_0 : \tau_0 \quad t_1 : \tau_1}{(t_0, t_1) : \tau_0 * \tau_1} \qquad \frac{t : \tau_0 * \tau_1}{\mathbf{fst}(t) : \tau_0} \qquad \frac{t : \tau_0 * \tau_1}{\mathbf{snd}(t) : \tau_1}$$

$$\frac{x : \tau_0 \quad t : \tau_1}{\lambda x.\ t : \tau_0 \to \tau_1} \qquad \frac{t_1 : \tau_0 \to \tau_1 \quad t_0 : \tau_0}{t_1\ t_0 : \tau_1}$$

$$\frac{x : \tau \quad t : \tau}{\mathbf{rec}\ x.\ t : \tau}$$

$$\mathbf{rec}\ rep.\ \lambda n.\ \lambda f.\ \lambda x.\ \mathbf{if}\ n\ \mathbf{then}\ x$$
$$\mathbf{else}\ f\ (rep\ (n-1)\ f\ x)$$

$$\tau_0 \to \tau_0$$

$$(\tau_1 \to \tau_0) \to \tau_0 \to \tau_0$$

$$int \to (\tau_1 \to \tau_0) \to \tau_0 \to \tau_0$$

$n : int$

$x : \tau_0$

$f : \tau_1 \to \tau_0$

$rep : int \to (\tau_1 \to \tau_0) \to \tau_0 \to \tau_1$ $\qquad \tau_0 = \tau_1$

Esercizio

$$\frac{}{x : \widehat{x}} \qquad \frac{}{n : int} \qquad \frac{t_0 : int \quad t_1 : int}{t_0 \ \mathbf{op}\ t_1 : int} \qquad \frac{t : int \quad t_0 : \tau \quad t_1 : \tau}{\mathbf{if}\ t\ \mathbf{then}\ t_0\ \mathbf{else}\ t_1 : \tau}$$

$$\frac{t_0 : \tau_0 \quad t_1 : \tau_1}{(t_0, t_1) : \tau_0 * \tau_1} \qquad \frac{t : \tau_0 * \tau_1}{\mathbf{fst}(t) : \tau_0} \qquad \frac{t : \tau_0 * \tau_1}{\mathbf{snd}(t) : \tau_1}$$

$$\frac{x : \tau_0 \quad t : \tau_1}{\lambda x.\, t : \tau_0 \to \tau_1} \qquad \frac{t_1 : \tau_0 \to \tau_1 \quad t_0 : \tau_0}{t_1\ t_0 : \tau_1}$$

$$\frac{x : \tau \quad t : \tau}{\mathbf{rec}\ x.\, t : \tau}$$

inferire il tipo del termine sotto

$$\lambda x. \left(\left(\begin{array}{l} \mathbf{rec}\ f.\ \lambda y.\ \mathbf{if}\ (x - y)\ \mathbf{then}\ 0 \\ \qquad\qquad\quad \mathbf{else\ if}\ (x + y)\ \mathbf{then}\ 1 \\ \qquad\qquad\quad \mathbf{else}\ f\ (y + 1) \end{array} \right)\ 0 \right)$$

$$int \to int$$

$$int$$

$$int \to int$$

int

$x : int$

$y : int$

$f : int \to int$

Substituzioni capture-avoiding
(di nuovo)

Variabili libere

$$\mathrm{fv}(n) \stackrel{\text{def}}{=} \varnothing$$

$$\mathrm{fv}(x) \stackrel{\text{def}}{=} \{x\}$$

$$\mathrm{fv}(t_0 \ \mathrm{op}\ t_1) \stackrel{\text{def}}{=} \mathrm{fv}(t_0) \cup \mathrm{fv}(t_1)$$

$$\mathrm{fv}(\mathbf{if}\ t\ \mathbf{then}\ t_0\ \mathbf{else}\ t_1) \stackrel{\text{def}}{=} \mathrm{fv}(t) \cup \mathrm{fv}(t_0) \cup \mathrm{fv}(t_1)$$

$$\mathrm{fv}((t_0, t_1)) \stackrel{\text{def}}{=} \mathrm{fv}(t_0) \cup \mathrm{fv}(t_1)$$

$$\mathrm{fv}(\mathbf{fst}(t)) \stackrel{\text{def}}{=} \mathrm{fv}(t)$$

$$\mathrm{fv}(\mathbf{snd}(t)) \stackrel{\text{def}}{=} \mathrm{fv}(t)$$

$$\mathrm{fv}(\lambda x.\ t) \stackrel{\text{def}}{=} \mathrm{fv}(t) \setminus \{x\}$$

$$\mathrm{fv}((t_0\ t_1)) \stackrel{\text{def}}{=} \mathrm{fv}(t_0) \cup \mathrm{fv}(t_1)$$

$$\mathrm{fv}(\mathbf{rec}\ x.\ t) \stackrel{\text{def}}{=} \mathrm{fv}(t) \setminus \{x\}$$

Substituzioni

$$n[{}^{t}/_{x}] = n$$

$$y[{}^{t}/_{x}] \stackrel{\text{def}}{=} \begin{cases} t & \text{if } y = x \\ y & \text{if } y \neq x \end{cases}$$

$$(t_0 \text{ op } t_1)[{}^{t}/_{x}] \stackrel{\text{def}}{=} t_0[{}^{t}/_{x}] \text{ op } t_1[{}^{t}/_{x}] \quad \text{with op} \in \{+,-,\times\}$$

$$(\textbf{if } t' \textbf{ then } t_0 \textbf{ else } t_1)[{}^{t}/_{x}] \stackrel{\text{def}}{=} \textbf{if } t'[{}^{t}/_{x}] \textbf{ then } t_0[{}^{t}/_{x}] \textbf{ else } t_1[{}^{t}/_{x}]$$

$$(t_0, t_1)[{}^{t}/_{x}] \stackrel{\text{def}}{=} (t_0[{}^{t}/_{x}], t_1[{}^{t}/_{x}])$$

$$\textbf{fst}(t')[{}^{t}/_{x}] \stackrel{\text{def}}{=} \textbf{fst}(t'[{}^{t}/_{x}])$$

$$\textbf{snd}(t')[{}^{t}/_{x}] \stackrel{\text{def}}{=} \textbf{snd}(t'[{}^{t}/_{x}])$$

$$(t_0\ t_1)[{}^{t}/_{x}] \stackrel{\text{def}}{=} (t_0[{}^{t}/_{x}]\ t_1[{}^{t}/_{x}])$$

$$(\lambda y.\ t')[{}^{t}/_{x}] \stackrel{\text{def}}{=} \lambda z.\ (\ t'[{}^{z}/_{y}][{}^{t}/_{x}]\ ) \quad \text{for } z \notin \text{fv}(\lambda y.\ t') \cup \text{fv}(t) \cup \{x\}$$

$$(\textbf{rec } y.\ t')[{}^{t}/_{x}] \stackrel{\text{def}}{=} \textbf{rec } z.\ (t'[{}^{z}/_{y}][{}^{t}/_{x}]) \quad \text{for } z \notin \text{fv}(\textbf{rec } y.\ t') \cup \text{fv}(t) \cup \{x\}$$

I tipi sono rispettati

TH. $x_0 : \tau_0$ $\quad t : \tau \quad \Rightarrow \quad t[{}^{t_0}/_{x_0}] : \tau$
$t_0 : \tau_0$

prova omessa
(per induzione strutturale)