Business Processes Modelling

MPB (6 cfu, 295AA)

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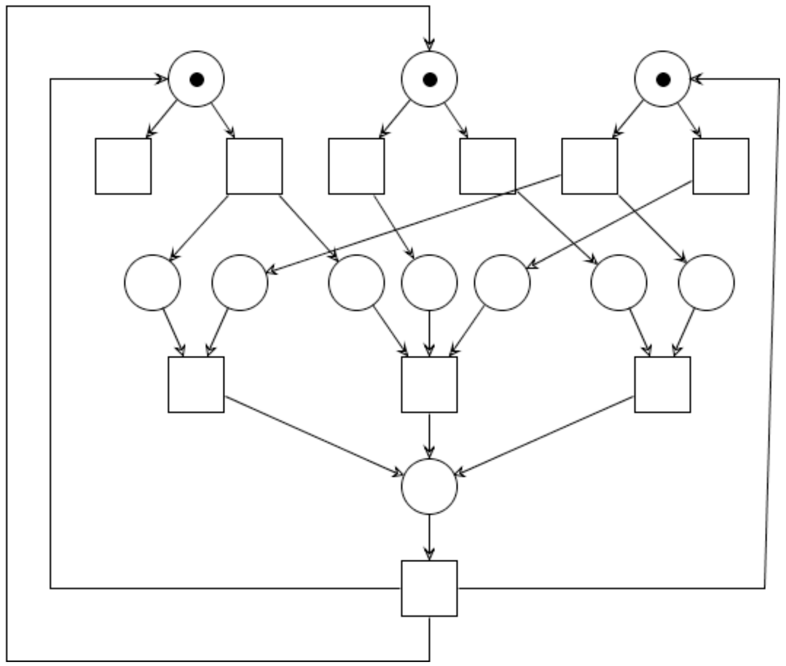
18 - Free-choice nets

Object

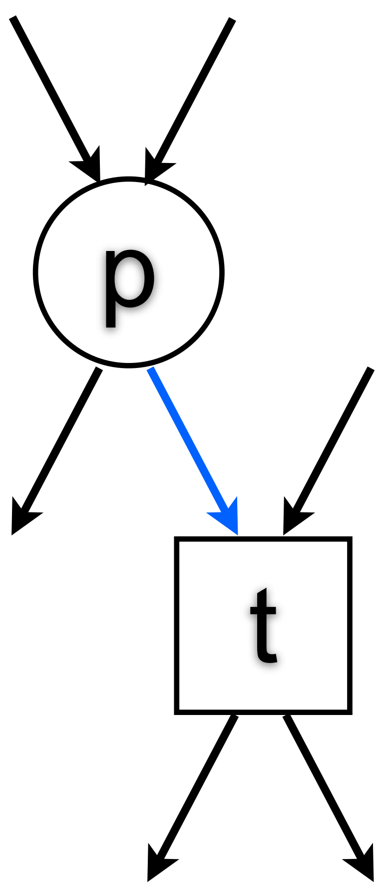


We study some “good” properties of free-choice nets

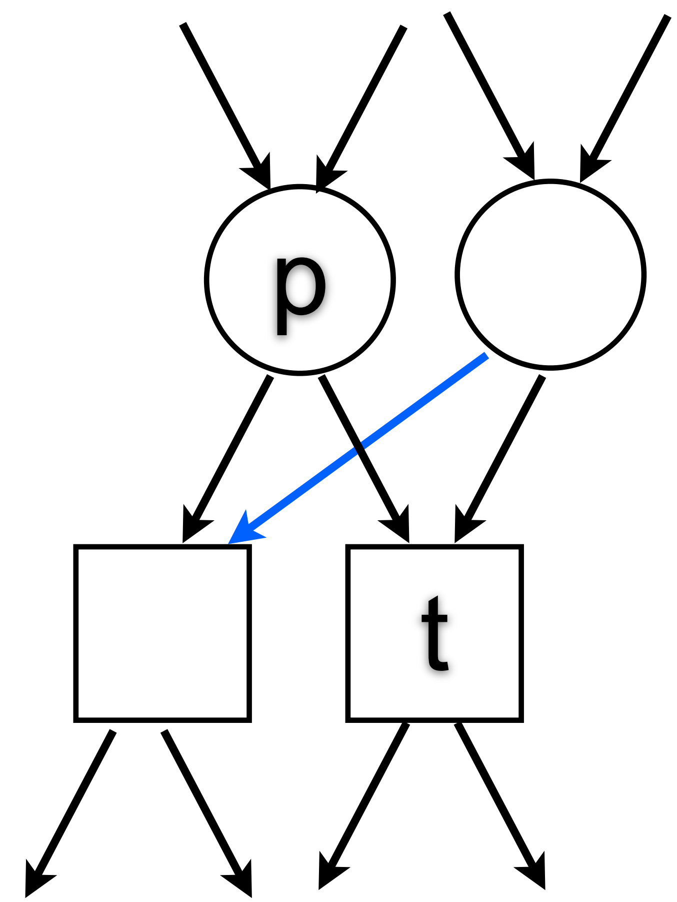
Free Choice Nets (book, optional reading)
https://www7.in.tum.de/~esparza/bookfc.html

Free-choice net

Definition: We recall that a net N is **free-choice** if whenever there is an arc (p,t), then there is an arc from any input place of t to any output transition of p



implies



Free-choice net: alternative definitions

Proposition: All the following definitions of free-choice net are equivalent.

1) A net (P, T, F) is free-choice if:
$\forall p \in P, \forall t \in T$, $(p, t) \in F$ implies $\bullet t \times p \bullet \subseteq F$.

2) A net (P, T, F) is free-choice if:
$\forall p, q \in P, \forall t, u \in T$, $\{(p, t), (q, t), (p, u)\} \subseteq F$ implies $(q, u) \in F$.

3) A net (P, T, F) is free-choice if:
$\forall p, q \in P$, either $p\bullet = q\bullet$ or $p \bullet \cap q\bullet = \emptyset$.

4) A net (P, T, F) is free-choice if:
$\forall t, u \in T$, either $\bullet t = \bullet u$ or $\bullet t \cap \bullet u = \emptyset$.

Free-choice net: my favourite definition

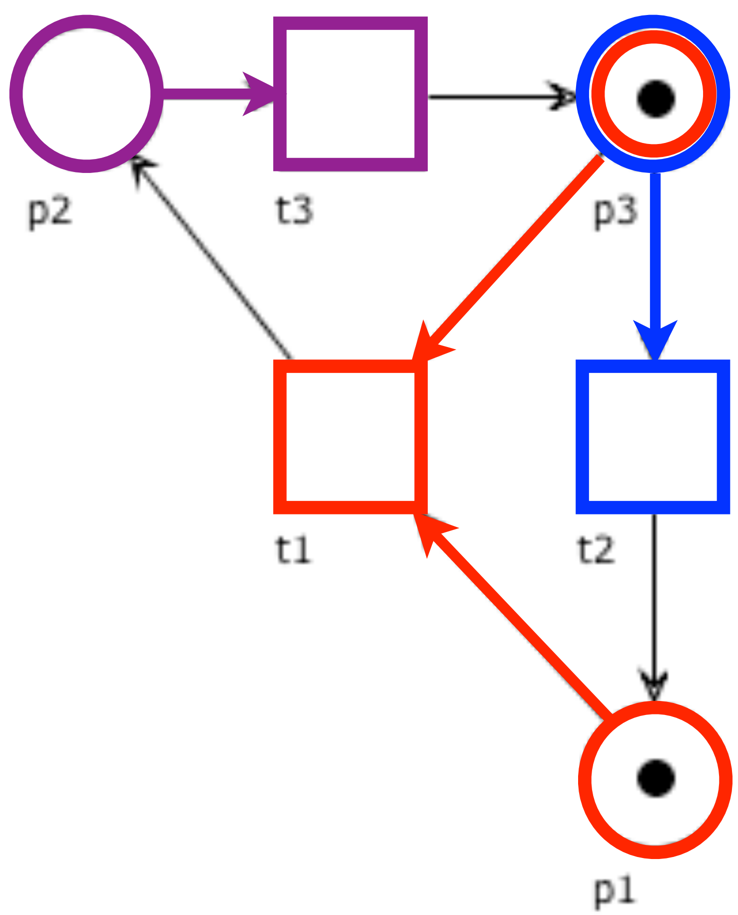
4) A net (P, T, F) is free-choice if:
$\forall t, u \in T$, either $\bullet t = \bullet u$ or $\bullet t \cap \bullet u = \emptyset$.

A system (N, M_0) is **free-choice** if N is free-choice

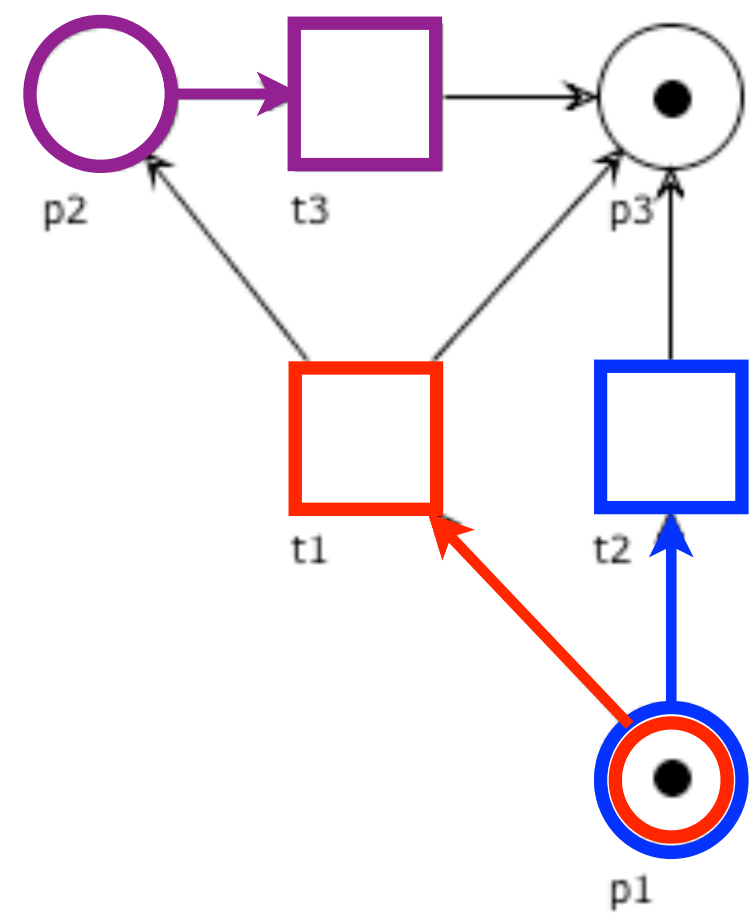
Example

$$\begin{aligned} \bullet t_1 &= \{\, p_1, p_3 \,\} \\ \bullet t_2 &= \{\, p_3 \,\} \\ \bullet t_1 &\neq \bullet t_2 \\ \bullet t_1 \cap \bullet t_2 &= \{\, p_3 \,\} \neq \emptyset \end{aligned}$$

$$\begin{aligned} \bullet t_1 &= \bullet t_2 \\ \bullet t_1 \cap \bullet t_3 &= \emptyset \\ \bullet t_2 \cap \bullet t_3 &= \emptyset \end{aligned}$$



non free-choice



free-choice

Fundamental property of free-choice nets

Proposition: Let (P, T, F, M_0) be free-choice.

If $M \xrightarrow{t}$ and $t \in p\bullet$, then $M \xrightarrow{t'}$ for every $t' \in p\bullet$.

The proof is trivial, by definition of free-choice net

$(t, t' \in p\bullet$ implies $\bullet t = \bullet t')$

Free-choice N*

Proposition: A workflow net N is free-choice
iff
N* is free-choice

N and N* differ only for the reset transition,
whose pre-set (o) is disjoint
from the pre-set of any other transition

Liveness = Place liveness (in Free Choice systems)

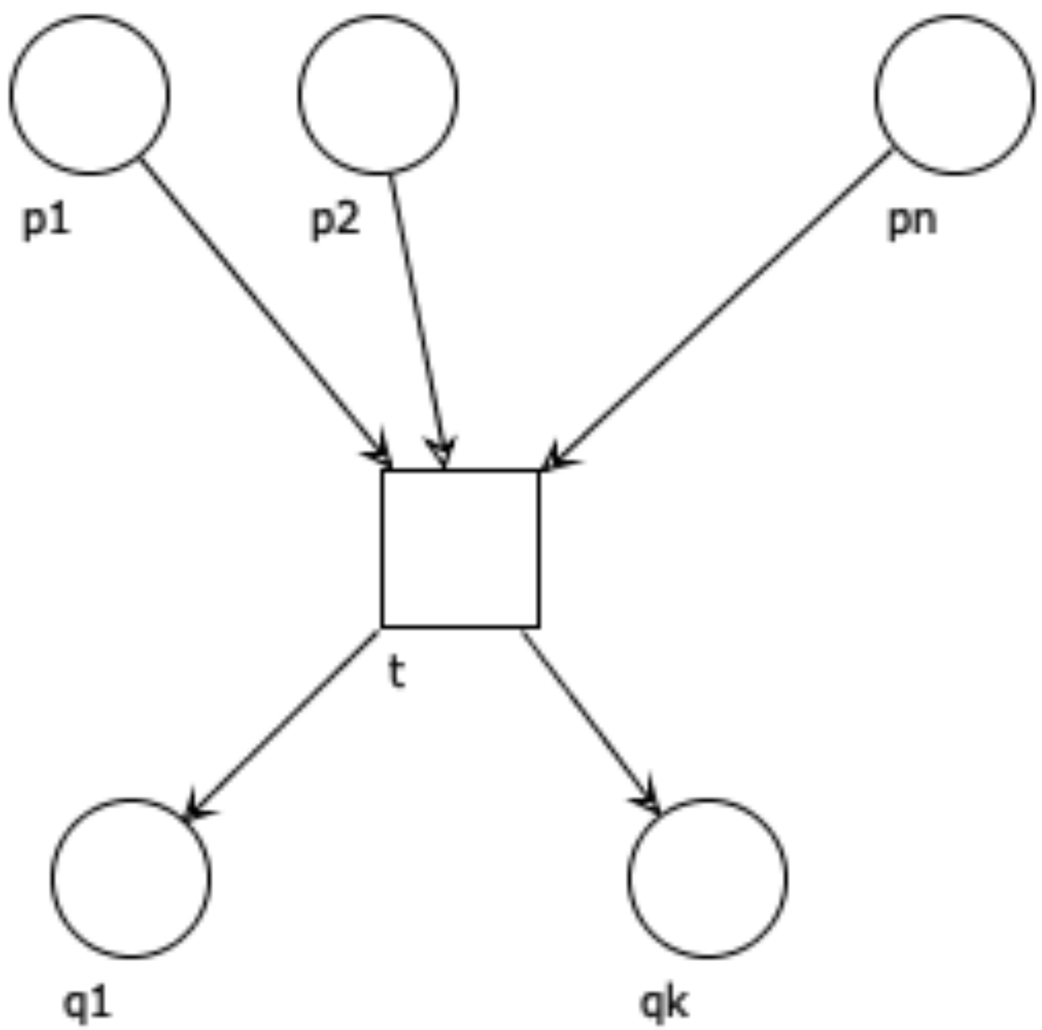
In any system:
liveness implies place-liveness
p dead implies any transition t in its pre/post-set is dead

It can be shown that
If a free-choice system is place-live, then it is live

Corollary:
A free-choice system is live **iff** it is place-live

FC Place-live implies FC Live (intuition)

From a reachable marking M we would like to enable t



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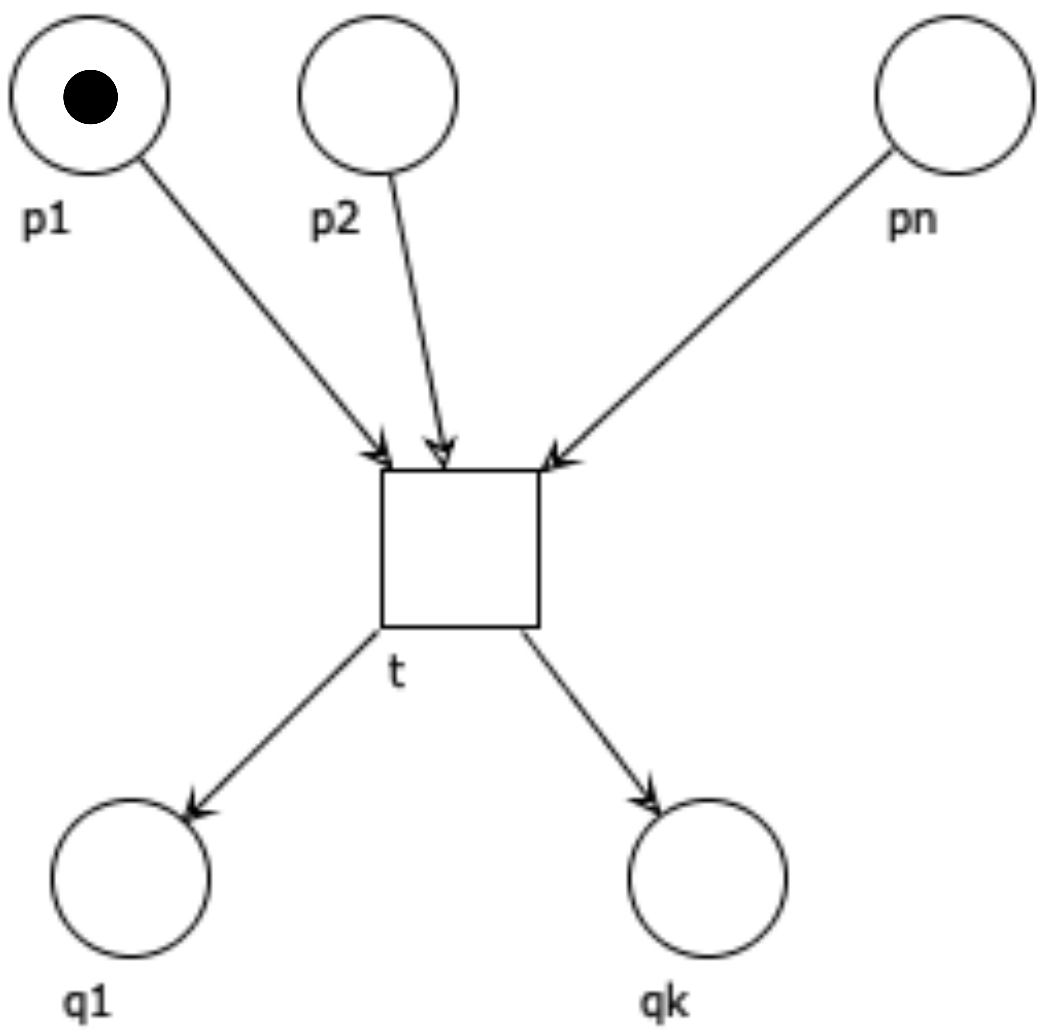

from M we can reach M_1 that marks p_1 (because place-live)

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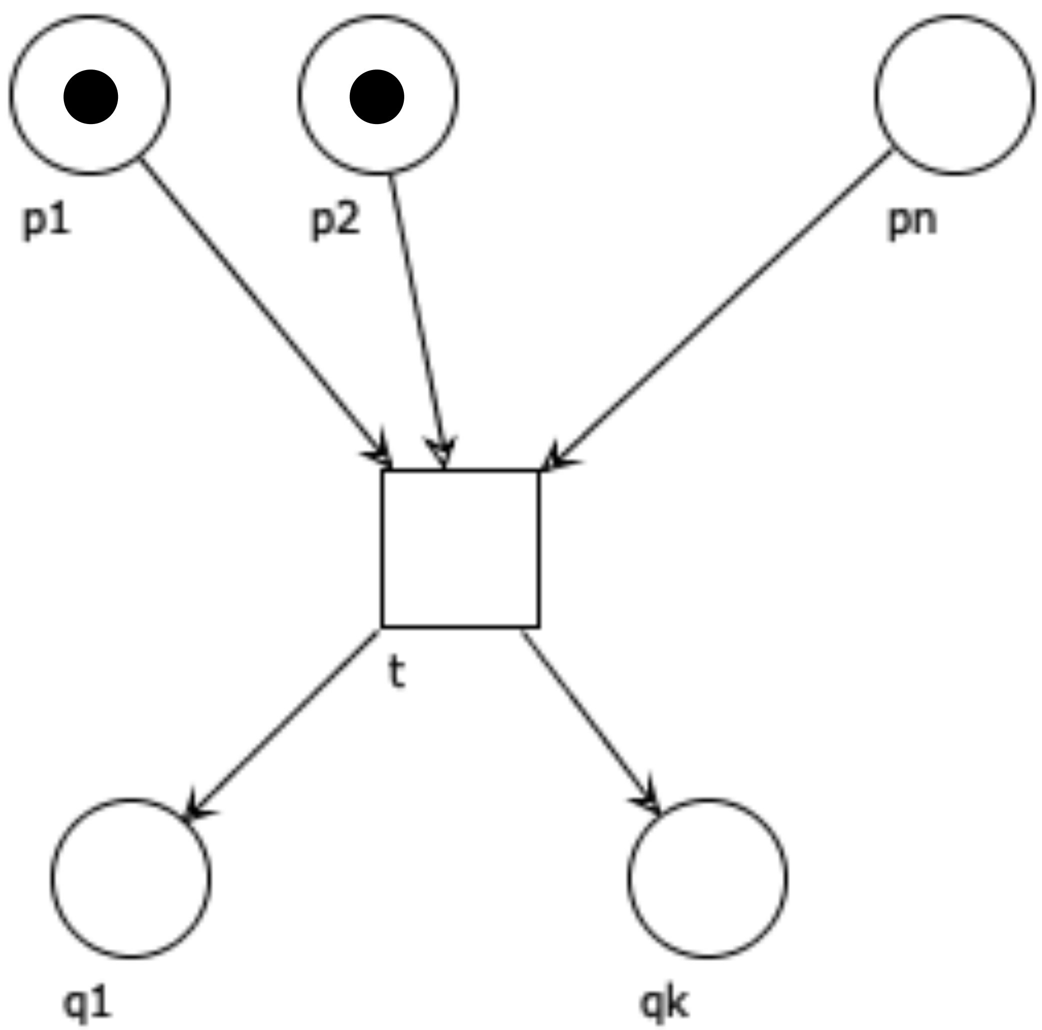


