Sorting atomic items

Chapter 5

Lower bounds

Sorting and permuting

In RAM model Sorting includes Permuting since we need to determine the sorted permutation and then permute the items. Sorting is $\Theta(n\log n)$ while permuting is $\Theta(n)$.

In disk model Sorting problem is equivalent to Permuting problem by the point of view of I/O complexity.

Moving elements is difficult as Sorting in this model. It is the real bottleneck: I/O bottelneck.

How to use Sort to Permute

Use Sort to Permute

Permute Sequence S, S[1,n] according Π[1,n], i.e.
Output S[Π(1)], S[Π(2)], …S[Π(n)]

RAM model: jump on the memory to read S[Π(i)] then O(n).
Same algorithm on 2-level model: O(n) I/O's: Too much!

Use Sort and Scan to Permute;

1) Create sequence P of pairs <i, Π(i)>
2) Sort according Π component
3) Parallel scan of of S and P and change S[i] with S[Π(i)]
4) Sort P on the first component

Use Sort to Permute

S: a, b, c, d Π : 2, 4, 1, 3

RESULT: b, d, a, c

1. Create P.

   P: <a, 2> <b, 4> <c, 1> <d, 3>

2. Sort on Π component

   P: <c, 1> <a, 2> <d, 3><b, 4>

3 Parallel scan of S and P to substitute in P to S[i] S[Π(i)]

S: a, b, c, d

P: <c, 1> <a, 2> <d, 3><b, 4>

P: <c, a> <a, b> <d, c> <b, d>

4 Sort on the first component

P: <a, b> <b, d> <c, a> <d,c>

Use Sort to Permute

Algorithm uses 2 Scan and 2 Sort. Hence:

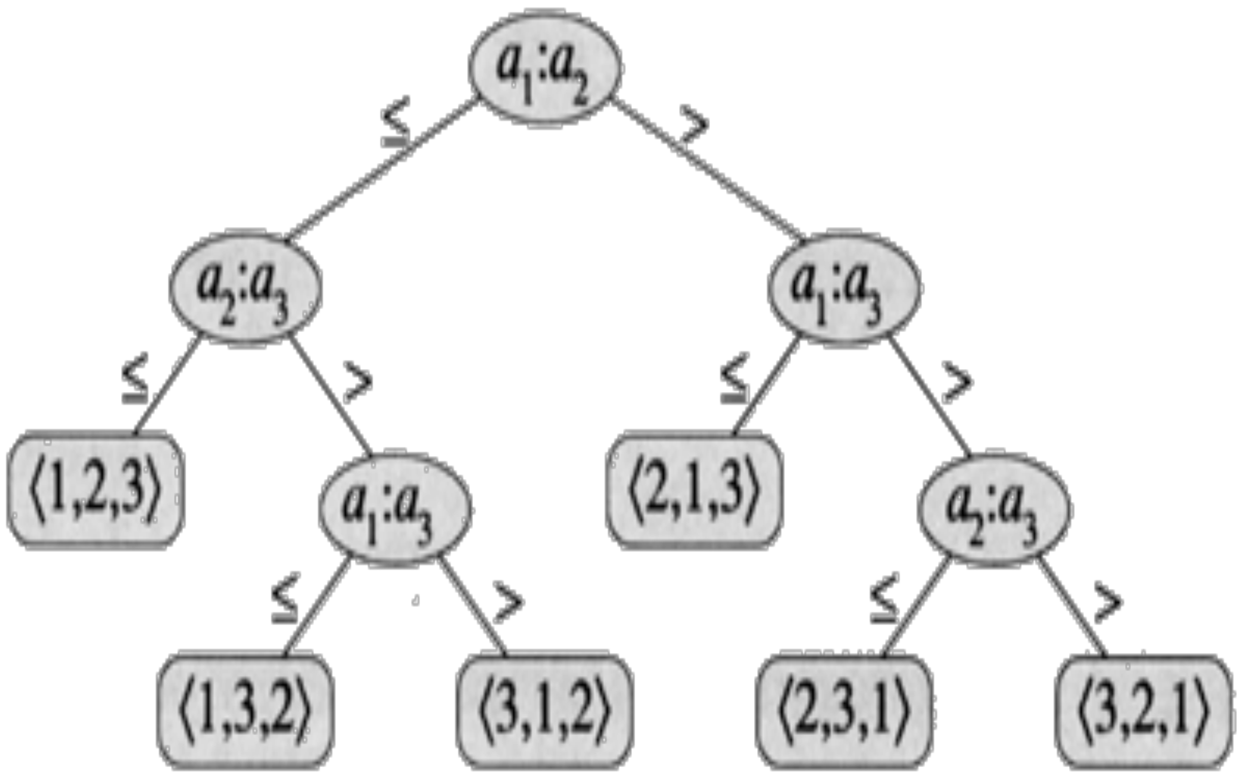
O(min{n, (n/B) log(n/M)}) I/O's

This bound and that for Sorting are optimal for I/O's.
The bounds are equal whenever n = Ω(n/B) log(n/M)

	time complexity (RAM model)	I/O complexity (two-level memory model)
Permuting	$O(n)$	$O(\min\{n, \frac{n}{B}\log_{M/B}\frac{n}{M}\})$
Sorting	$O(n\log_2 n)$	$O(\frac{n}{B}\log_{\frac{M}{B}}\frac{n}{M})$

Lower bound for sorting

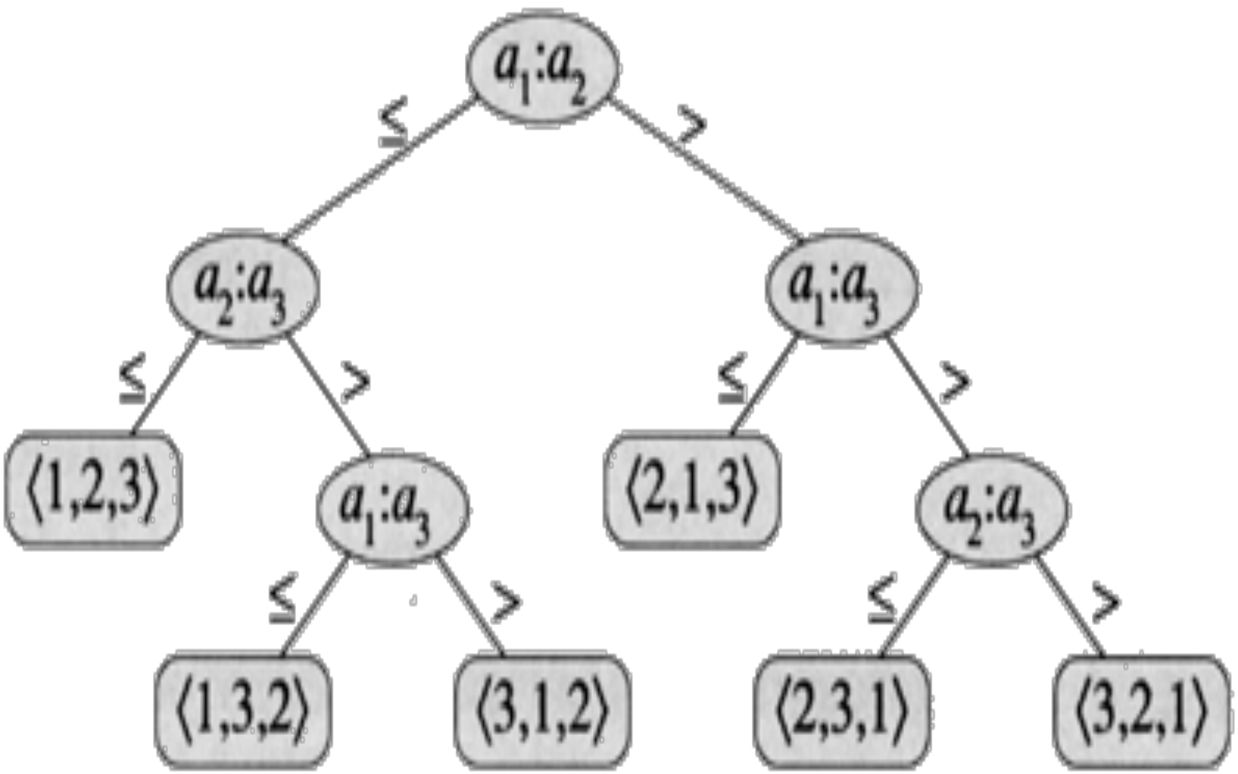
RAM model: Comparison tree to prove lower bound.
Node: comparison. Leaf: solution. Root-to-leaf path t: execution of the alg. on specific data.



