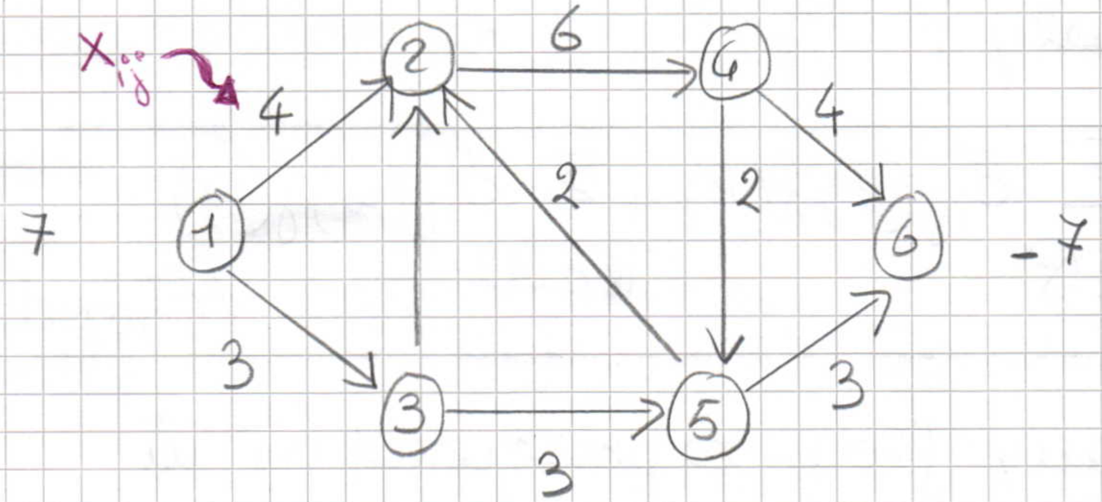
Flow decomposition

(Ahuja - Magnanti - Orlin : Chapter 3 (3.5))

<u>Observation</u> : any flow x can be defined in terms of flows on arcs (as previously formulated) or in terms of flows on paths and cycles

<u>example</u>



< flows on arcs >

$x = (x_{ij})$

<u>or</u>

4 units along the path $(1, 2, 4, 6) = P_1$

3 units along the path $(1, 3, 5, 6) = P_2$

2 units along the cycle $(2, 4, 5) = W$

< flows on paths and cycle >

Let $\mathcal{P}$: set of all paths

$f(P)$: flow on $P \in \mathcal{P}$

$\mathcal{W}$: set of all cycles

$f(w)$: flow on $w \in \mathcal{W}$

$$\delta_{ij}(P) = \begin{cases} 1 & \text{if } (i,j) \in P \ (w) \\ 0 & \text{otherwise} \end{cases}$$
(w)

Then : any flow representation in terms of path and cycle flows determines arc flows uniquely :

$$\boxed{x_{ij} = \sum_{P \in \mathcal{P}} \delta_{ij}(P) f(P) + \sum_{w \in \mathcal{W}} \delta_{ij}(w) f(w) \quad \forall (i,j) \in A}$$

Viceversa: any flow representation in terms of arc flows "decomposes" into path and cycle flow (not uniquely)

$\Downarrow$

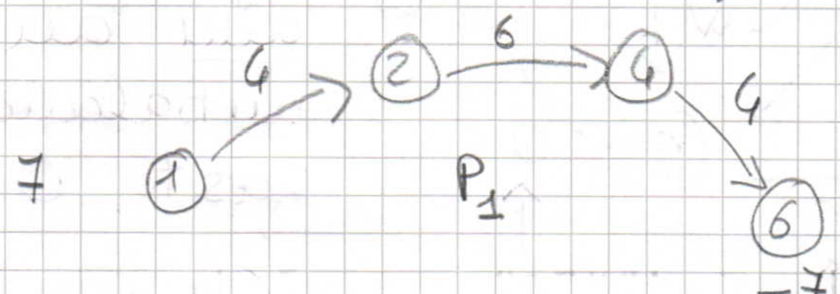
Flow decomposition Theorem : each path and cycle flow has a unique representation in terms of (nonnegative) arc flows. Conversely, each arc flow x can be represented as path and cycle flow (not uniquely) s.t.:

a) each directed path with a positive flow connects a source to a destination

b) at most $n+m$ paths and cycles have a positive flow; out of these, at most m cycles have positive flow

<u>Proof</u> (intuition) < example cont. >

<u>select a source node</u> (1):



$\min\{r(P_1), b(1), -b(7)\} = 4$

subtract 4 units of flow along P_1:

$f(P_1) = 4$

at least one flow arc goes to 0