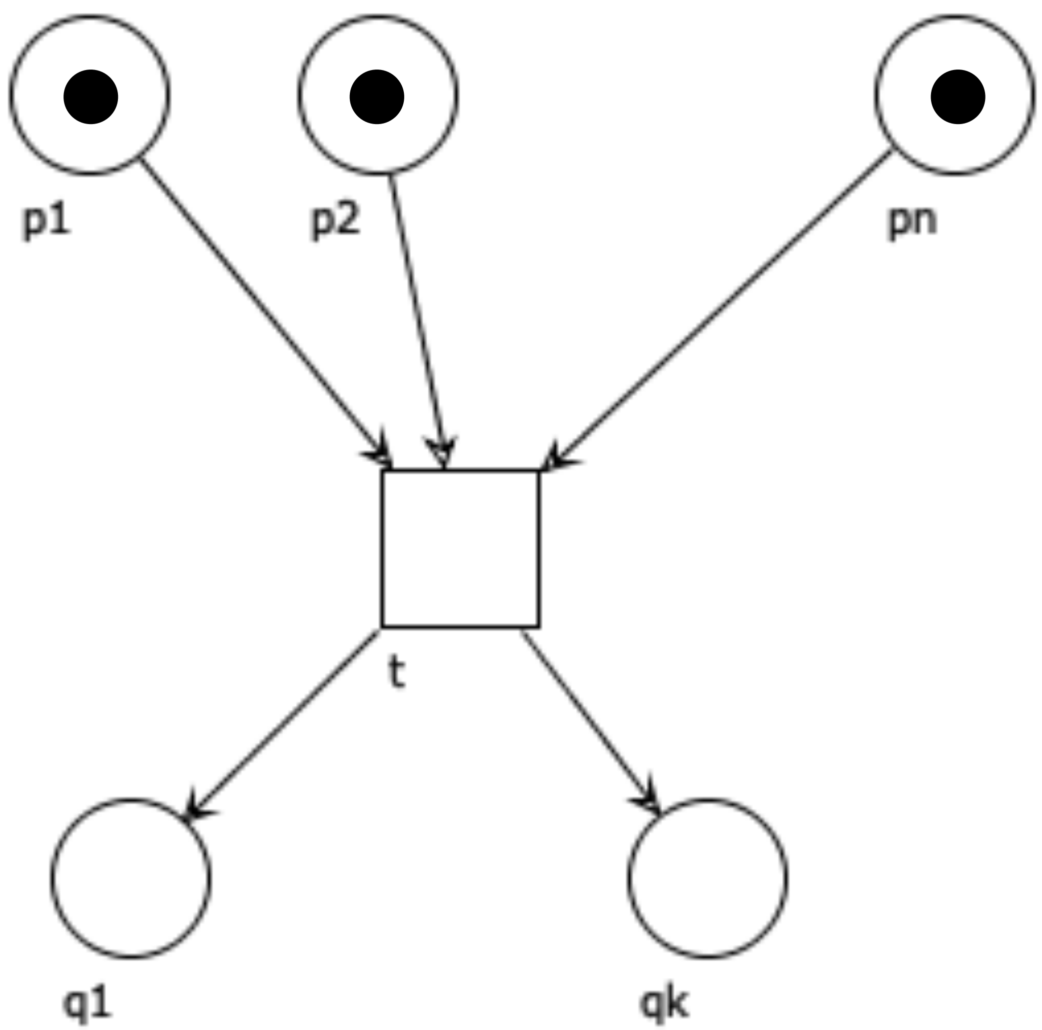
from M we can reach M_1 that marks p_1 (because place-live)

from M_1 we can reach M_2 that marks p_2 (because place-live)

Note: the token remains in p_1 (fundamental property of FC: if t' can remove a token from p_1, then t' has the same preset as t)

FC Place-live implies FC Live (intuition)

From a reachable marking M we would like to enable t



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from M_{n-1} we can reach M_n that marks p_n (because place-live)

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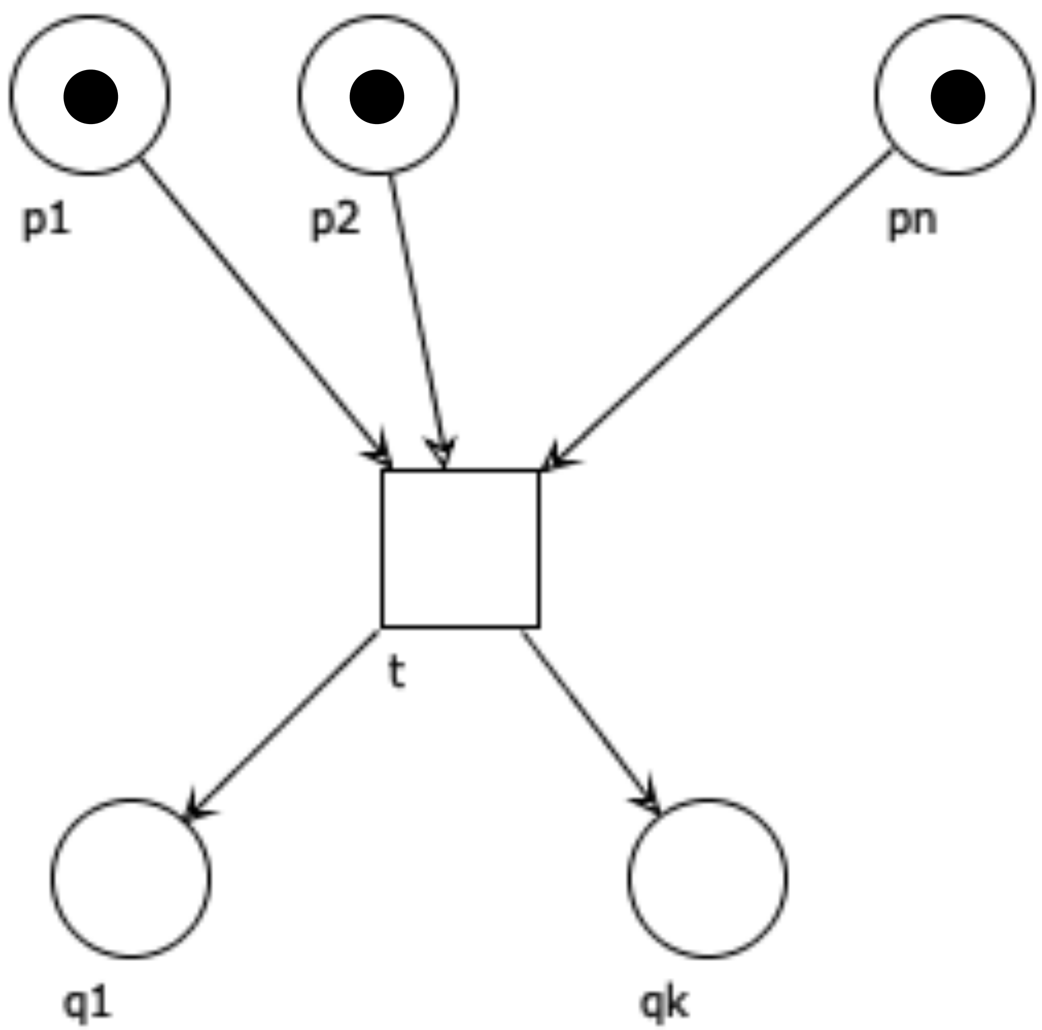
From a reachable marking M we would like to enable t



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from M_{n-1} we can reach M_n that marks p_n (because place-live)

from M we reach M_n that enables t !

Commoner's theorem (proof omitted)

Theorem:
A free-choice system is live
iff
every proper siphon includes an initially marked trap

Rank Theorem (main result, proof omitted)

Theorem:

A free-choice system (P,T,F,M_0) is live and bounded

iff

1. it has at least one place and one transition
2. it is connected
3. M_0 marks every proper siphon
4. it has a positive S-invariant
5. it has a positive T-invariant
6. $rank(N) = |C_N| - 1$

(where C_N is the set of clusters)

Coming next

What is a cluster?
What is a siphon?
What is a trap?

Is it hard to show that a free-choice net is live?

Is it hard to show that a free-choice net is live and bounded?

What is a cluster?

Cluster

Let x be the node of a net $N = (P, T, F)$
(not necessarily free-choice)

Definition:
The **cluster** of x, written $[x]$, is the least set s.t.

1. $x \in [x]$

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1. $x \in [x]$
2. if $p \in [x] \cap P$ then $p\bullet \subseteq [x]$

(if a place p is in the cluster, then all transitions in the post-set of p are in the cluster)

Cluster

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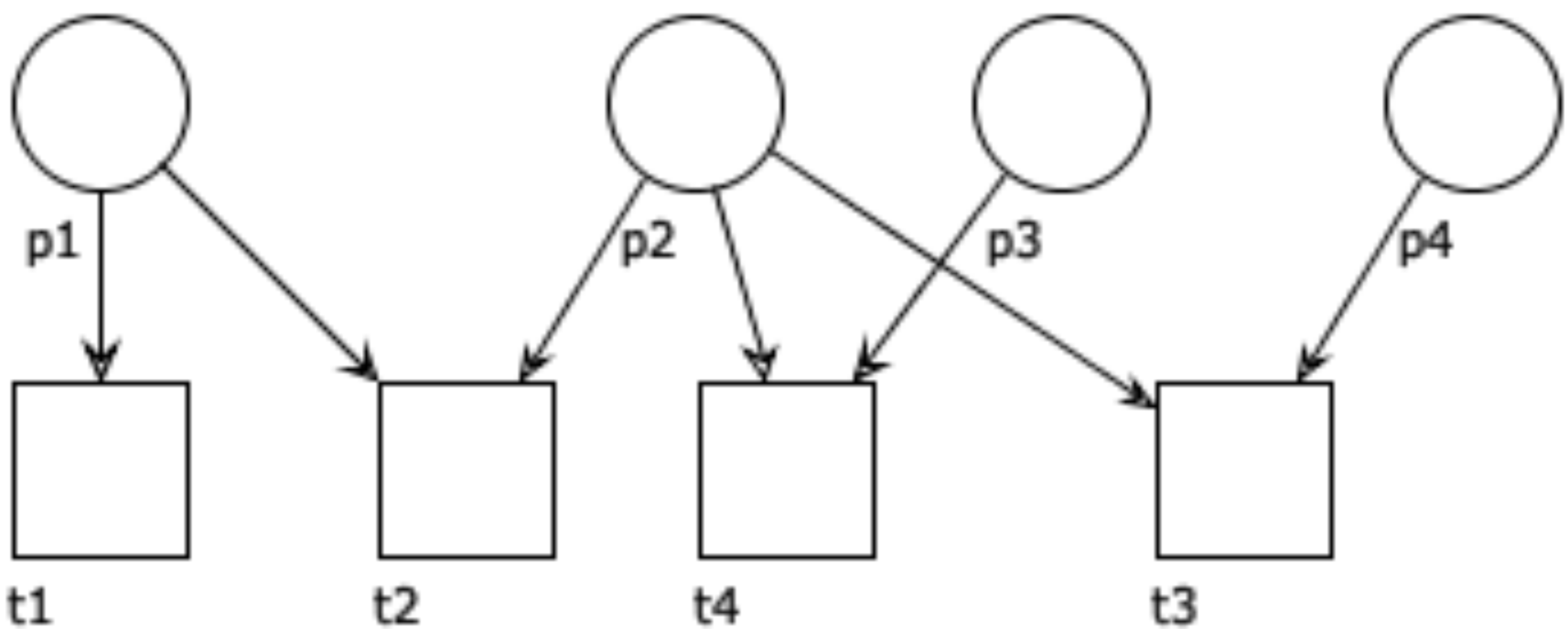
The **cluster** of x, written $[x]$, is the least set s.t.

1. $x \in [x]$
2. if $p \in [x] \cap P$ then $p\bullet \subseteq [x]$ (if a place p is in the cluster, then all transitions in the post-set of p are in the cluster)
3. if $t \in [x] \cap T$ then $\bullet t \subseteq [x]$ (if a transition t is in the cluster, then all places in the pre-set of t are in the cluster)

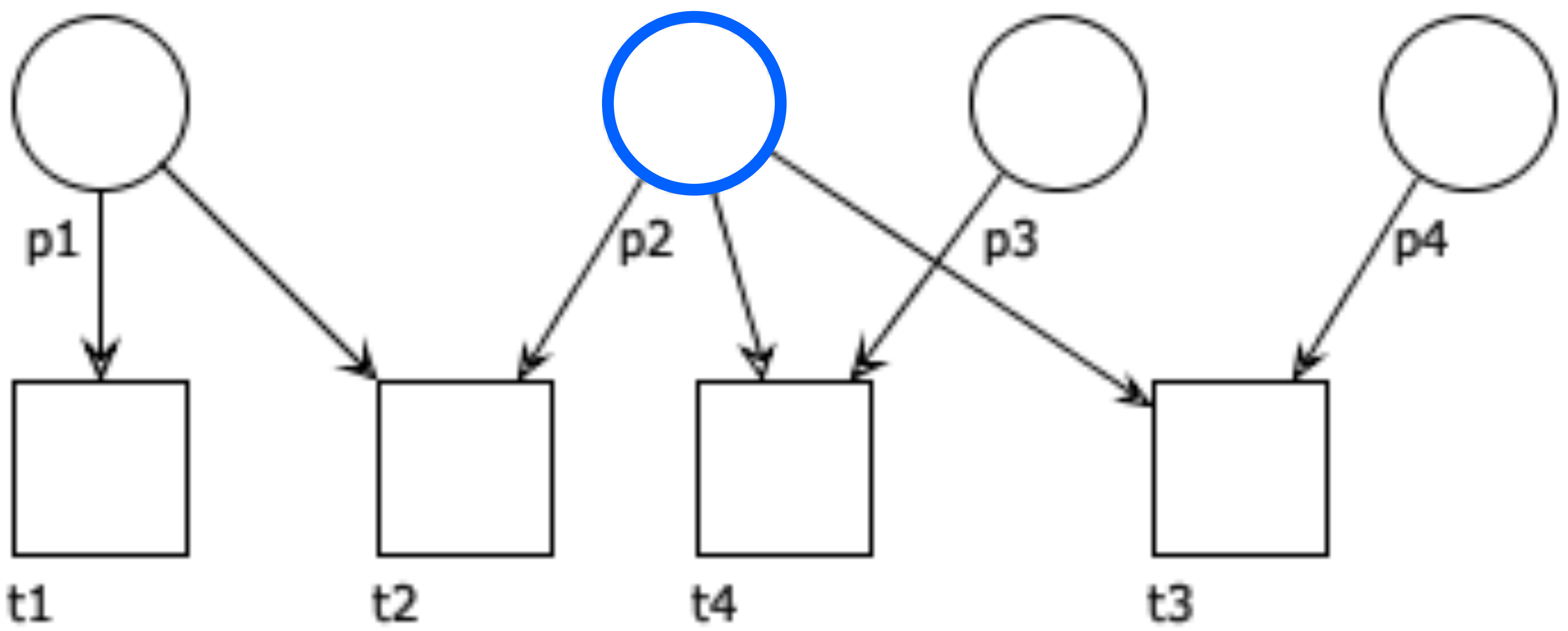
Cluster: intuition

[p2] = ?



Cluster: intuition

[p2] = { p2 ,… }



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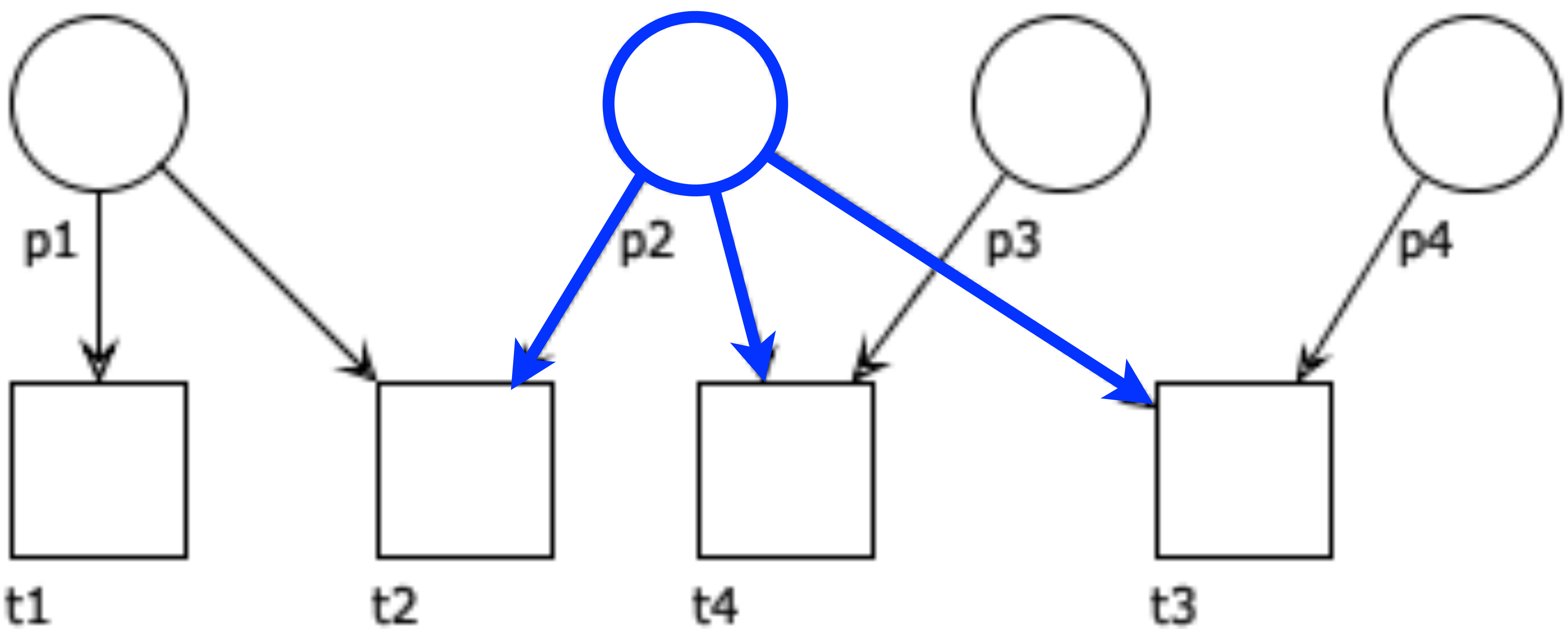

Cluster: intuition

[p2] = { p2 , t2, t4, t3,… }

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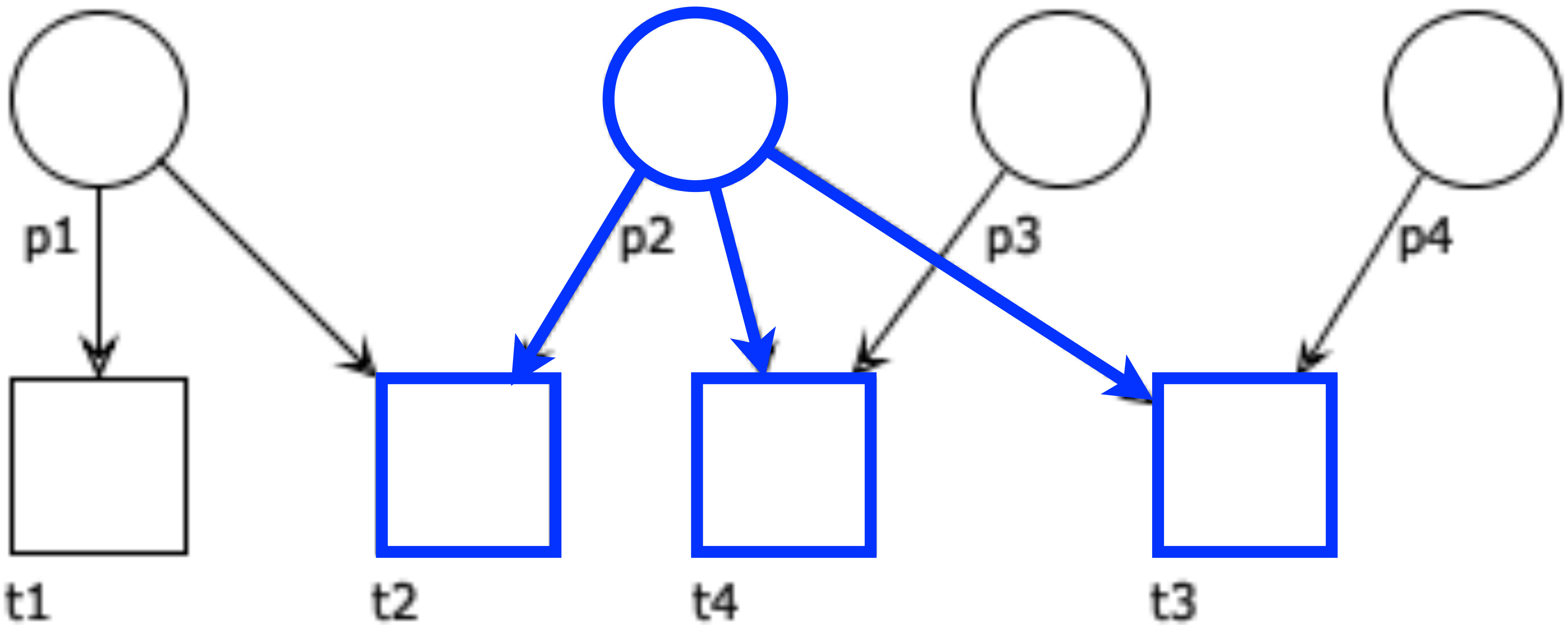

