Integer Linear Programming

(Wolsey, Chap. 1)

Formulations

Let A : $m \times n$ matrix

c : n-dimensional vector

b : m-dimensional vector

x : n-dimensional vector of variables

- If some but not all variables are restricted to be integer:

$$
\text{(MILP)} \quad
\begin{aligned}
\max \; & cx + hy \\
& Ax + Gy \leq b \\
& x \geq 0 \text{ integer} \\
& y \geq 0
\end{aligned}
$$

where y are continuos variables

Mixed Integer Linear Program (Problem)

- If all variables are integer:

$$\text{(ILP)} \quad \begin{aligned} \max\ & cx \\ & Ax \leq b \\ & x \geq 0 \text{ integer} \end{aligned}$$

Integer Linear Program

- If all variables are restricted in $\{0,1\}$:

$$\text{(BIP)} \quad \begin{aligned} \max\ & cx \\ & Ax \leq b \\ & x \in \{0,1\}^n \end{aligned}$$

0-1 or Binar Integer Program

- Combinatorial optimization problems:

$$\text{(COP)} \quad \min \sum_{j \in S} c_j$$
$$S \in F$$

where $N = \{1, \dots n\}$ basic "objects"

c_j : cost of j, $j = 1, \dots n$

F : feasible subsets of N

Example 1

0-1 Knapsack problem

$N = \{1, \dots n\}$ objects that can be put into a knapsack

c_j : profit of j, $j = 1, \dots n$

a_j : weight of j, $j = 1 \dots n$

b : capacity of the knapsack

COP:
$$\max \sum_{j \in S} c_j$$
$$\underbrace{S \subseteq N : \sum_{j \in S} a_j \le b}_{\text{feasible set } F}$$

- Often a COP can be formulated as an ILP or a BIP:

e.g.
$$x_j = \begin{cases} 1 & \text{if } j \text{ is selected} \\ 0 & \text{otherwise} \end{cases} \qquad j = 1, \dots n$$

Knapsack formulation

$$\max \sum_{j=1}^{n} c_j \cdot x_j \qquad (KP_1)$$
$$F \begin{cases} \sum_{j=1}^{n} a_j \cdot x_j \le b \\ x_j \in \{0, 1\} \quad j = 1, \dots n \end{cases}$$