Lower bound for sorting

Sorting: binary tree. The possible solutions (n! for sorting) must be allocated on the leaves. $2^t \geq n!$

$T \geq \log(n!)$ $t = \Omega(n\log n)$



Lower bound sorting in 2 level model

We assume items are only moved not copied, created or destroyed.

Consequence: Exactly one copy of each item exist during the sorting (or permuting).

Items cannot be broken up into pieces (indivisibility).

Consequence: hashing is not allowed.

Lower bound sorting in 2 level model

Comparison tree.

Account for I/O operations.

Operations in internal memory can be used for free.

Every node of the decision tree corresponds to one I/O.

The fan-out corresponds to the result of the comparisons after an I/O:

A block of B new elements is fetched to M. M-B elements are old , B are new.

The B elements can occupy $\binom{M}{B}$ positions of M .

Lower bound sorting in 2 level model

An I/O operations can generate $\binom{M}{B}$ different results.

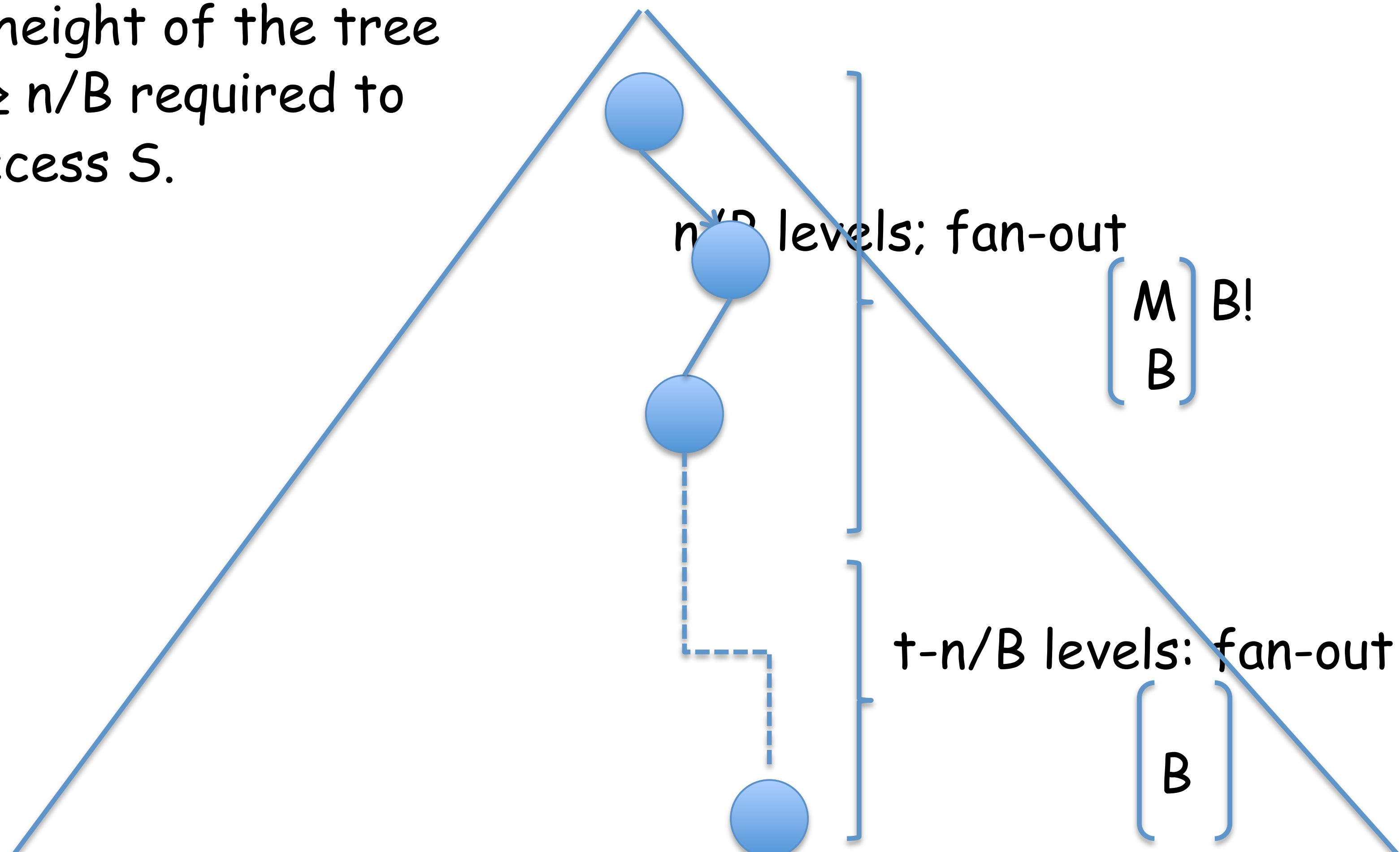
Un addition, we have to consider B! different permutations of B. (the other M-B items have already been considered in previous I/O operations).