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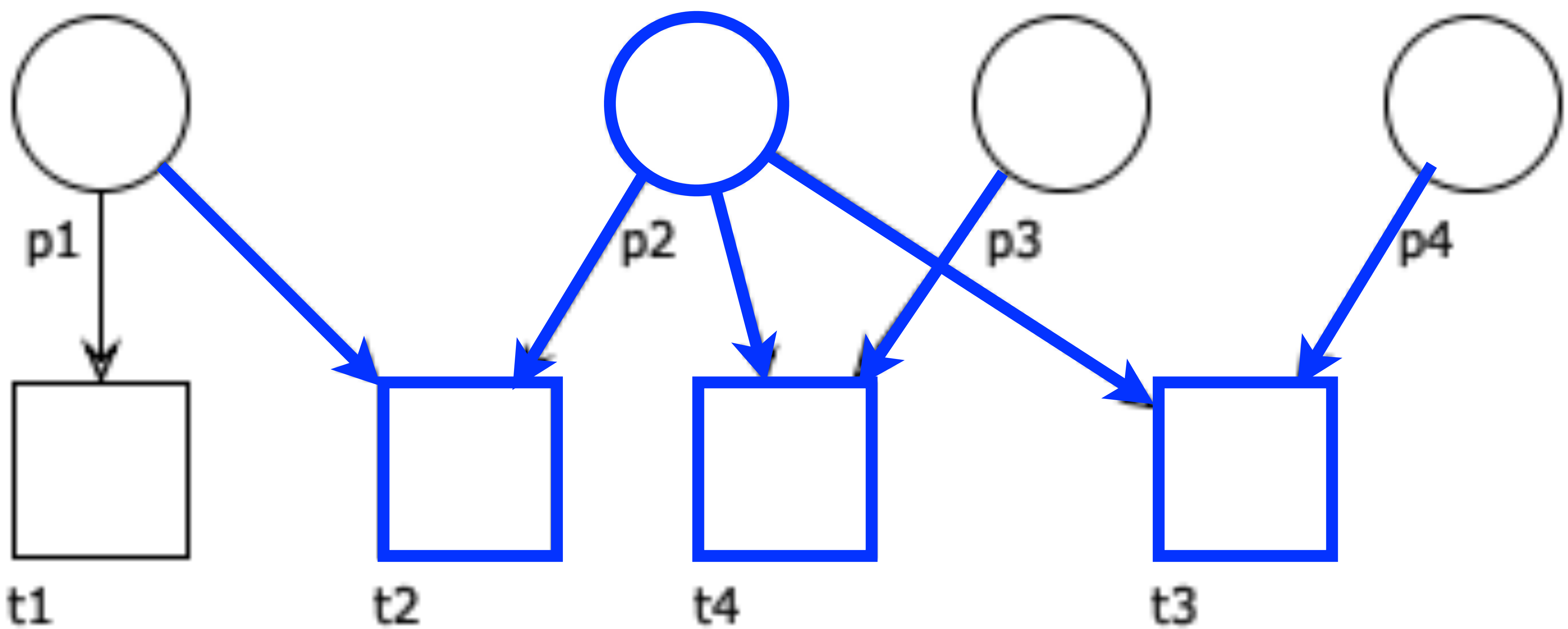

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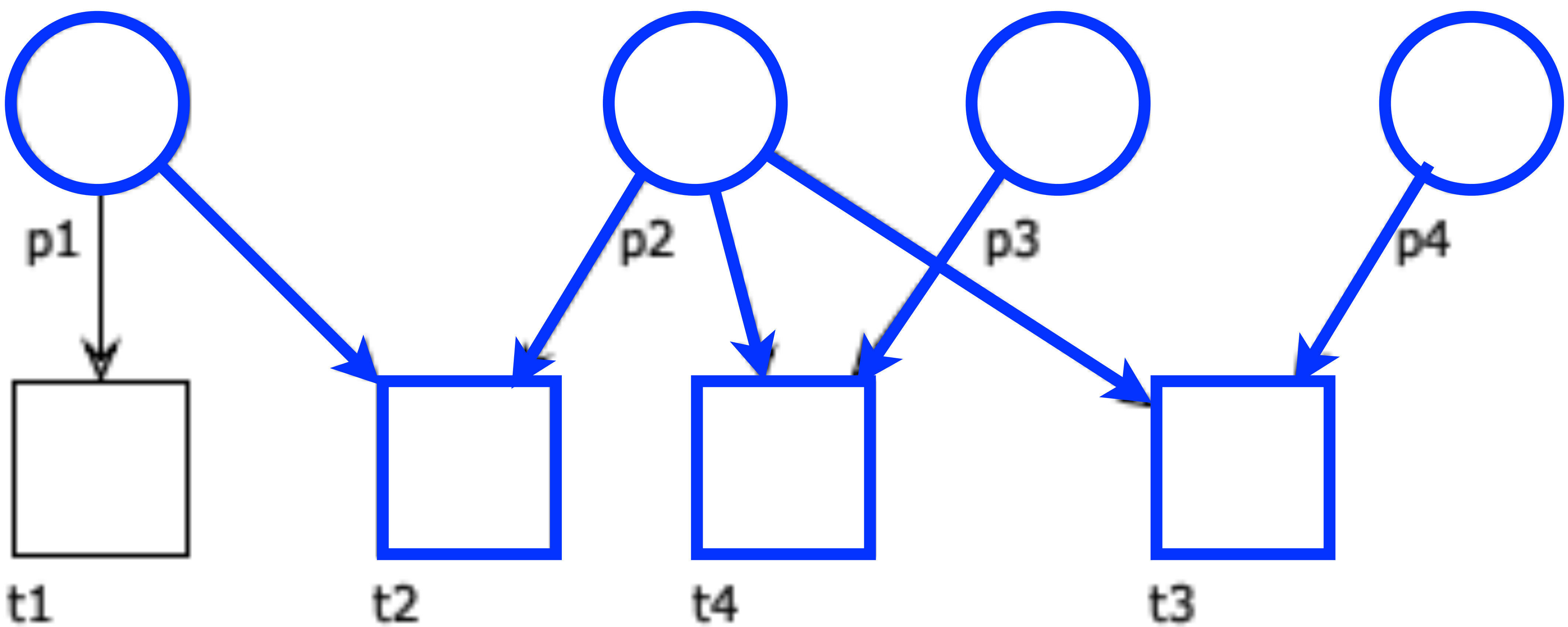

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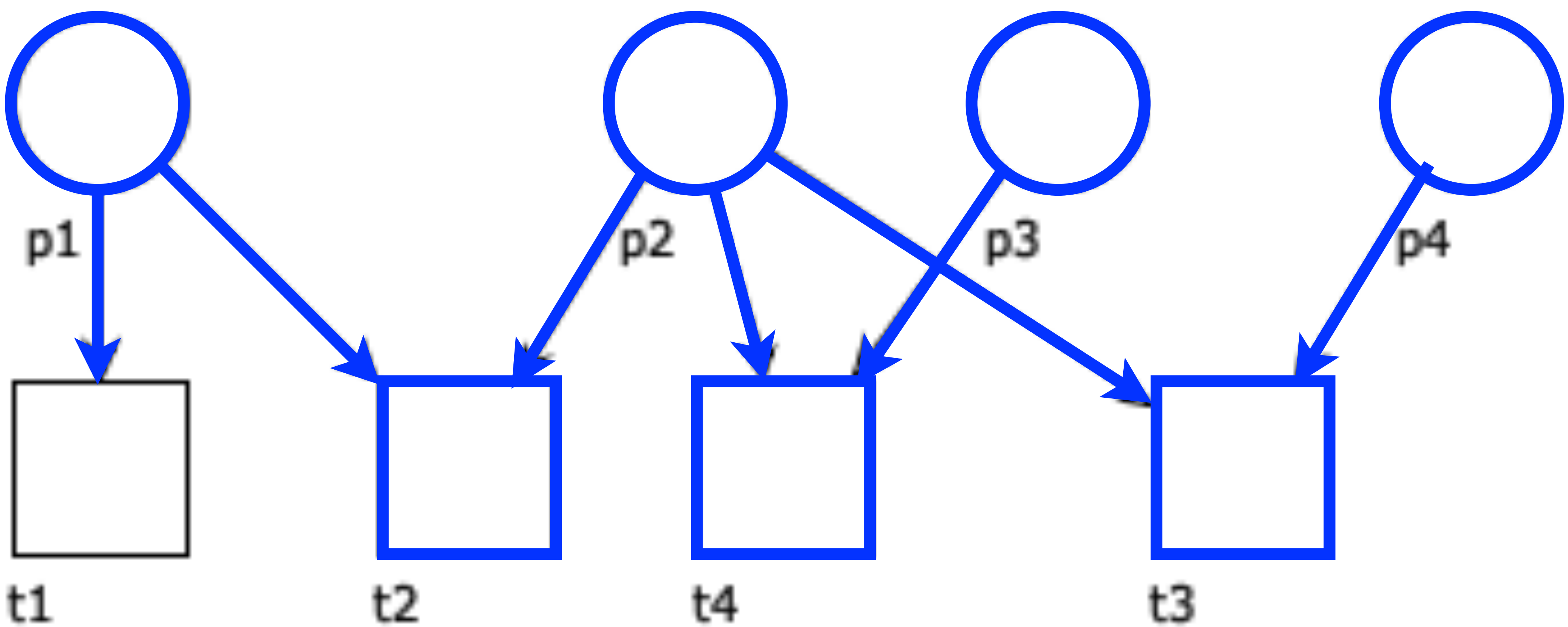

Cluster: intuition

[p2] = { p2 , t2, t4, t3, p1, p3, p4, t1 }

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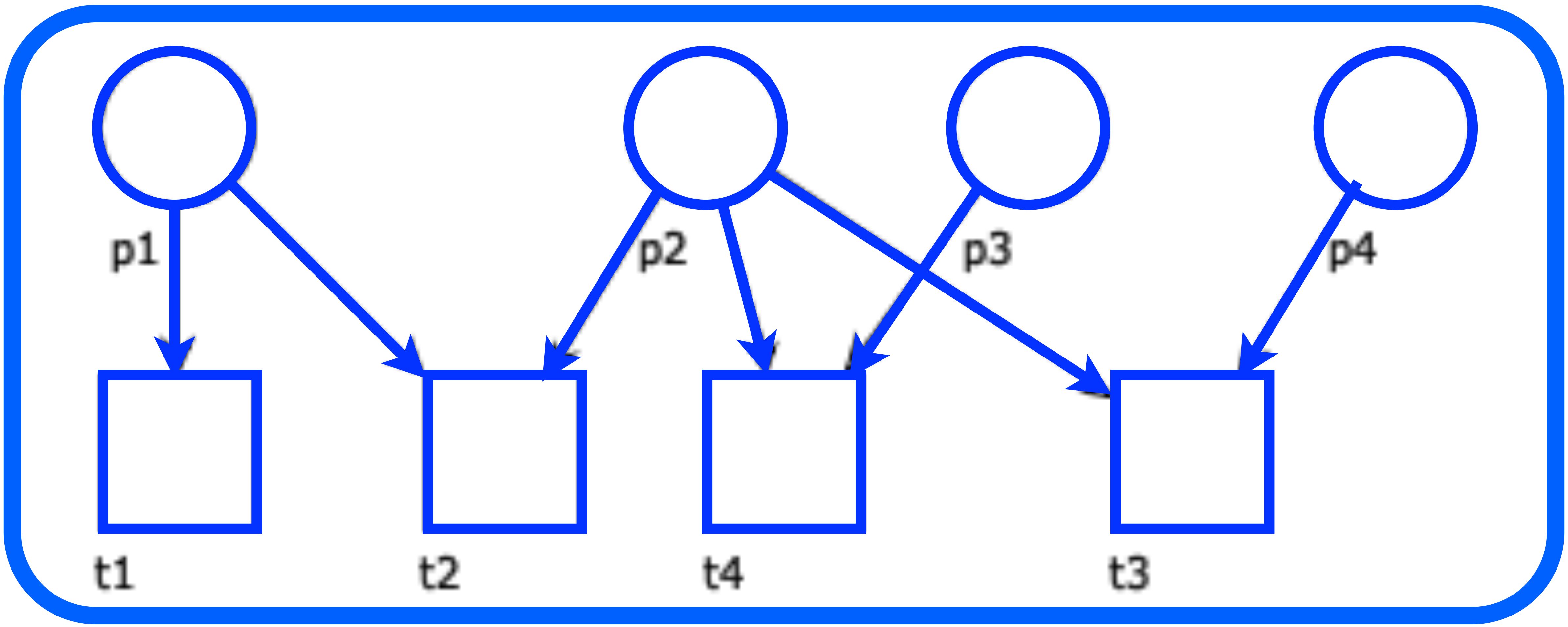
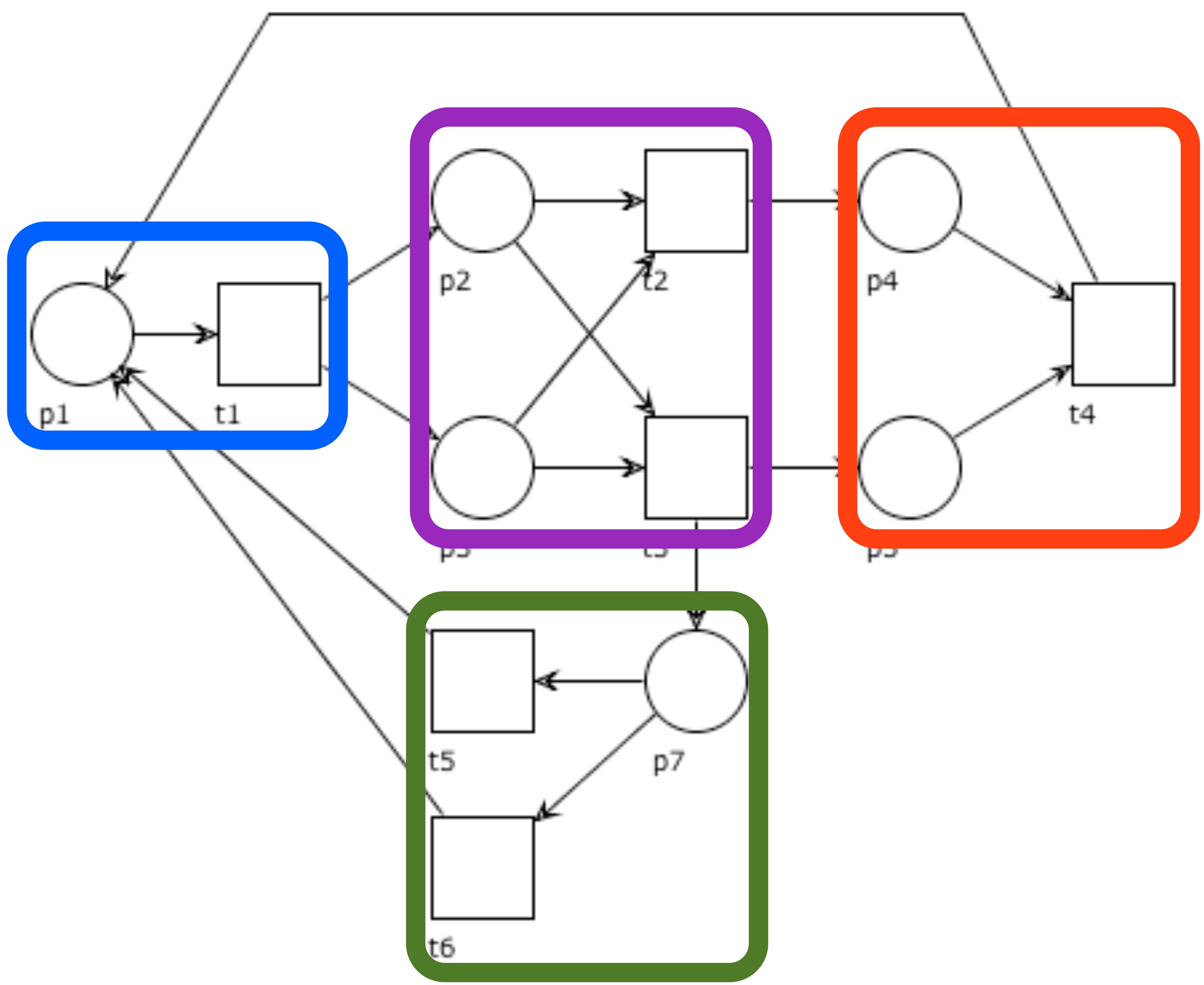
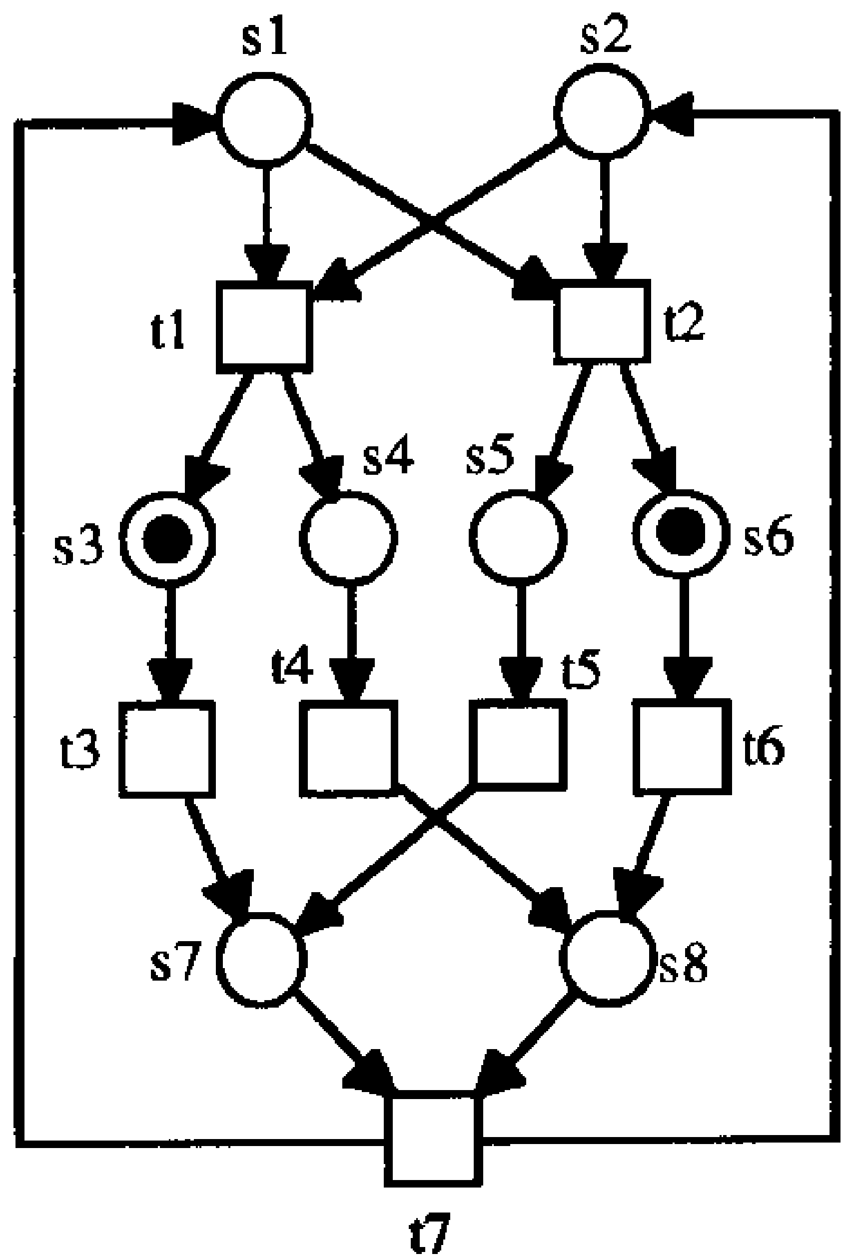


Clusters: example



Exercise

Draw all clusters in the free-choice net below



Clusters and Rank Theorem

Theorem:

A free-choice system (P,T,F,M_0) is live and bounded

iff

1. it has at least one place and one transition
2. it is connected
3. M0 marks every proper siphon
4. it has a positive S-invariant
5. it has a positive T-invariant
6. $rank(N) = |C_N| - 1$

(where C_N is the set of clusters)

A convenient concept: Stable set of markings

Stable set of markings

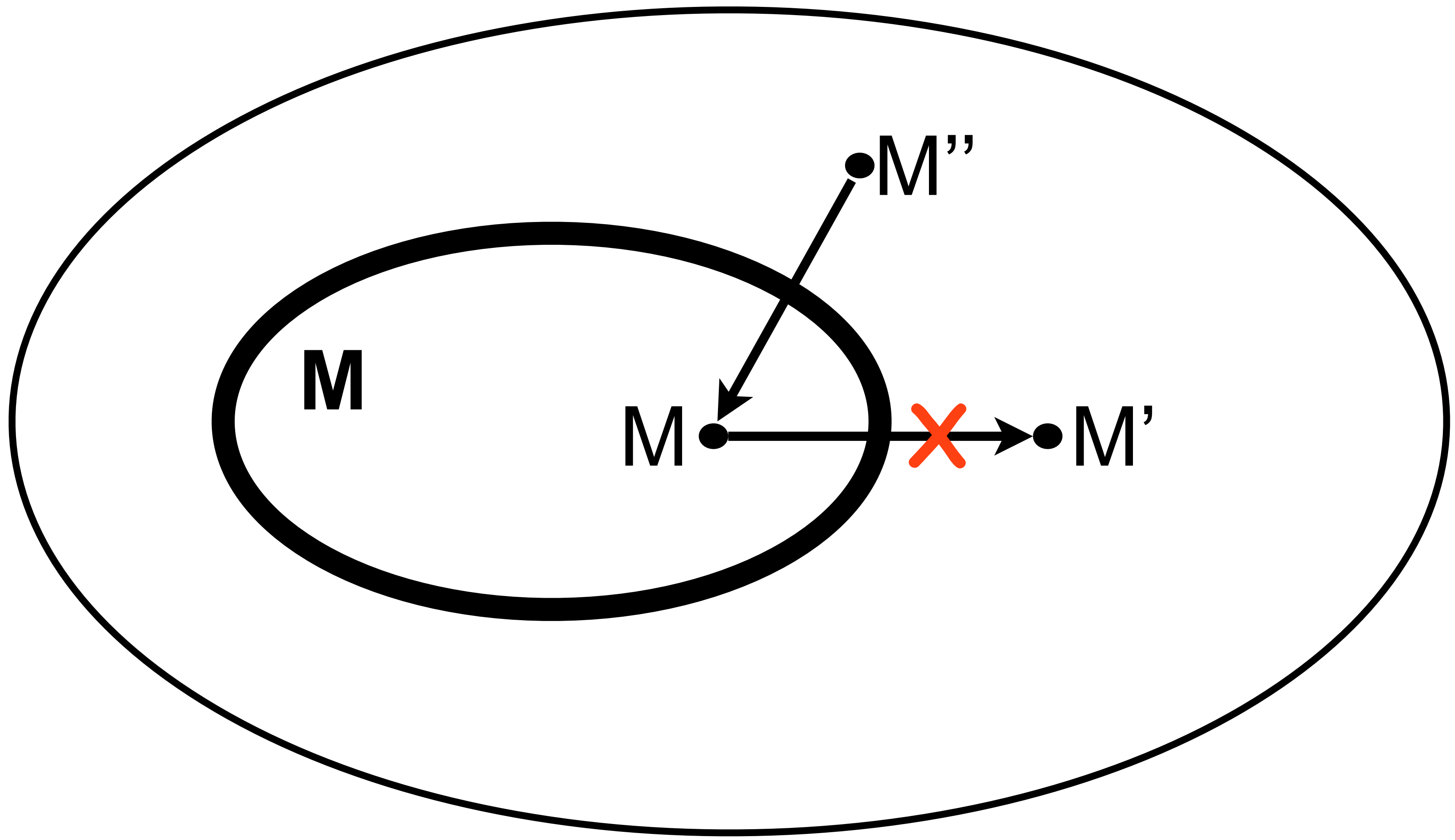
Definition: A set of markings $\mathbf{M}$ is called **stable** if

$$M \in \mathbf{M} \qquad \text{implies} \qquad [\, M \,\rangle \subseteq \mathbf{M}$$

(starting from any marking in the stable set **M**,
no marking outside **M** is reachable)

$[M_0\rangle$ is the least stable set that includes the marking M_0

Stable set of markings



(starting from any marking M in the stable set **M**,
no marking M' outside **M** is reachable)

Stability check

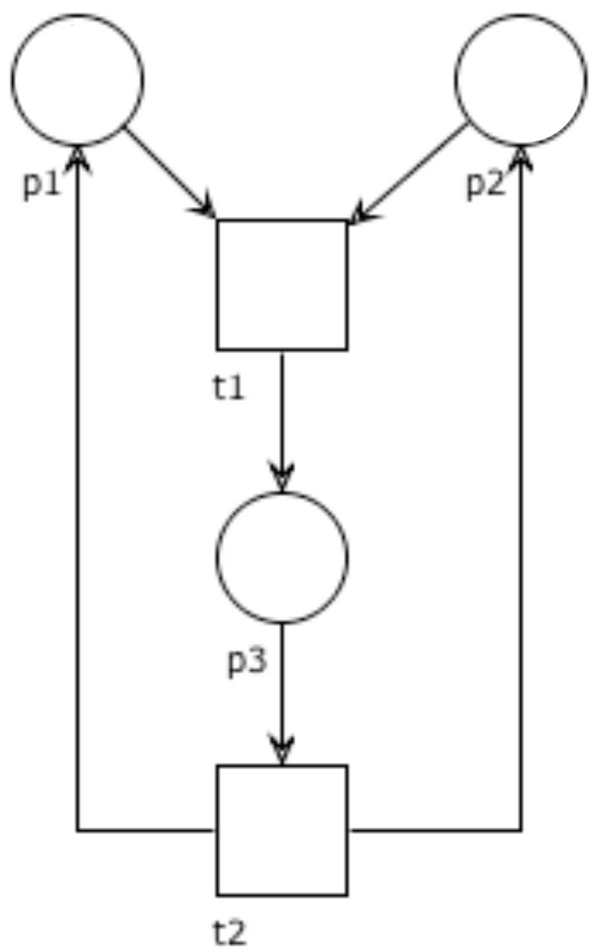
$\mathbf{M}$ is stable iff

$$\forall M, t, M'. \, (M \in \mathbf{M} \ \wedge \ M \xrightarrow{t} M' \text{ implies } M' \in \mathbf{M})$$

Example

Which of the following is a stable set of markings?

$$\forall M, t, M'. (M \in \mathbf{M} \ \wedge \ M \xrightarrow{t} M' \text{ implies } M' \in \mathbf{M})$$



$\{ 2p_1+p_2 \}$

$\{ 2p_1+p_2 , p_1+2p_3 \}$

$\{ p_1 , \ p_2 \}$

$\{ p_1+p_2 , \ p_3 \}$

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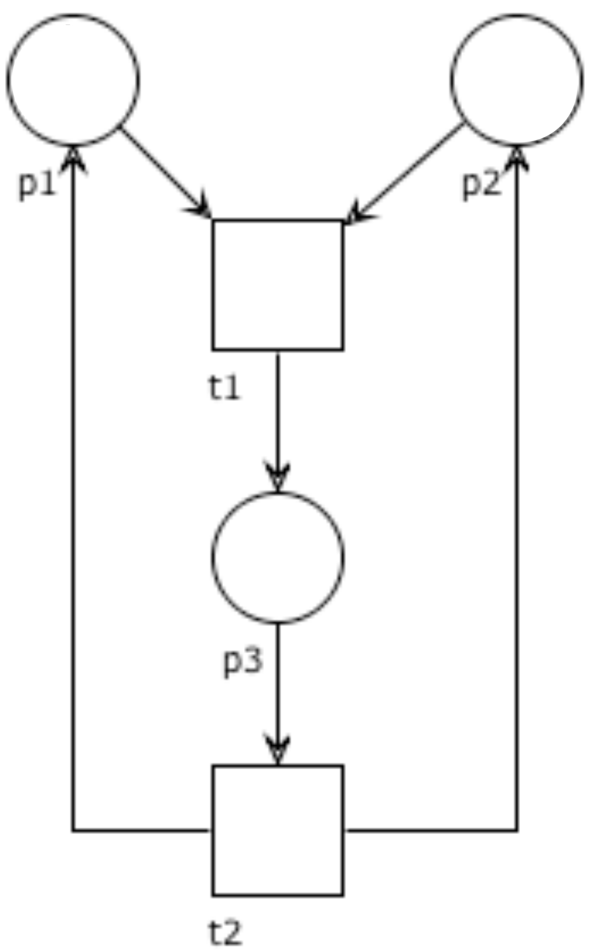

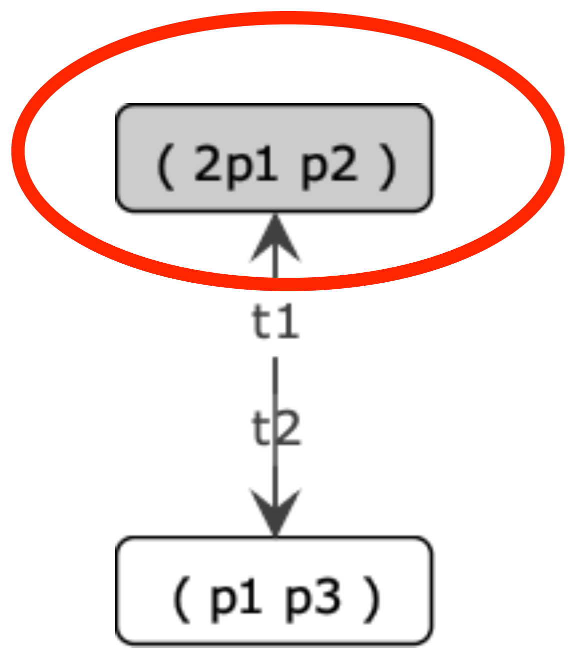

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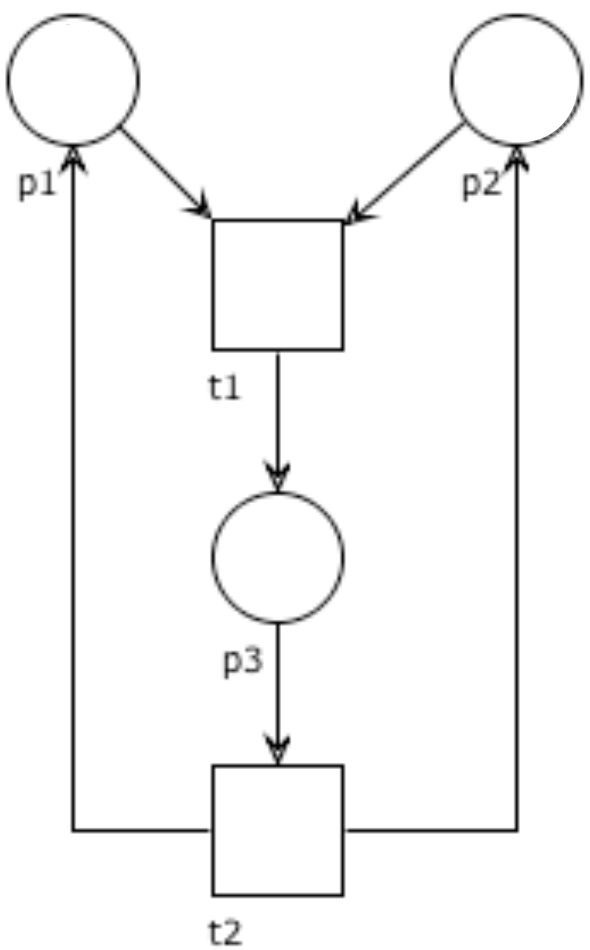

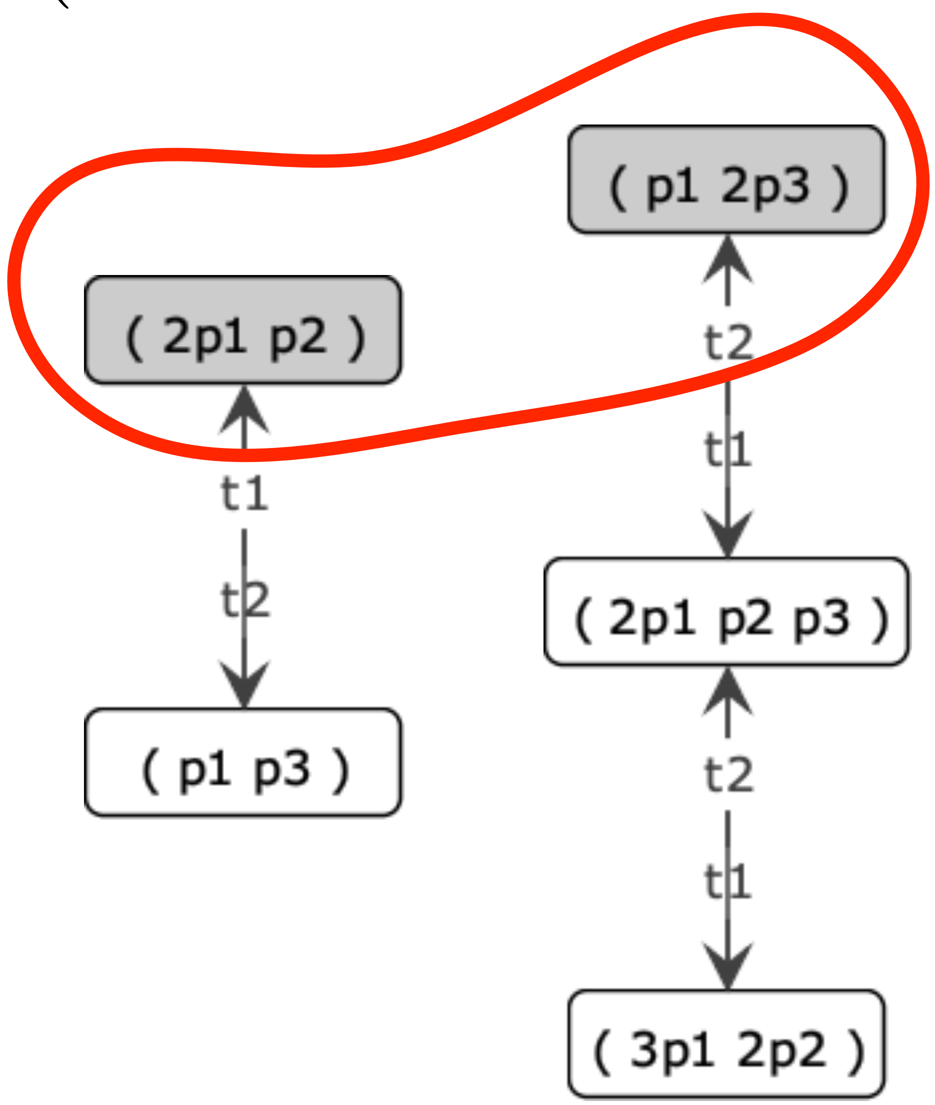

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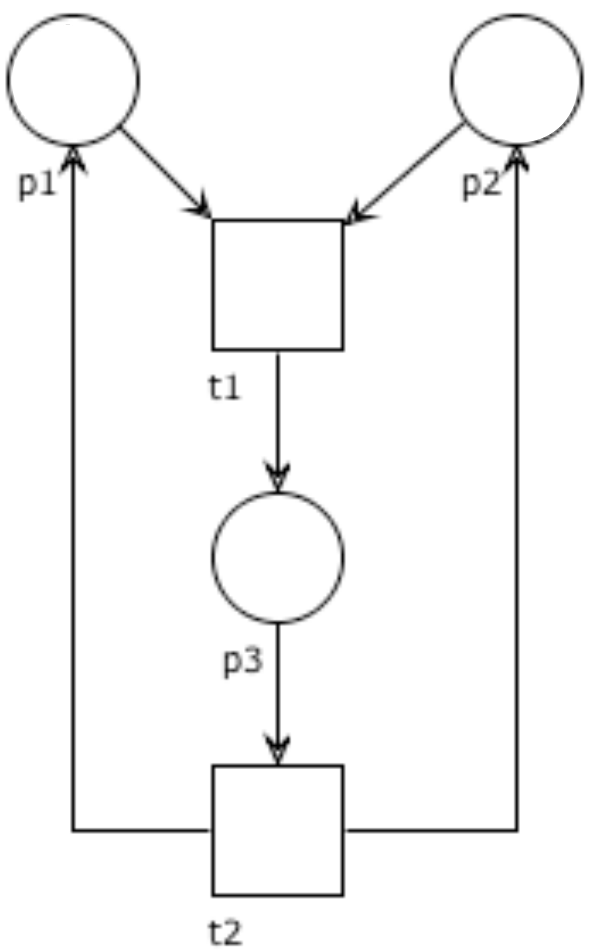

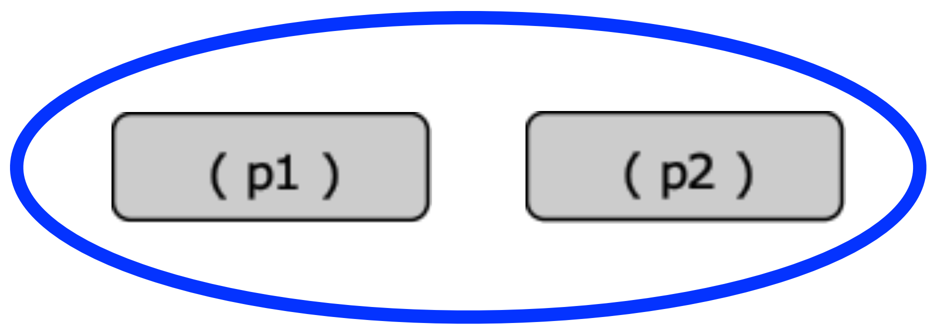

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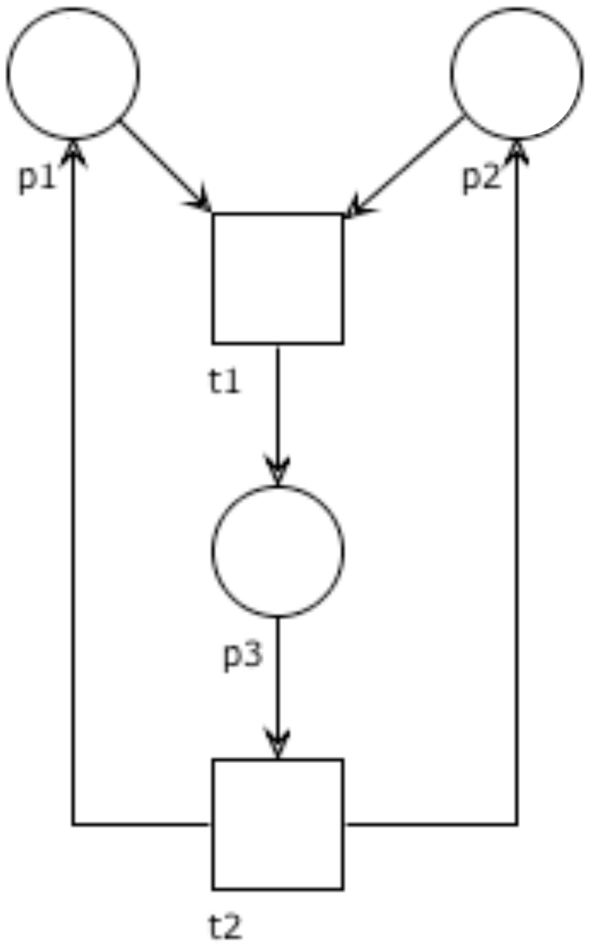

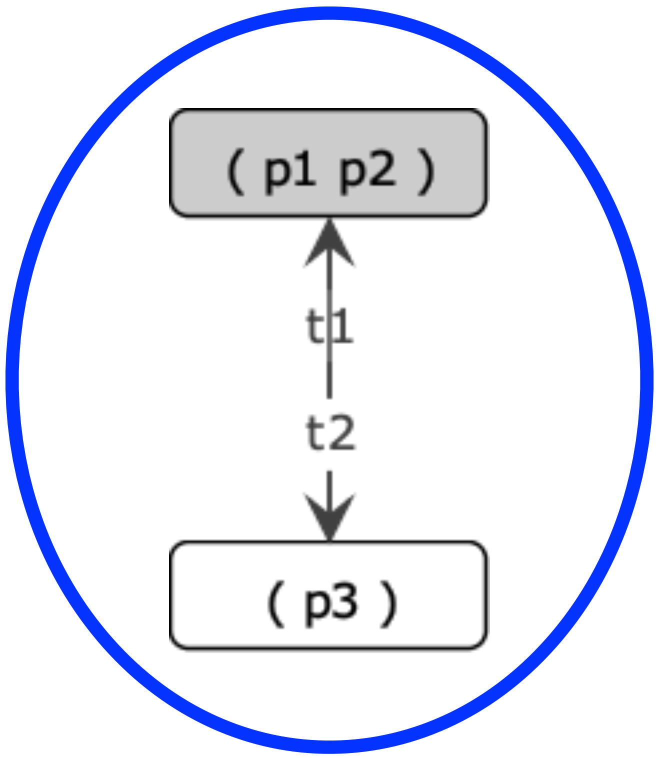

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Example

$$\forall M, t, M'. (M \in \mathbf{M} \;\wedge\; M \xrightarrow{t} M' \text{ implies } M' \in \mathbf{M})$$

Given a net system:

empty marking

Is the singleton set { $\mathbf{0}$ } a stable set?

Is the set of all markings a stable set?

Is the set of live markings a stable set?

Is the set of deadlock markings a stable set?

Example

$$\forall M, t, M'. \, (M \in \mathbf{M} \ \wedge \ M \xrightarrow{t} M' \text{ implies } M' \in \mathbf{M})$$

Given a net system:

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Is the singleton set { $\mathbf{0}$ } a stable set?

YES: no firing is possible

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Given a net system:

Is the singleton set { $\mathbf{0}$ } a stable set?

YES

Is the set of all markings a stable set?

YES: it is not possible to leave the set of all markings

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Is the set of deadlock markings a stable set?

Example

$$\forall M, t, M'. (M \in \mathbf{M} \ \wedge \ M \xrightarrow{t} M' \text{ implies } M' \in \mathbf{M})$$

Given a net system:

Is the singleton set { $\mathbf{0}$ } a stable set?

YES

Is the set of all markings a stable set?

YES

Is the set of live markings a stable set?

YES: liveness is an invariant

Is the set of deadlock markings a stable set?

Example

$$\forall M, t, M'. (M \in \mathbf{M} \ \wedge \ M \xrightarrow{t} M' \text{ implies } M' \in \mathbf{M})$$

Given a net system:

Is the singleton set { **0** } a stable set?

YES

Is the set of all markings a stable set?

YES

Is the set of live markings a stable set?

YES

Is the set of deadlock markings a stable set?

YES: no firing is possible

Exercises

Given a net (P,T,F):

Show that the set $\{ M \mid M(P)=1 \}$ is not necessarily stable.

Show that the set $\{ M \mid M(P)<k \}$ is not necessarily stable.

Exercises

Let **I** be an S-invariant for (P,T,F,M_0)

Is the set $\{ M \mid \mathbf{I} \cdot M = \mathbf{I} \cdot M_0 \}$ a stable set?

Is the set $\{ M \mid \mathbf{I} \cdot M \neq \mathbf{I} \cdot M_0 \}$ a stable set?

Is the set $\{ M \mid \mathbf{I} \cdot M = 1 \}$ a stable set?

Is the set $\{ M \mid \mathbf{I} \cdot M = 0 \}$ a stable set?

What is a siphon?