In total $\binom{M}{B} B!$ possible orderings generated by an I/O operation and by the internal comparisons.

$\binom{M}{B} B!$ is the fan-out of each node.

Lower bound sorting in 2 level model

t height of the tree
$t \geq n/B$ required to access S.



M

The number of leaves is $\binom{M}{B}^{t} (B!)^{n/B}$

Lower bound sorting in 2 level model

$$\binom{M}{B}^{t} (B!)^{n/B} \geq n!$$

There must be enough leaves to contain all the possible solution of the problem.

Use Stirling approximation : $Z! \cong Z \log Z$

and the formulas: $\log \binom{M}{B} = B \log (M/B)$

$$\log_b a = \log_c a / \log_c b$$

Sorting: lower bound

$$
\begin{aligned}
\binom{M}{B}^{t} (B!)^{\frac{N}{B}} &\geq N! \\
t \log \binom{M}{B} + \frac{N}{B} \log(B!) &\geq \log(N!) \\
tB \log(\frac{M}{B}) + \frac{N}{B} B \log B &\geq N \log N \\
tB \log(\frac{M}{B}) &\geq N \log(\frac{N}{B}) \\
t &\geq \frac{N}{B} \frac{\log(\frac{N}{B})}{\log(\frac{M}{B})}
\end{aligned}
$$

$t = \Omega\,(N/B)\log_{M/B}(N/B) = \Omega\,(N/B)\log_{M/B}(N/M)$

Lower bound for the D disks model

- Parallel D disks model : computer + D disks
- Input and output are on disks
- I/O operation: 1 block of B data is fetched to the core memory of size M from each one disk. DB data are fetched in parallel.
- Evaluate the number of parallel I/O's

Lower bound for the D disks model

The previous bound can be easily extended to D disks.

A comparison-based Sorting algorithm must execute:

$\Omega((n/DB) \log_{M/B} (n/DB))$ I/O operations

Observation: D does not appear in the base of log. If this would be the case, it will increase the bound, so penalizing the sorting algorithm which uses D disks!

MergeSort is optimal for 1 disks but it is not for D disks.

The merging should be O(n/DB) I/O's, that is at each step D pages are fetched one per disk, with an I/O.

Merging is not parallel: after a comparison more than B items have to be possibly fetched from the same disk.

Lower bound for the D disks model

We must design a different algorithm.
In the following:

- Disk Striping technique: data layout on disks
- Greed Sort: elegant and complex new algorithm achieving a close to optimal upper bound.

Lower bound for Permuting

1 disk model:

If $B \log(M/B) \leq \log n$ then $\Omega(n)$

otherwise $\Omega((n/B)\log_{M/B}(n/M))$

The previous algorithm was optimal!

D disks model: $\Omega(\min\{n/D, (n/DB)\log_{M/B}(n/DB)\})$

The bounds for sorting and permuting are the same except for the case $B \log(M/B) \leq \log n$. This inequality holds for $n = \Omega(2^B)$ (since B and M are few k bytes and $\log(M/B)$ can be neglected) and this is unreasonable situation!

In practice, Sorting = Permuting