Proper siphon

Definition:

A set of places R is a **siphon** if $\bullet R \subseteq R \bullet$

It is a **proper siphon** if $R \neq \emptyset$

Siphons, intuitively

A set of places R is a siphon if

all transitions that can produce tokens in the places of R

$$\bullet R \subseteq R \bullet$$

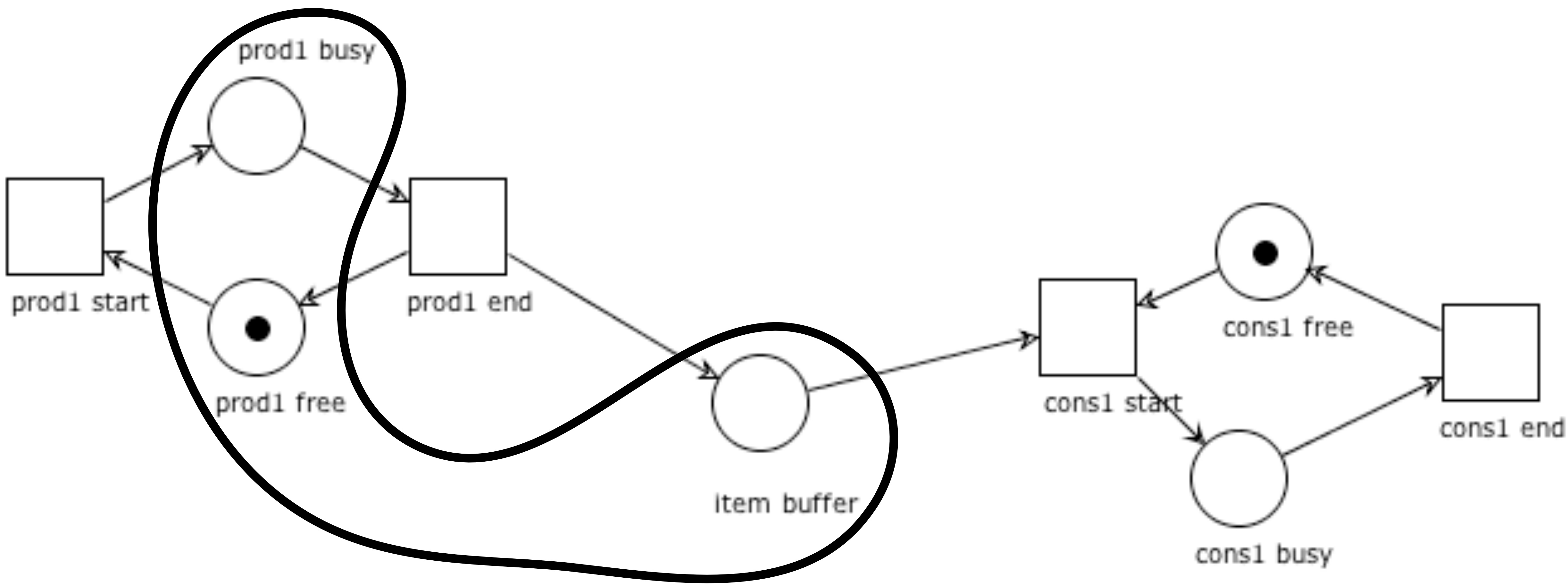
require some place in R to be marked

Therefore:
if no token is present in R,
then no token will ever be produced in R

$\bullet R \subseteq R \bullet$

Siphon check: example

Is R = { prod1busy, prod1free, itembuffer} a siphon?



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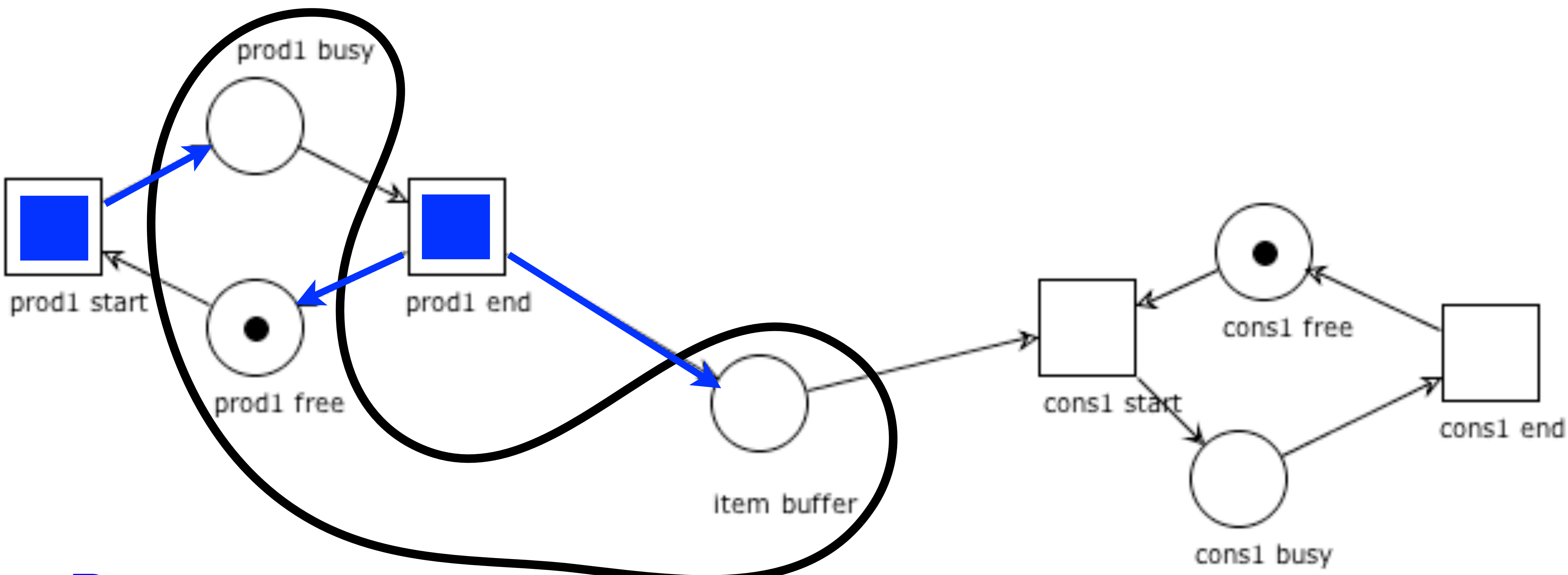

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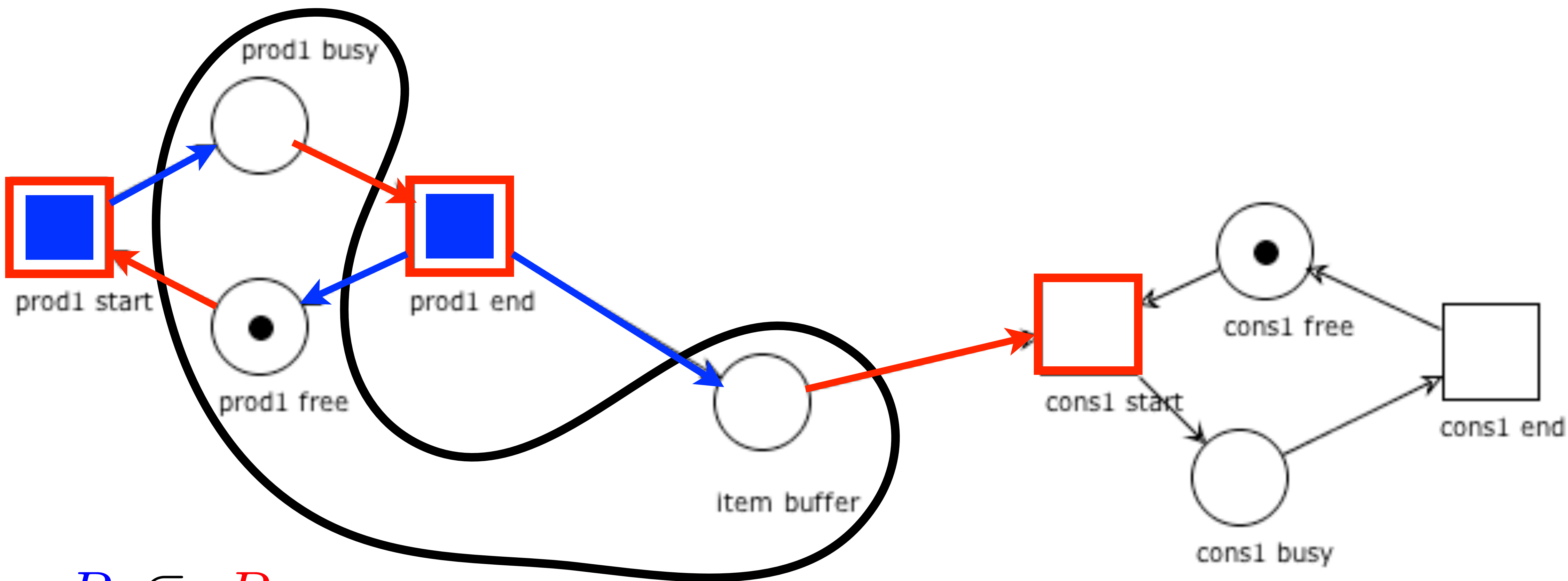

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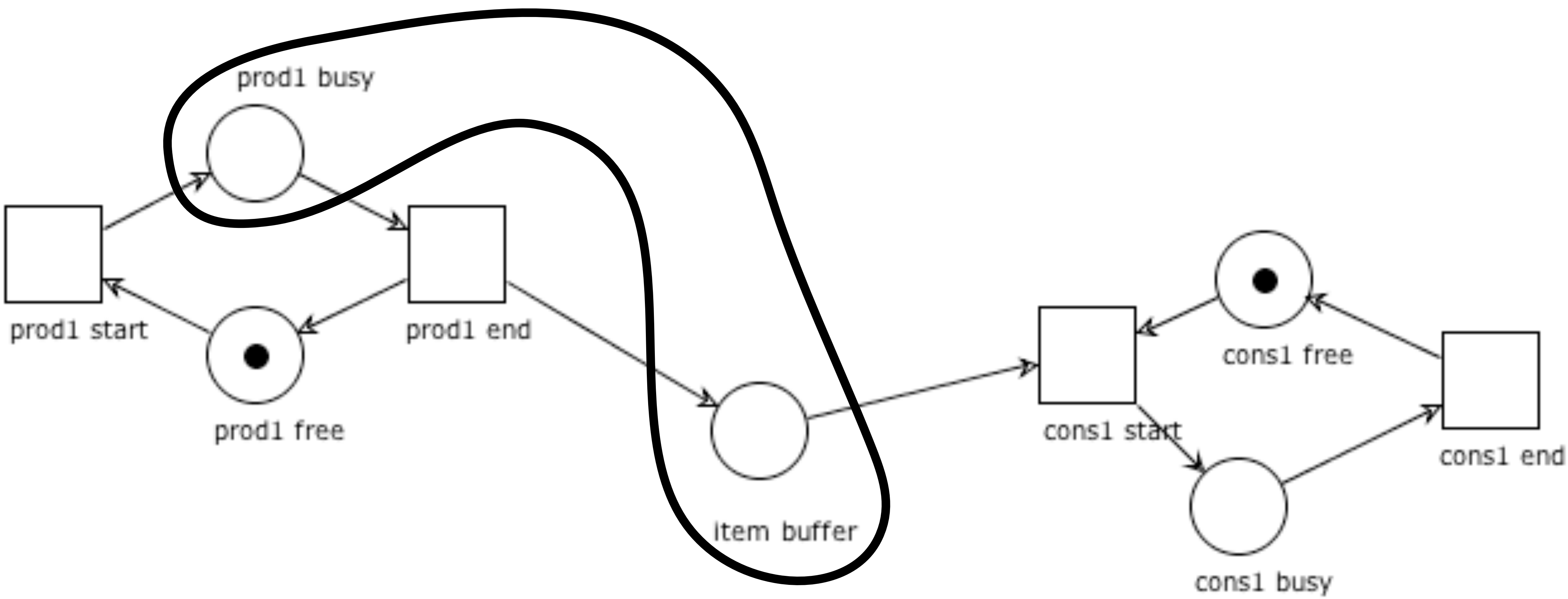

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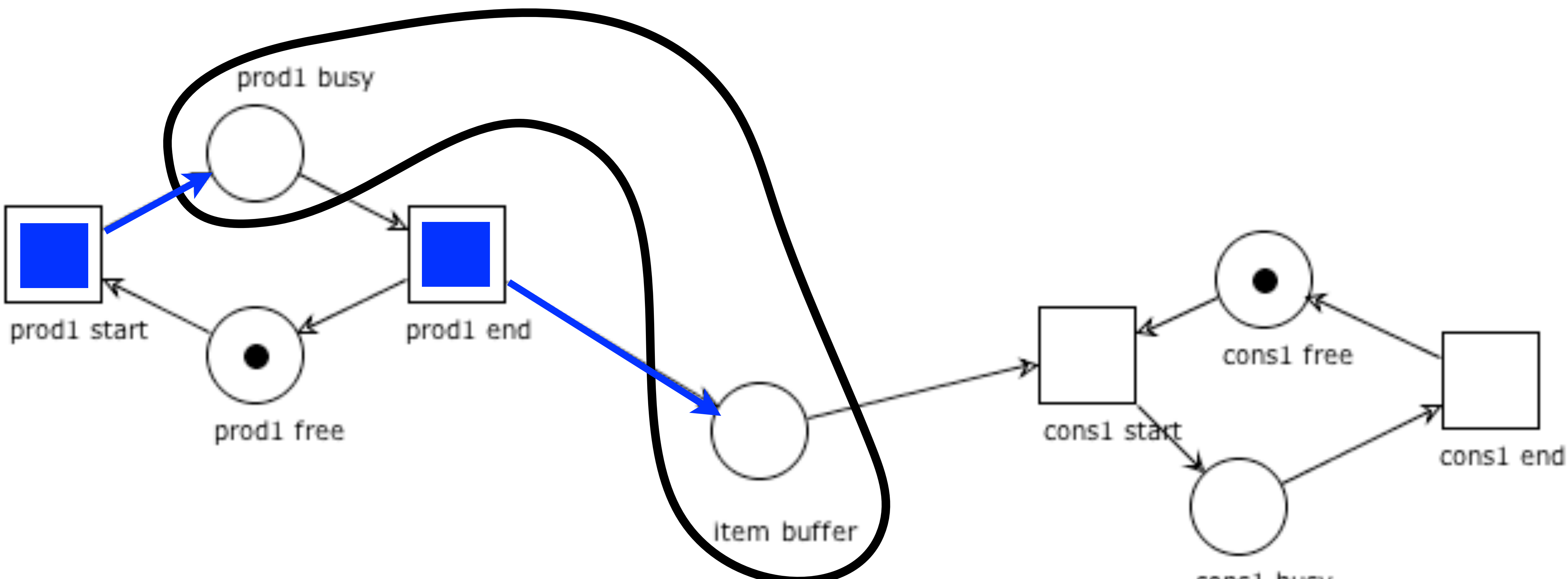

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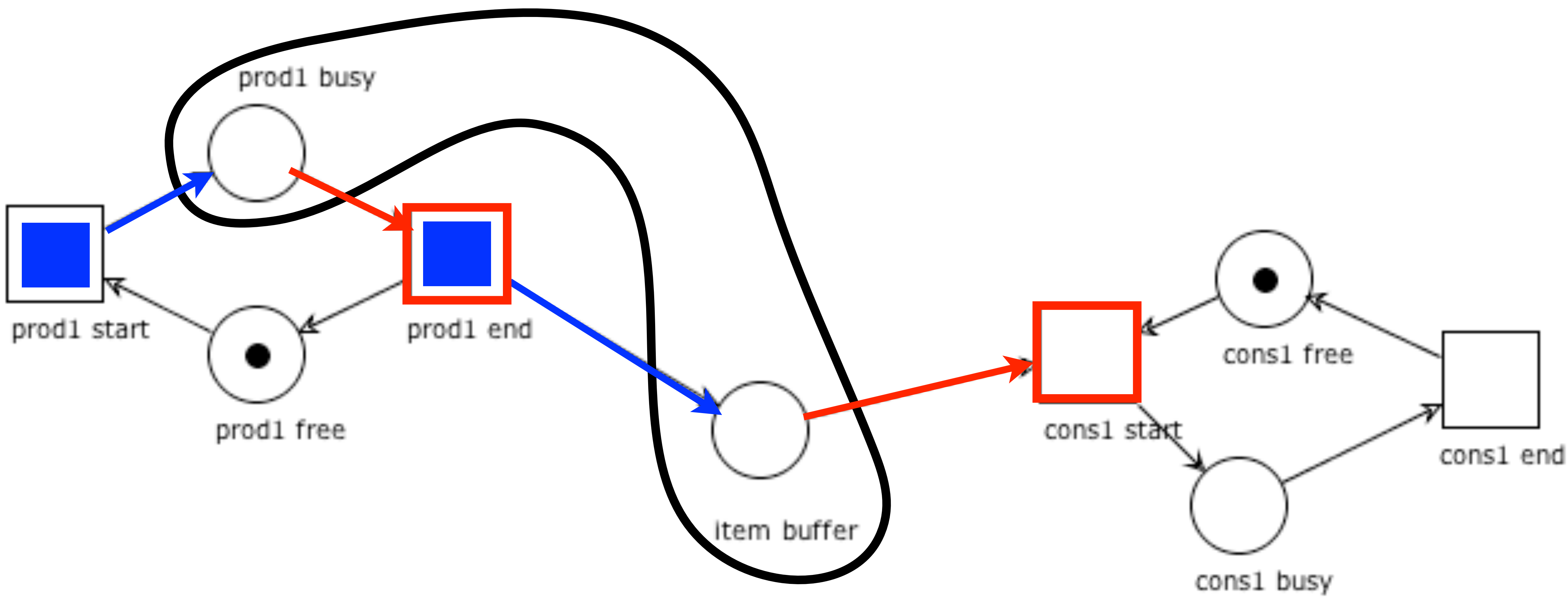
Siphon check: example

Is R = { prod1busy, itembuffer} a siphon?



$\bullet R \not\subseteq R \bullet$

Fundamental property of siphons

Proposition: Unmarked siphons remain unmarked

Take a siphon R.
We just need to prove that the set of markings
M = { M | M(R)=0 }
is stable, which is immediate by definition of siphon

Corollary:
If a siphon R is marked at some reachable marking M, then it was initially marked at M_0

Siphons and liveness

Prop.: If a system is live any proper siphon R is marked

Take $p \in R$ and let $t \in \bullet p \cup p \bullet$

Since the system is live, then there are $M, M' \in [\, M_0 \,\rangle$ such that

$$M \xrightarrow{t} M'$$

Therefore p is marked at either M or M'
Therefore R is marked at either M or M'
Therefore R was initially marked (at M_0)

Siphons and liveness

Corollary: If a system has an unmarked proper siphon then it is not live

Siphons and Rank Theorem

Theorem:

A free-choice system (P,T,F,M_0) is live and bounded

iff

1. it has at least one place and one transition
2. it is connected
3. M_0 marks every proper siphon
4. it has a positive S-invariant
5. it has a positive T-invariant
6. $rank(N) = |C_N| - 1$

(where C_N is the set of clusters)

Siphons and Commoner's theorem

Theorem:
A free-choice system is live
iff
every proper siphon includes an initially marked trap

What is a trap?

Proper trap

Definition:

A set of places R is a **trap** if $\bullet R \supseteq R \bullet$

It is a **proper trap** if $R \neq \emptyset$

Traps, intuitively

A set of places R is a trap if

all transitions that can consume tokens from R

$$\bullet R \supseteq R\bullet$$

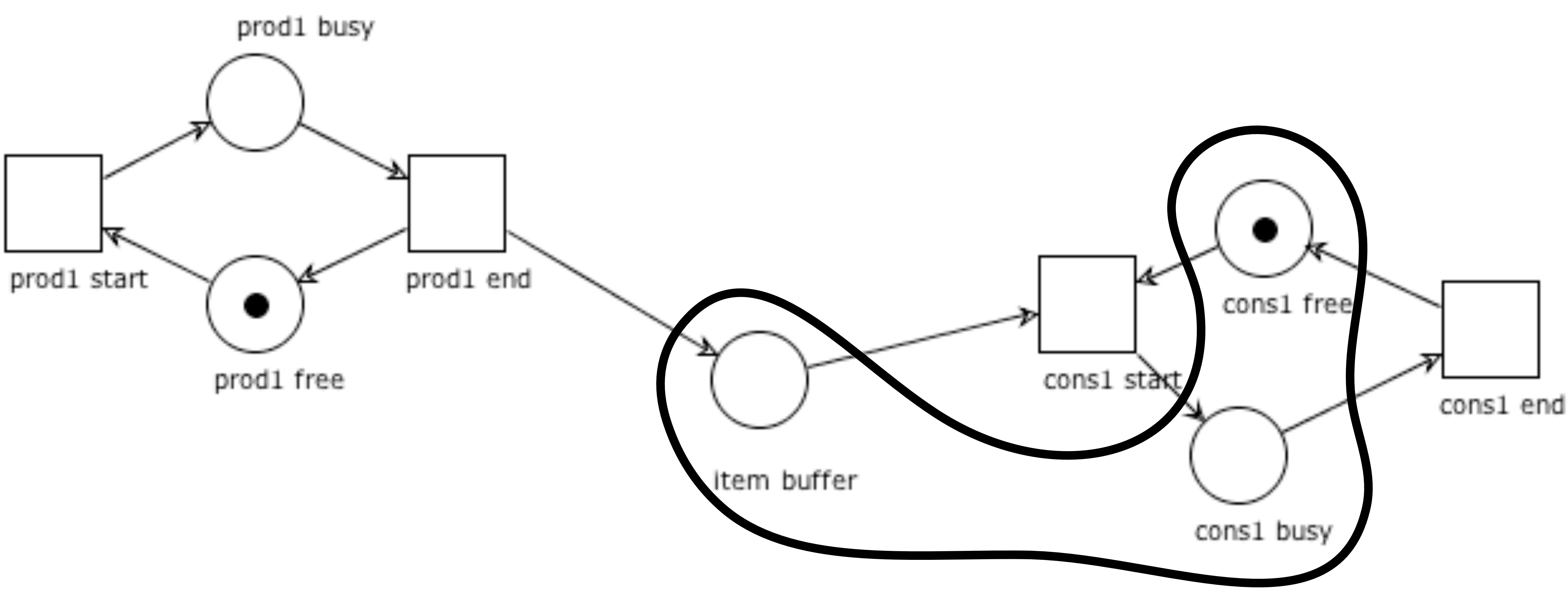
produce some token in some place of R

Therefore:
if some token is present in R,
then it is never possible for R to become empty

$\bullet R \supseteq R\bullet$

Trap check: example

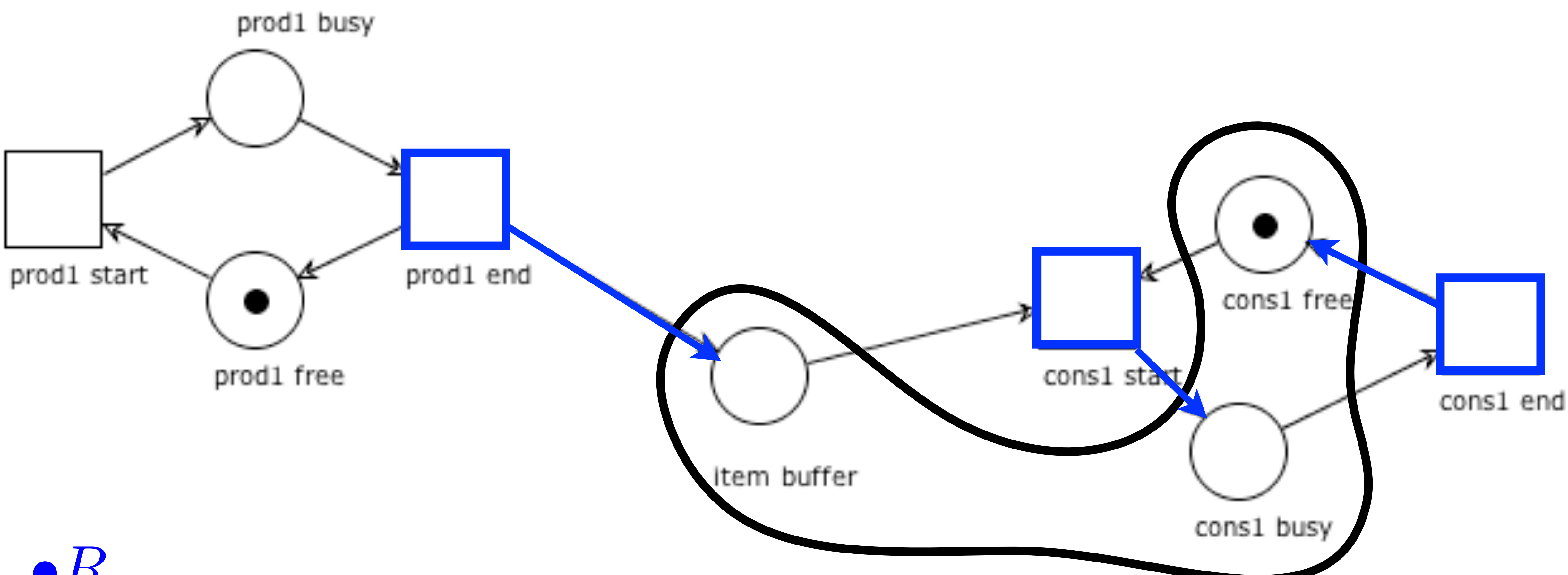
Is R = { itembuffer, cons1busy, cons1free} a trap?



$\bullet R \supseteq R \bullet$

Trap check: example

Is R = { itembuffer, cons1busy, cons1free} a trap?

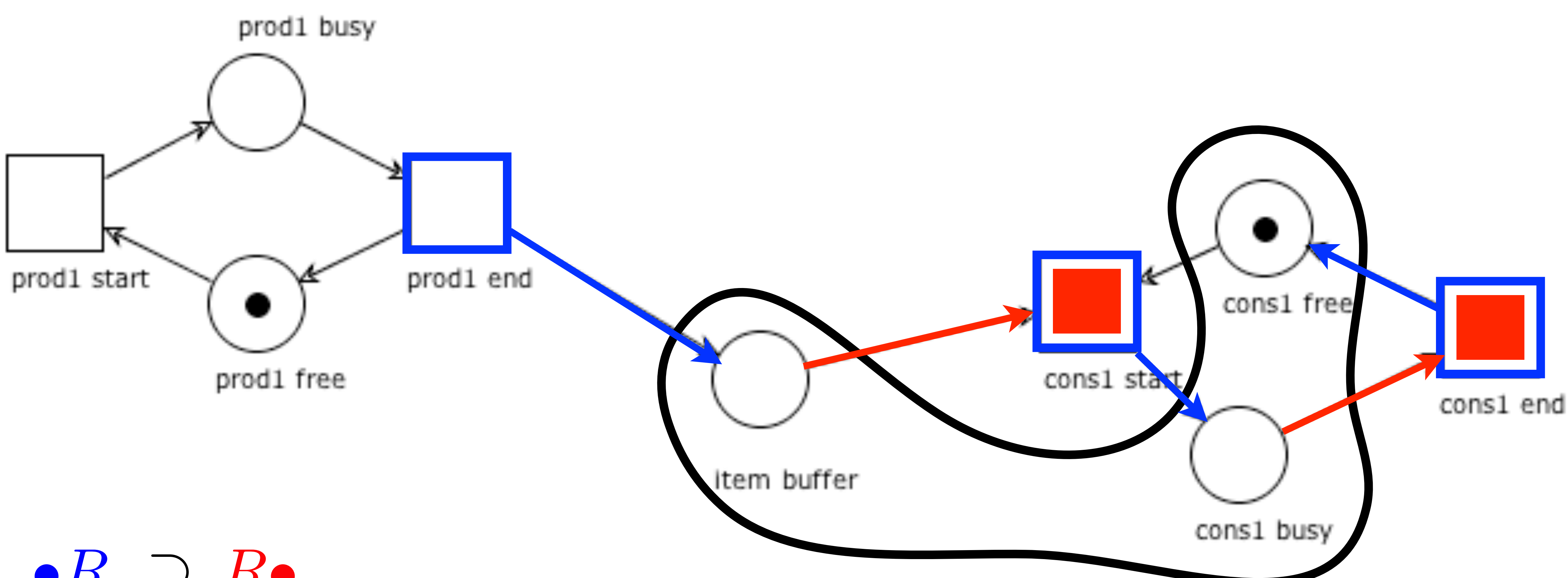


$\bullet R$

$\bullet R \supseteq R\bullet$

Trap check: example

Is R = { itembuffer, cons1busy, cons1free} a trap?

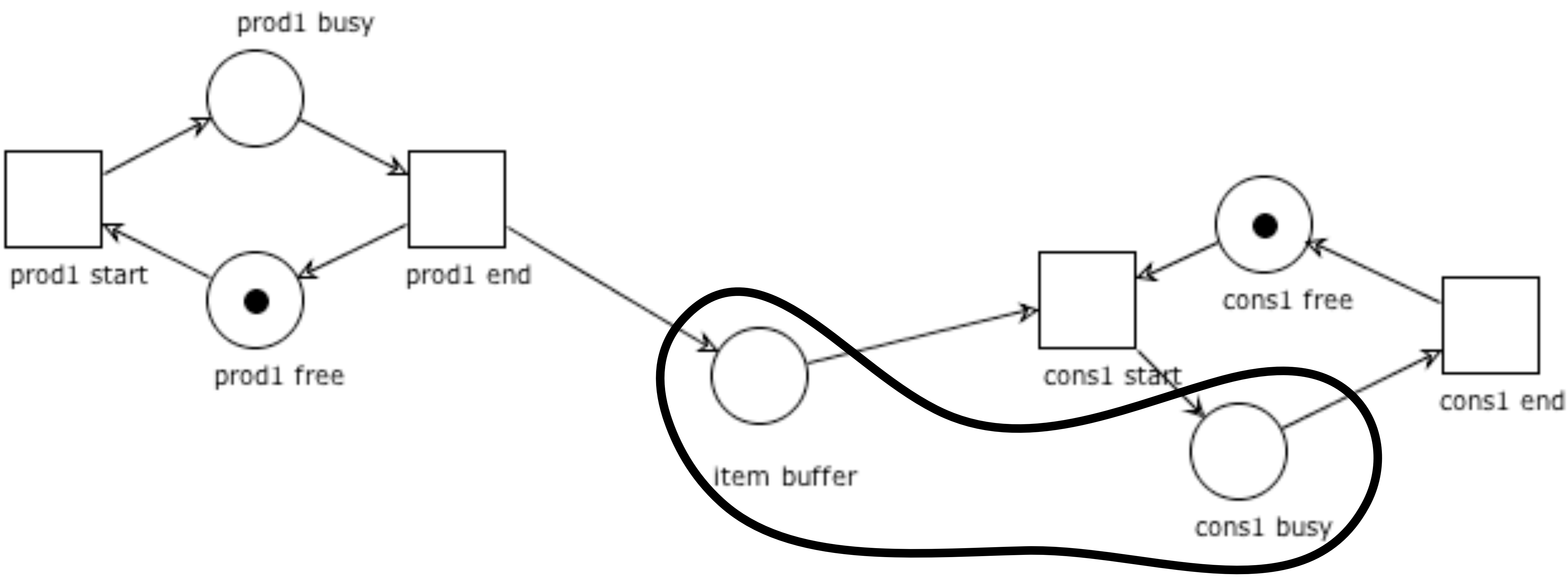


$\bullet R \supseteq R\bullet$

$\bullet R \supseteq R \bullet$

Trap check: example

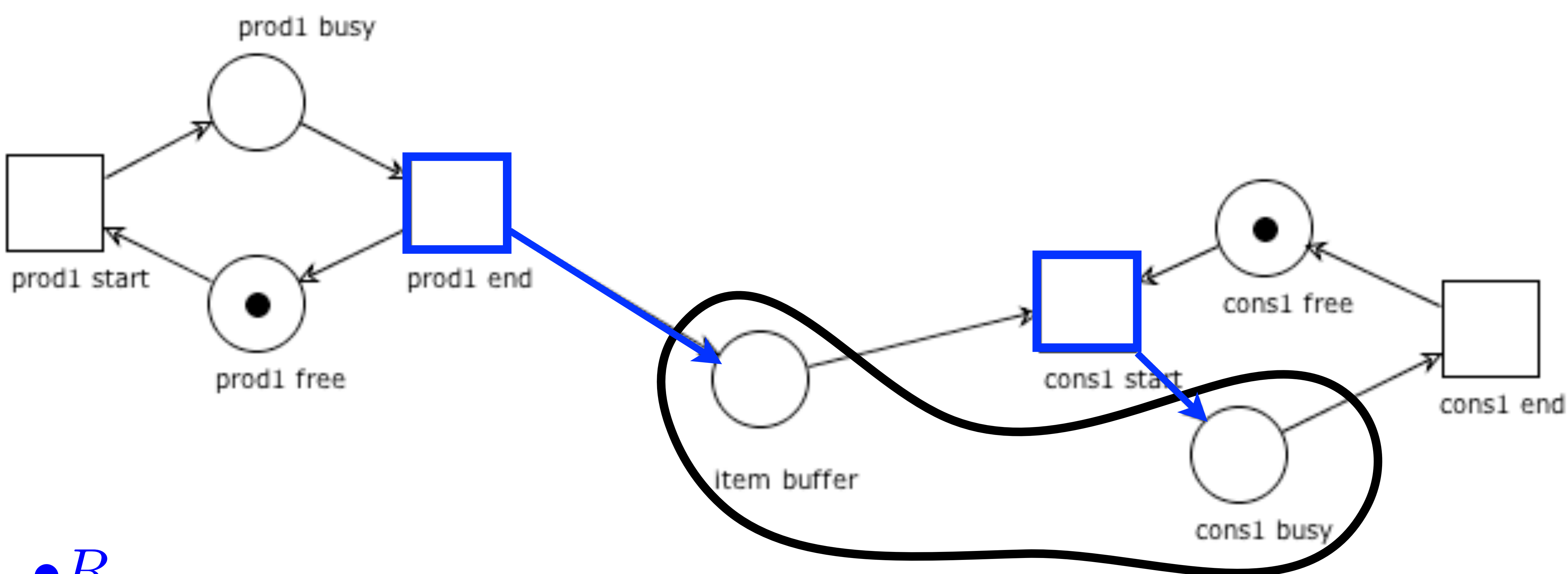
Is R = { itembuffer, cons1busy} a trap?



$\bullet R \supseteq R\bullet$

Trap check: example

Is R = { itembuffer, cons1busy} a trap?

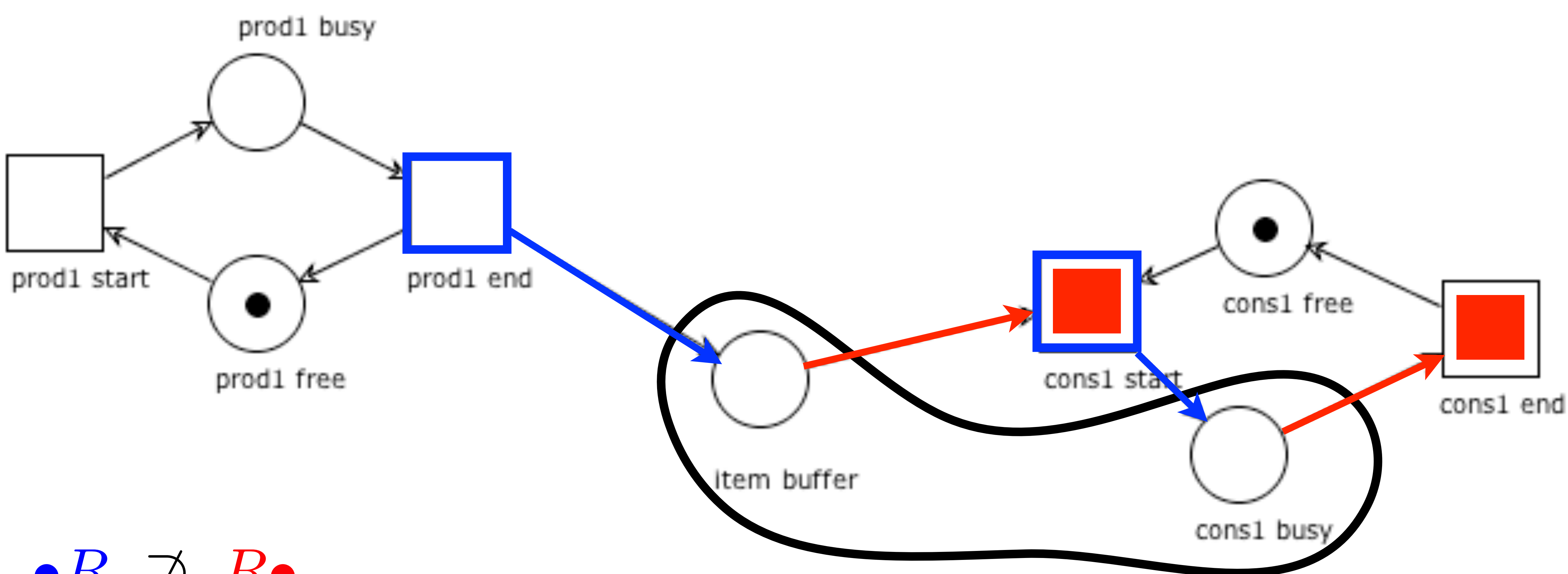


$\bullet R$

$\bullet R \supseteq R\bullet$

Trap check: example

Is R = { itembuffer, cons1busy} a trap?



$\bullet R \not\supseteq R\bullet$

Fundamental property of traps

Proposition: Marked traps remain marked

Take a trap R.
We just need to prove that the set of markings
M = { M | M(R)>0 }
is stable, which is immediate by definition of trap

Corollary:
If a trap R is unmarked at some reachable marking M,
then it was initially unmarked at M_0

Complexity issues 1:
Is it hard to show that a free-choice net is live?

Commoner's theorem

Theorem:
A free-choice system is live
iff
every proper siphon includes an initially marked trap

We show that
the non-liveness problem for free-choice systems

is NP-complete

Note

It is easy to observe that every siphon includes a (possibly empty) unique maximal trap with respect to set inclusion (the union of traps is a trap)

Moreover, a siphon includes a marked trap iff its maximal trap is marked

Traps are closed under union

Lemma. The union of traps is a trap

Let X_1, X_2 be traps.
From $X_1\bullet \subseteq \bullet X_1$ and $X_2\bullet \subseteq \bullet X_2$ we have:

$$(X_1 \cup X_2)\bullet = X_1 \bullet \cup X_2\bullet \subseteq \bullet X_1 \cup \bullet X_2 = \bullet(X_1 \cup X_2)$$

Commoner's theorem

Theorem:
A free-choice system is live
iff
every proper siphon includes an initially marked trap

Theorem:
A free-choice system is non-live
iff
there is a proper siphon that only includes initially unmarked traps

A non-deterministic algorithm for non-liveness

1. guess a set of places R
   [polynomial time, non-deterministic step]

2. check if R is a siphon ($\bullet R \subseteq R \bullet$)
   [polynomial time]

3. if R is a siphon, compute the maximal trap $Q \subseteq R$
   [complexity?]

4. if $M_0(Q)=0$, then answer “non-live”, otherwise "live"
   [polynomial time]

A polynomial algorithm for maximal trap in a siphon

$\bullet R \subseteq R \bullet$ $\bullet Q \supseteq Q \bullet$

3. if R is a siphon, compute the maximal trap Q ⊆ R

Input: A net $N = (P, T, F)$ and $R \subseteq P$
Output: $Q \subseteq R$ maximal trap in R

$Q := R$
while $(\exists p \in Q,\ \exists t \in p\bullet,\ t \notin \bullet Q)$
$\quad Q := Q \setminus \{p\}$
return Q

A polynomial non-det. algorithm for non-liveness

1. guess a set of places R
   [polynomial time, non-deterministic step]

2. check if R is a siphon ($\bullet R \subseteq R\bullet$)
   [polynomial time]

3. if R is a siphon, compute the maximal trap $Q \subseteq R$
   [polynomial time]

4. if $M_0(Q)=0$, then answer “non-live”, otherwise "live"
   [polynomial time]

Non-liveness for f.c. nets is in NP

The non-liveness problem for free-choice systems is in NP

Is the same problem in P?

The corresponding deterministic algorithm cannot make the guess in step 1

It has to explore all possible subsets of places
$2^{|P|}$ cases!

NP-completeness

We next sketch the proof of the reduction to non-liveness in a free-choice net of the CNF-SAT problem

(SATisfiability problem
for propositional formulas in Conjunctive Normal Form)

CNF-SAT is an NP-complete problem

CNF-SAT decision problem

Variables: $x_1, x_2, ..., x_n$

Literals: $x_1, \bar{x}_1, x_2, \bar{x}_2, ..., x_n, \bar{x}_n$

Clause: disjunction of literals

Formula: conjunction of clauses

Example: $\phi = (x_1 \vee \bar{x_3}) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_2 \vee \bar{x}_3)$

Is there an assignment of boolean values to the variables such that $\phi = true$?

The free-choice net of a formula

Given a formula ϕ, the idea is

to construct a free-choice system (P,T,F,M_0)

and show that

the formula ϕ is satisfiable

iff

(P,T,F,M_0) is not live

The free-choice net of a formula

Given a formula ϕ, the idea is
to construct a free-choice system (P,T,F,M_0)
and show that

the formula ϕ is not satisfiable
iff
(P,T,F,M_0) is live

CNF-SAT formulas

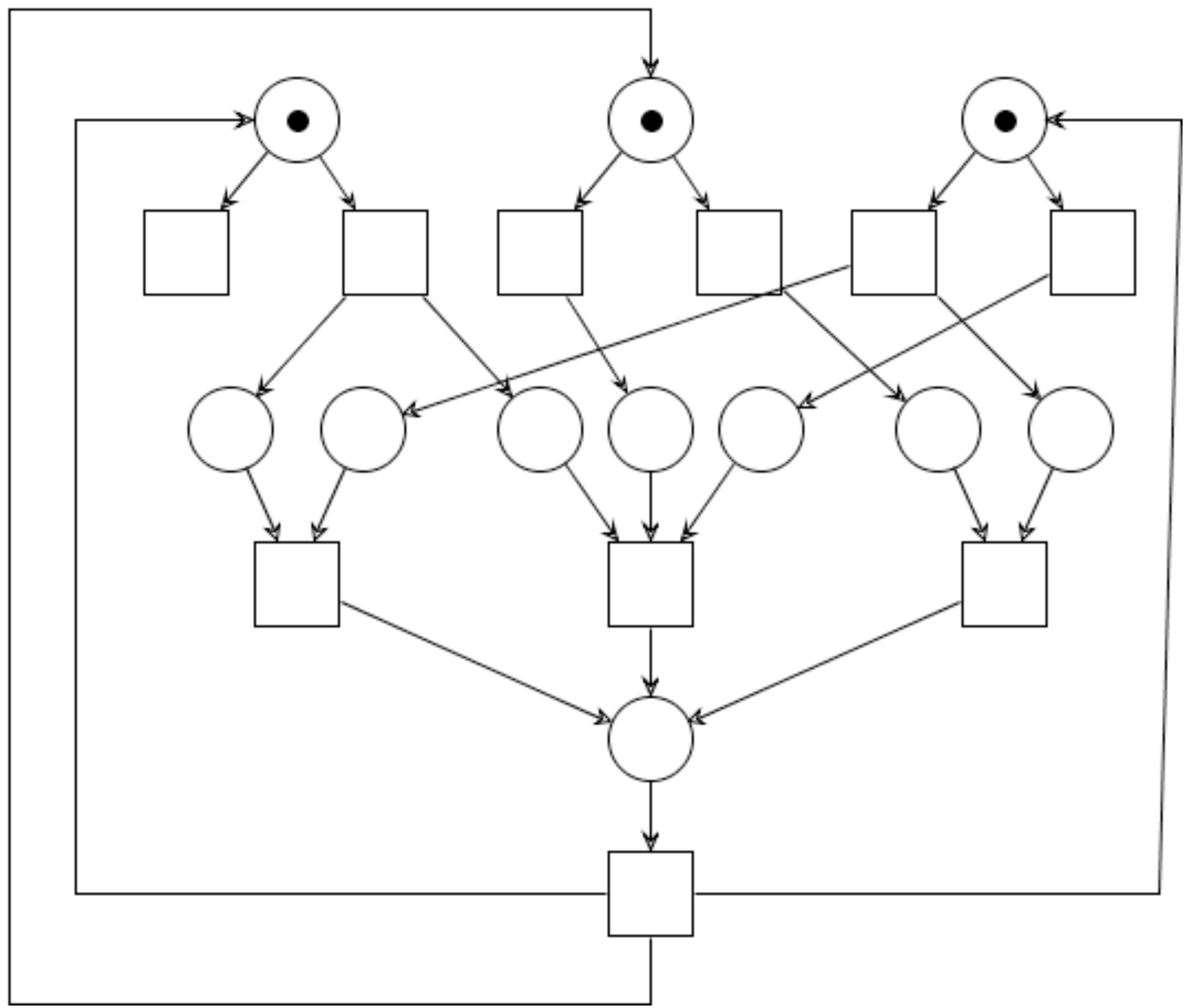
Is there an assignment of boolean values to the variables such that $\phi = true$?

Is there an assignment of boolean values to the variables such that $\neg\phi = false$?

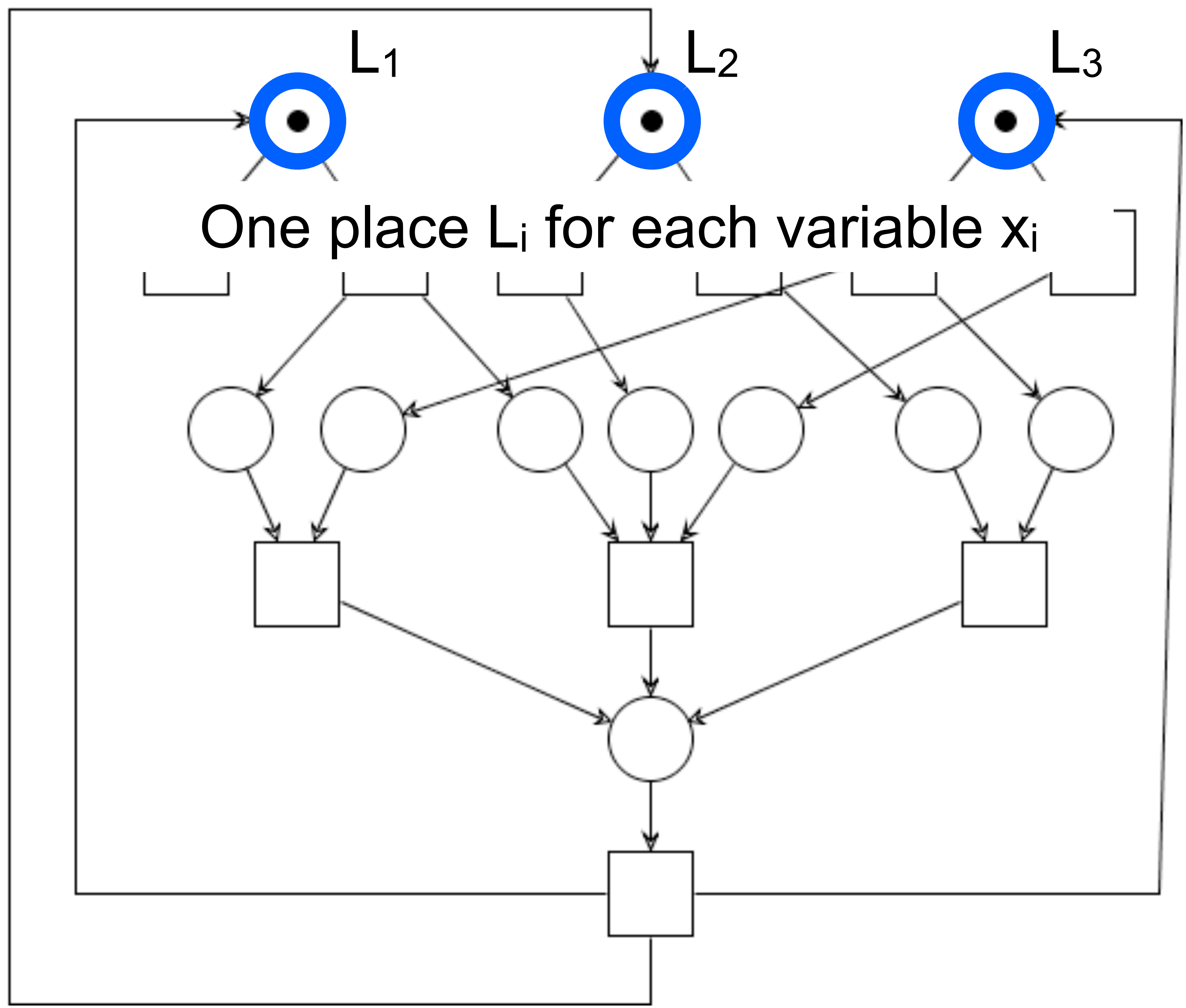
$$\phi = (x_1 \vee \overline{x}_3) \wedge (x_1 \vee \overline{x}_2 \vee x_3) \wedge (x_2 \vee \overline{x}_3)$$

$$\neg\phi = (\overline{x}_1 \wedge x_3) \vee (\overline{x}_1 \wedge x_2 \wedge \overline{x}_3) \vee (\overline{x}_2 \wedge x_3)$$

$$\neg\phi = (\overline{x}_1 \land x_3) \lor (\overline{x}_1 \land x_2 \land \overline{x}_3) \lor (\overline{x}_2 \land x_3)$$



$$\neg\phi = (\overline{x}_1 \wedge x_3) \vee (\overline{x}_1 \wedge x_2 \wedge \overline{x}_3) \vee (\overline{x}_2 \wedge x_3)$$



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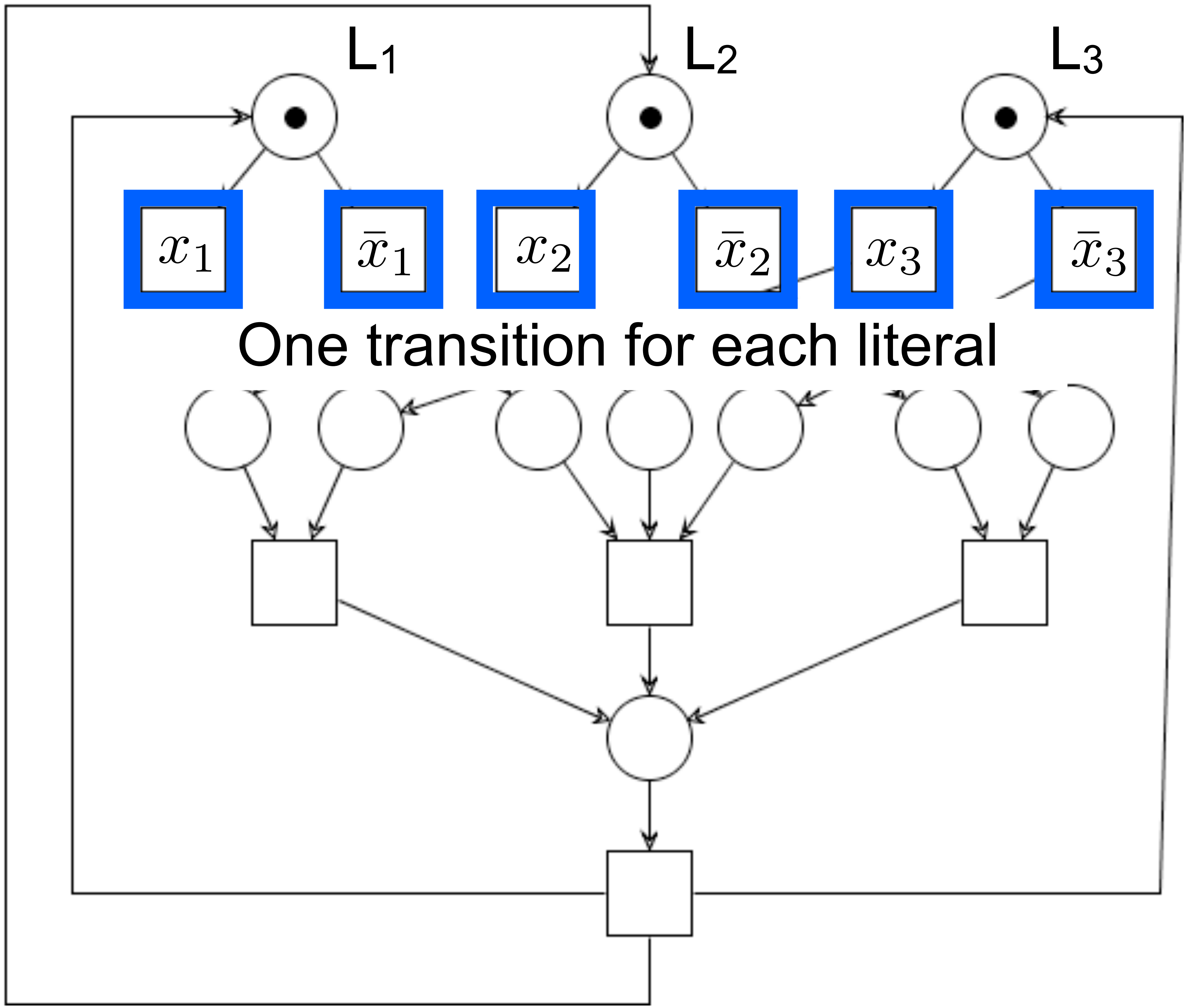

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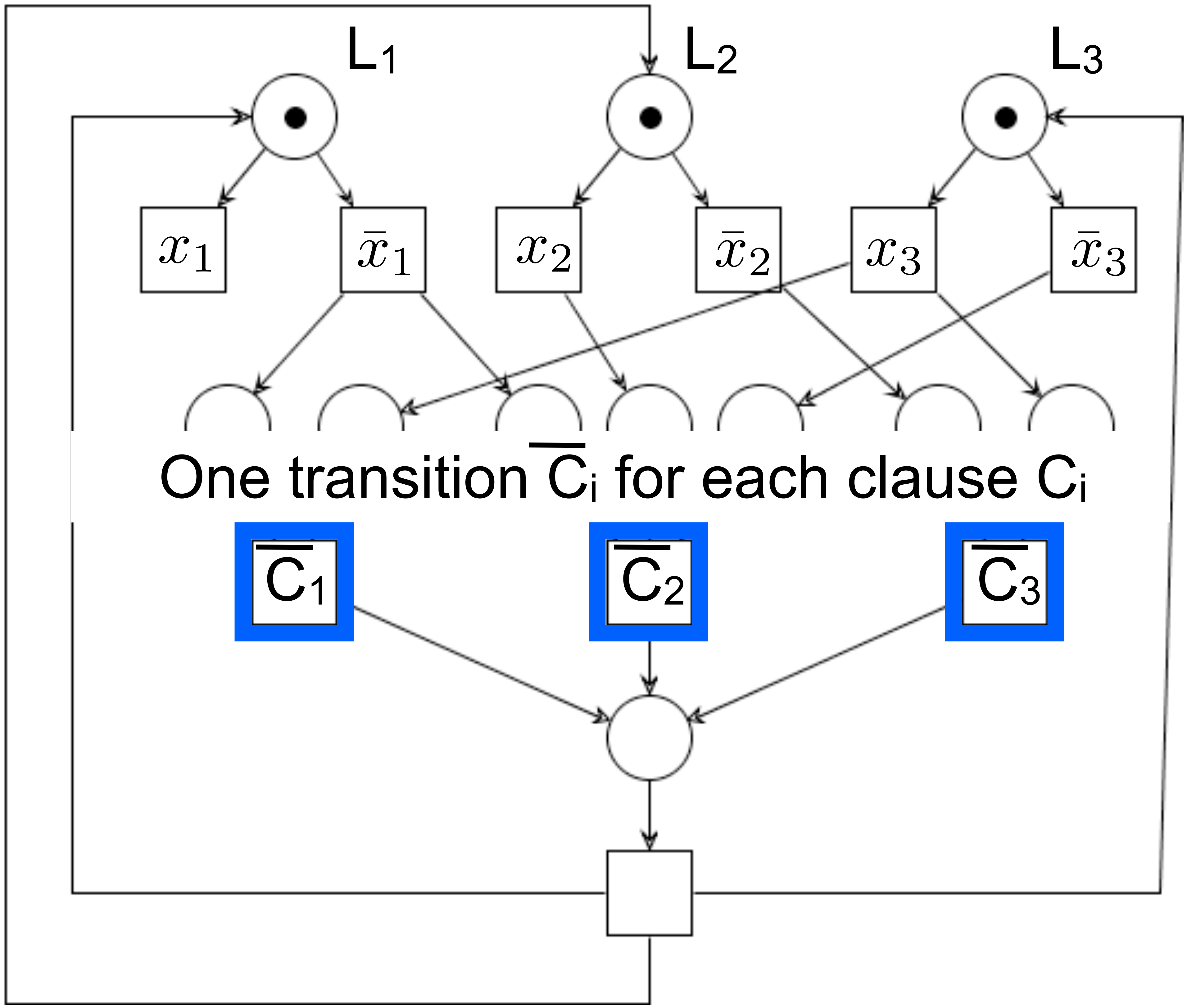
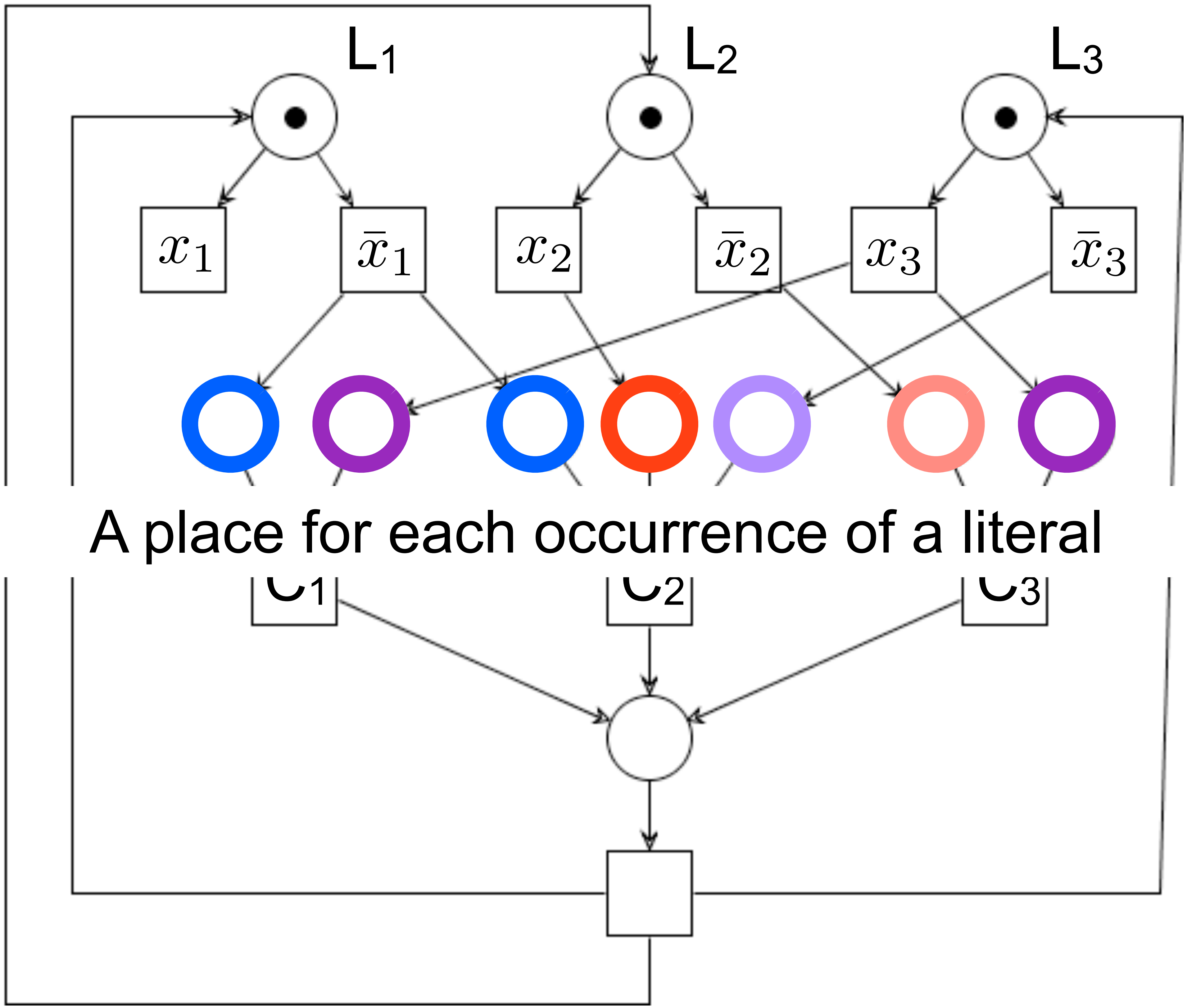
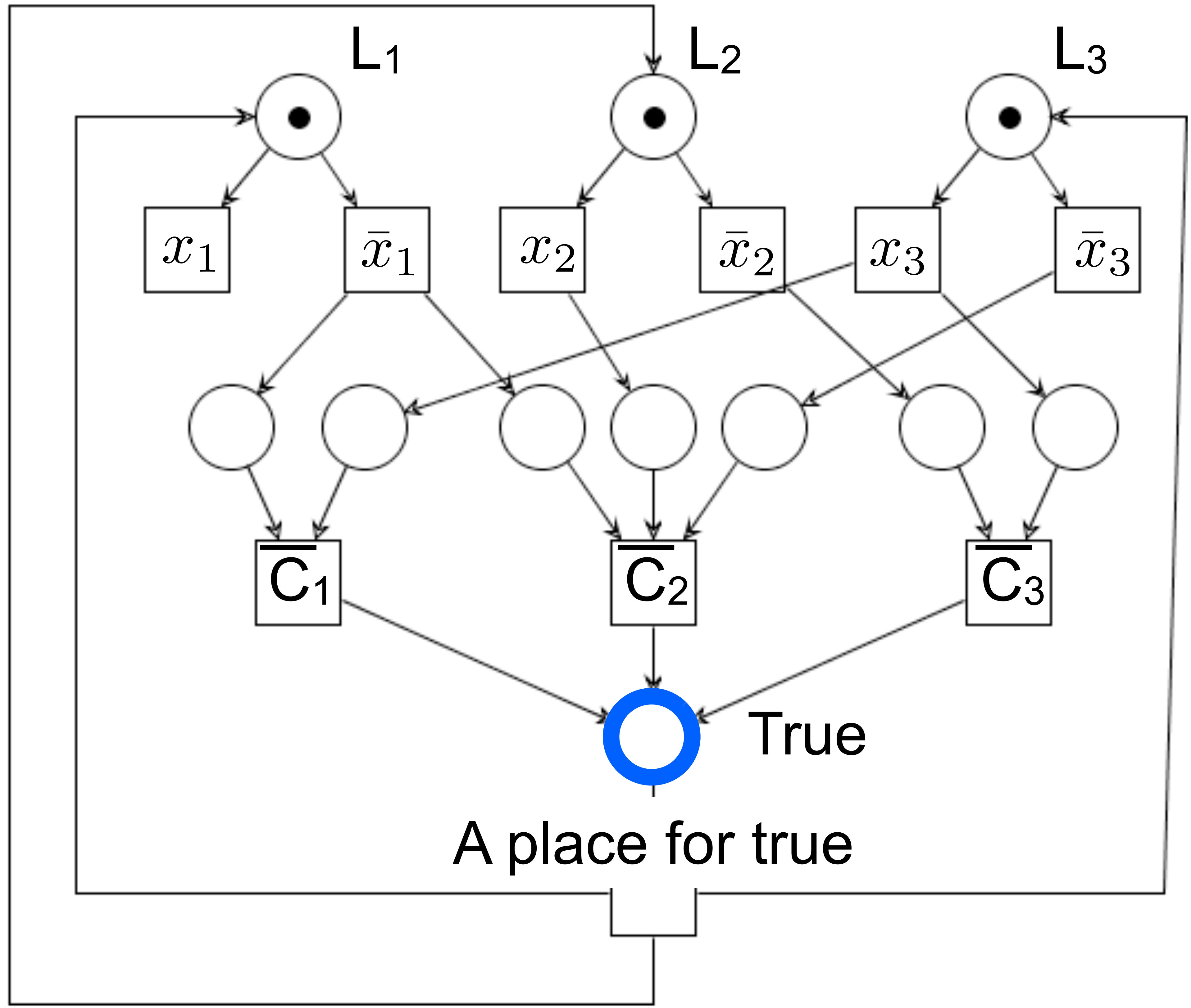


$$\neg\phi = (\overline{x}_1 \wedge x_3) \vee (\overline{x}_1 \wedge x_2 \wedge \overline{x}_3) \vee (\overline{x}_2 \wedge x_3)$$

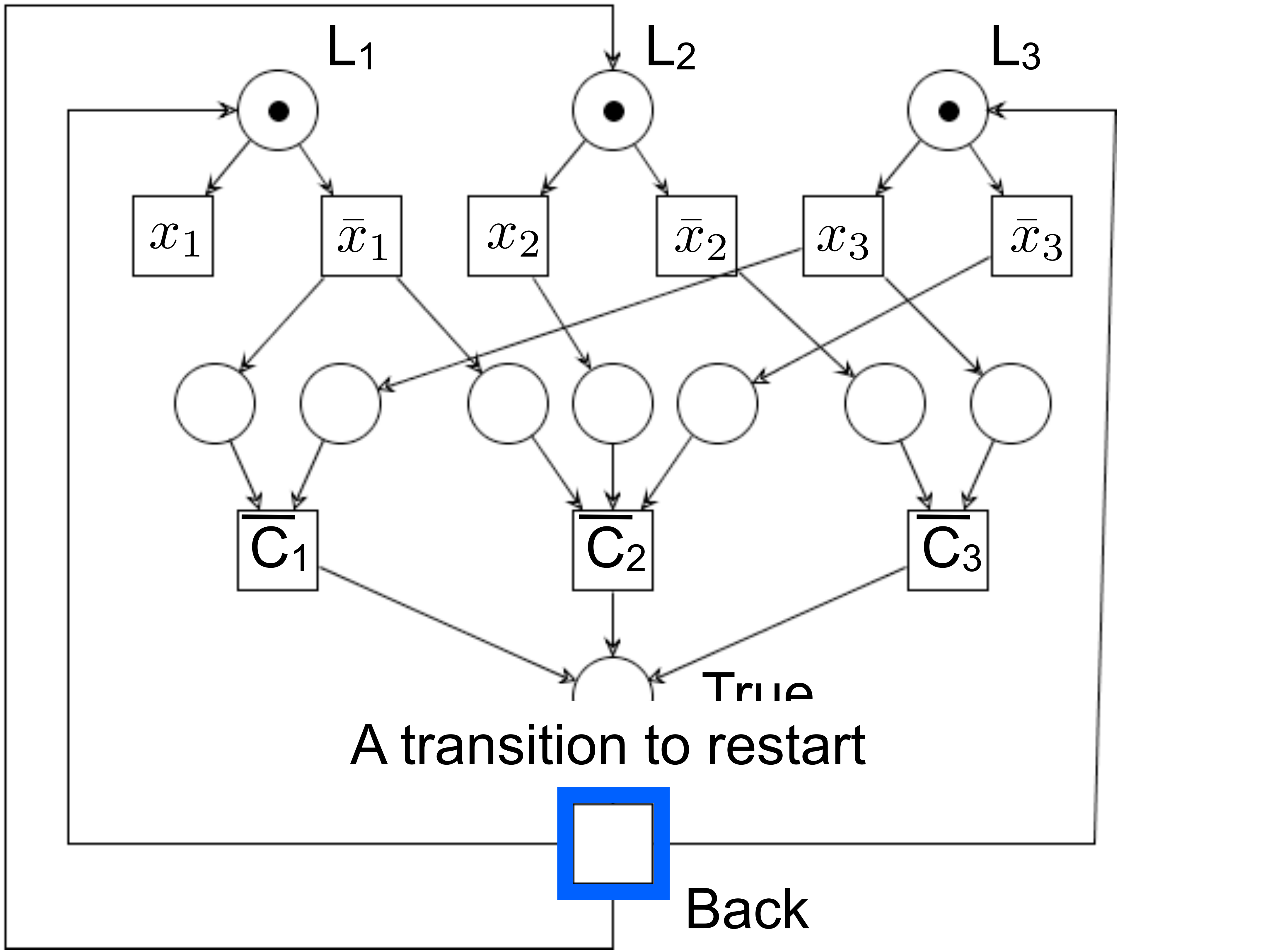


A place for each occurrence of a literal

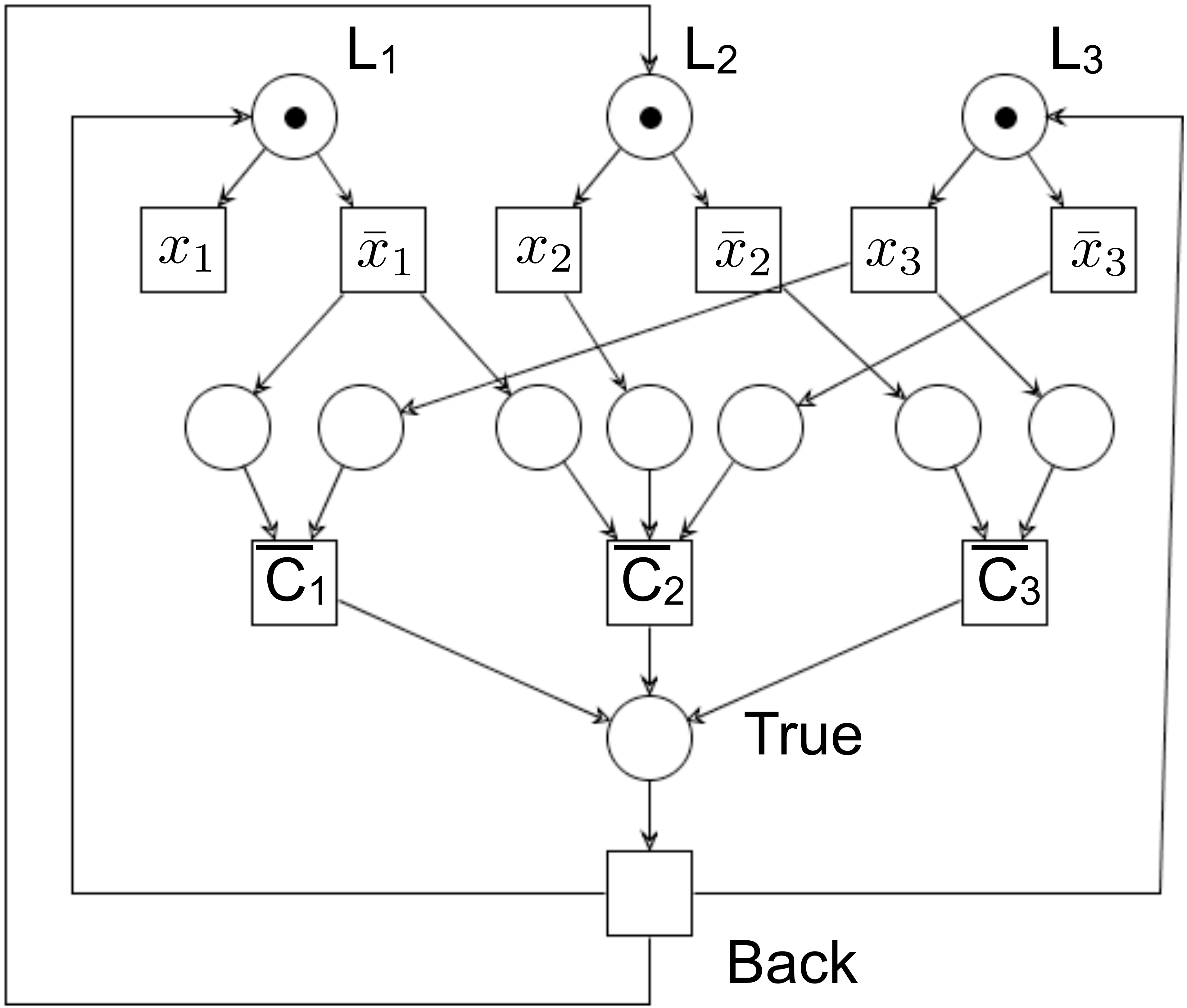
$$\neg\phi = (\overline{x}_1 \wedge x_3) \vee (\overline{x}_1 \wedge x_2 \wedge \overline{x}_3) \vee (\overline{x}_2 \wedge x_3)$$



$$\neg\phi = (\overline{x}_1 \wedge x_3) \vee (\overline{x}_1 \wedge x_2 \wedge \overline{x}_3) \vee (\overline{x}_2 \wedge x_3)$$



$$\neg\phi = (\overline{x}_1 \wedge x_3) \vee (\overline{x}_1 \wedge x_2 \wedge \overline{x}_3) \vee (\overline{x}_2 \wedge x_3)$$



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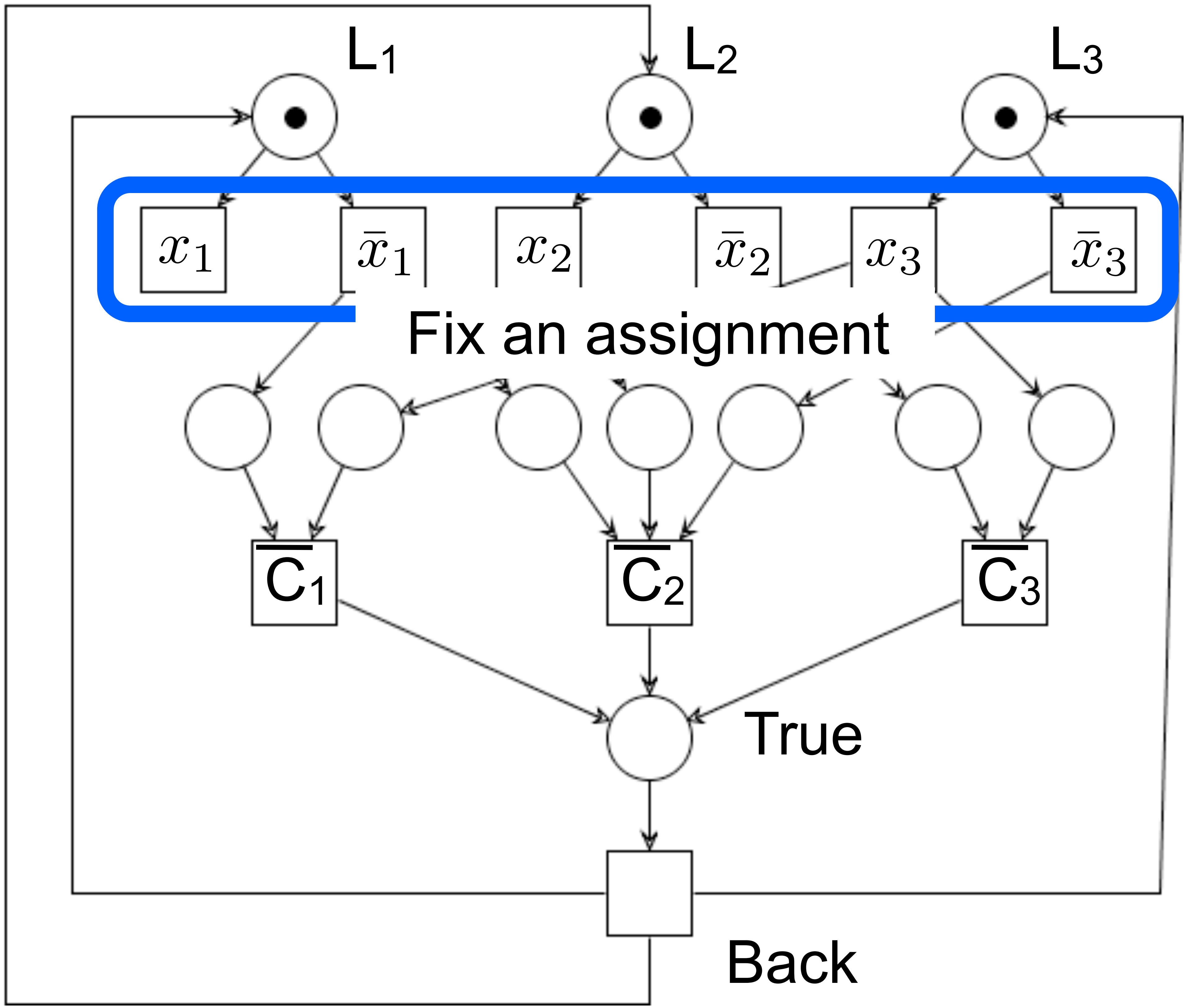

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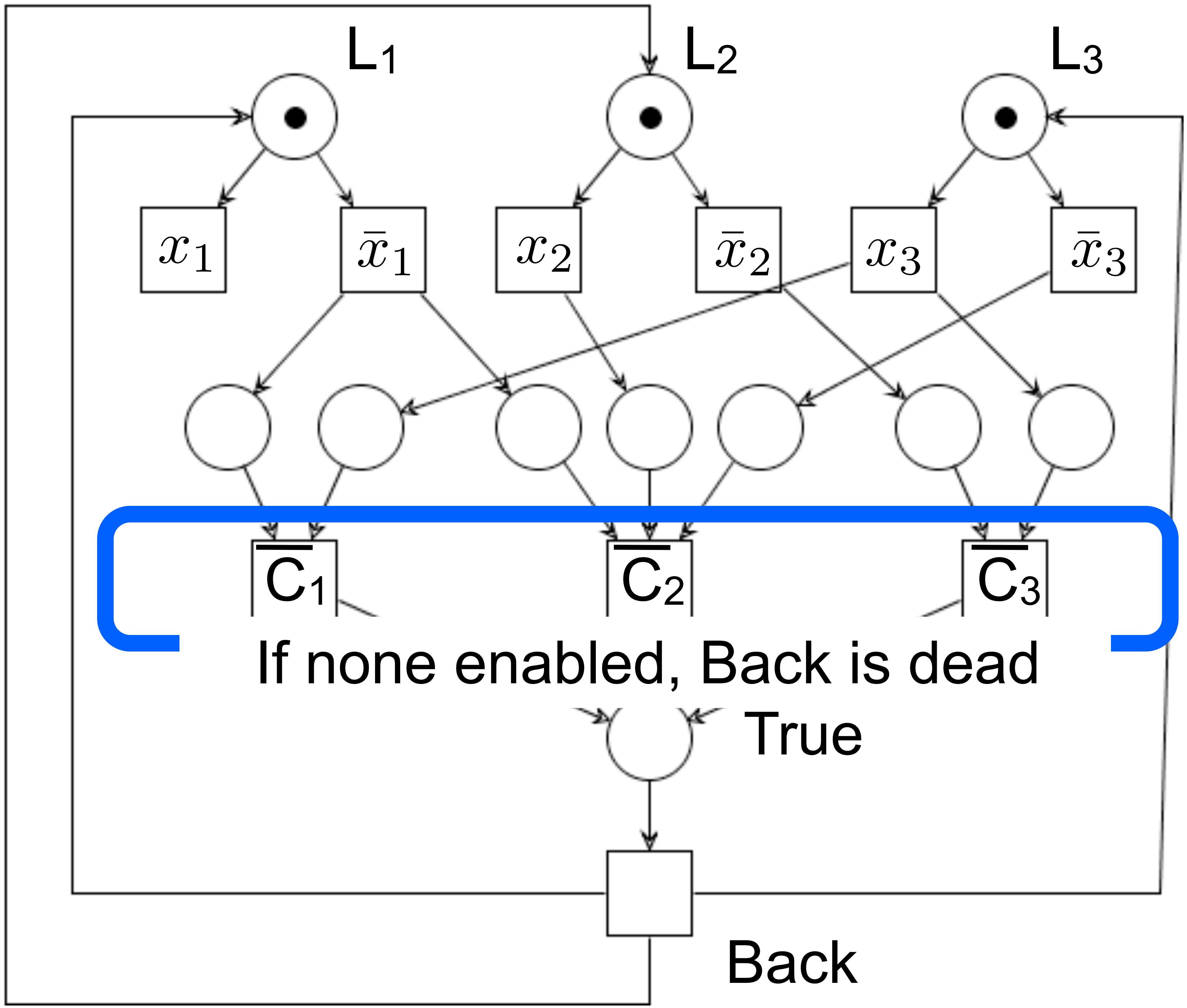
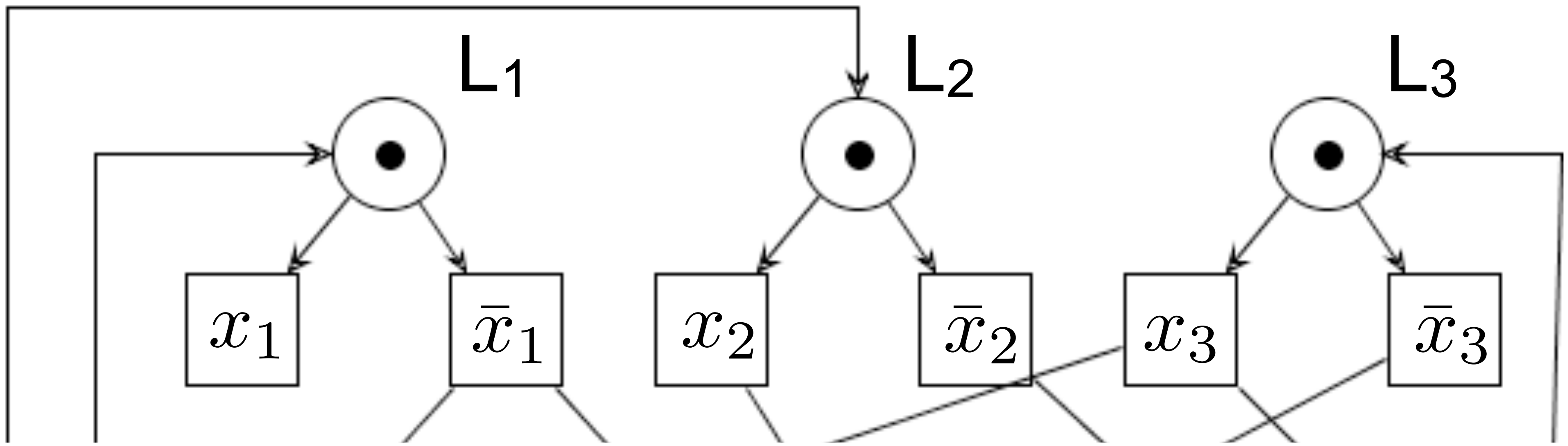
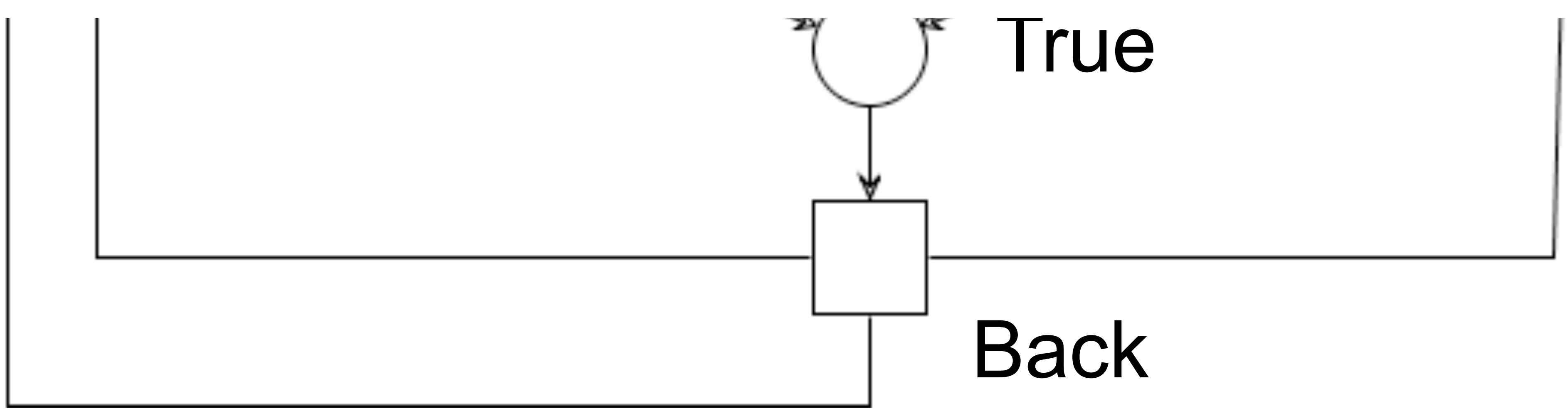


$$\neg\phi = (\overline{x}_1 \wedge x_3) \vee (\overline{x}_1 \wedge x_2 \wedge \overline{x}_3) \vee (\overline{x}_2 \wedge x_3)$$

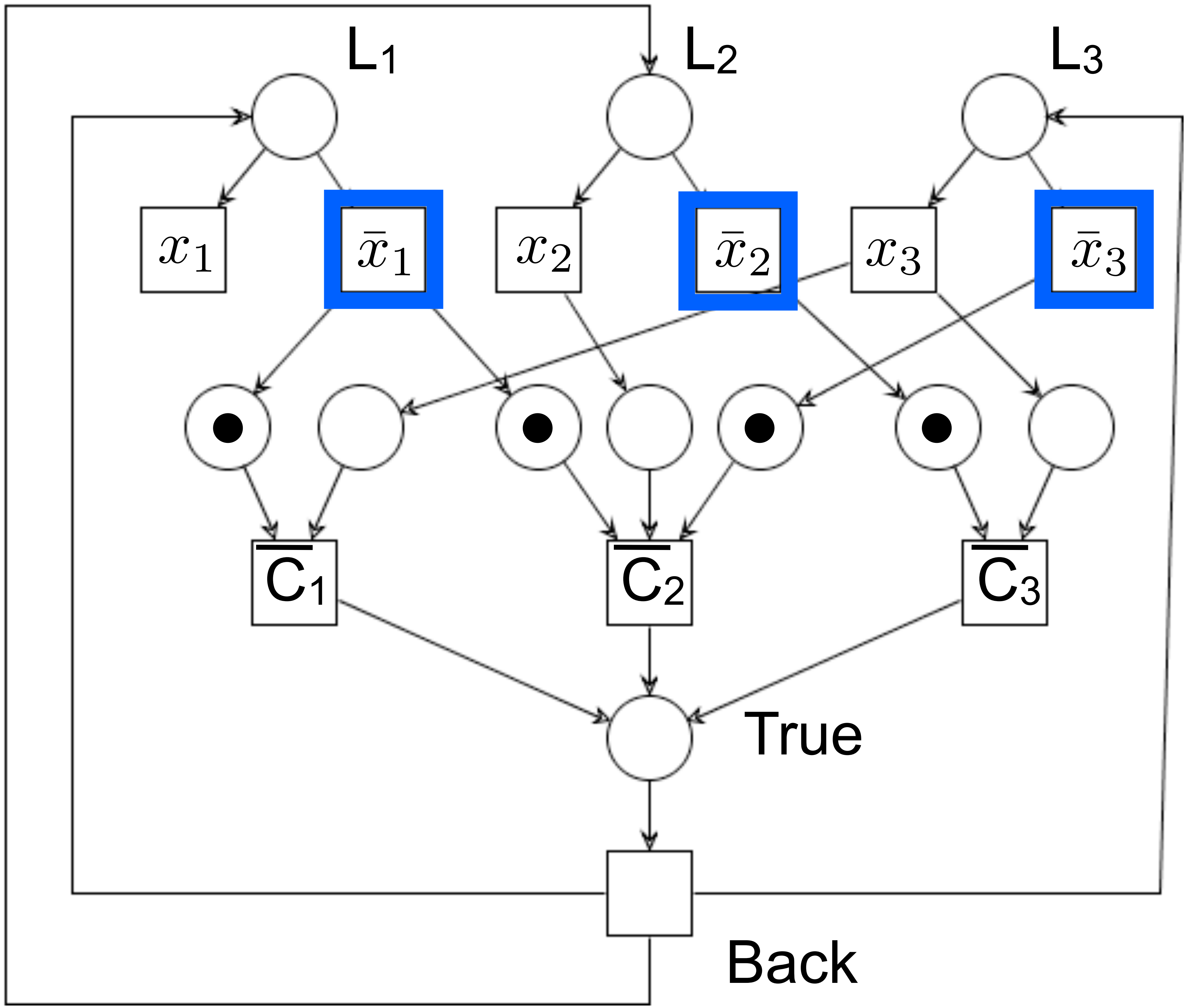


If ϕ is satisfiable, then the net is not live

If the net is not live, then ϕ is satisfiable



$$\neg\phi = (\overline{x}_1 \wedge x_3) \vee (\overline{x}_1 \wedge x_2 \wedge \overline{x}_3) \vee (\overline{x}_2 \wedge x_3)$$



Main consequence

No deterministic polynomial algorithm to decide liveness of a free-choice system is currently available

(unless P=NP)

Complexity issues 2: Is it hard to show that a free-choice net is live and bounded?

Rank Theorem: a polynomial decision algorithm

Theorem:

A free-choice system (P,T,F,M_0) is live and bounded

iff

1. it has at least one place and one transition polynomial
2. it is connected polynomial
3. M_0 marks every proper siphon
4. it has a positive S-invariant polynomial
5. it has a positive T-invariant polynomial
6. $rank(N) = |C_N| - 1$ polynomial

(where C_N is the set of clusters)

A polynomial algorithm for maximal unmarked siphon

3. M_0 marks every proper siphon **polynomial**

Input: A net $N = (P, T, F, M_0)$, $R = \{ p \mid M_0(p) = 0 \}$

Output: $Q \subseteq R$ maximal unmarked siphon

$(\bullet Q \subseteq Q \bullet)$

$Q := R$

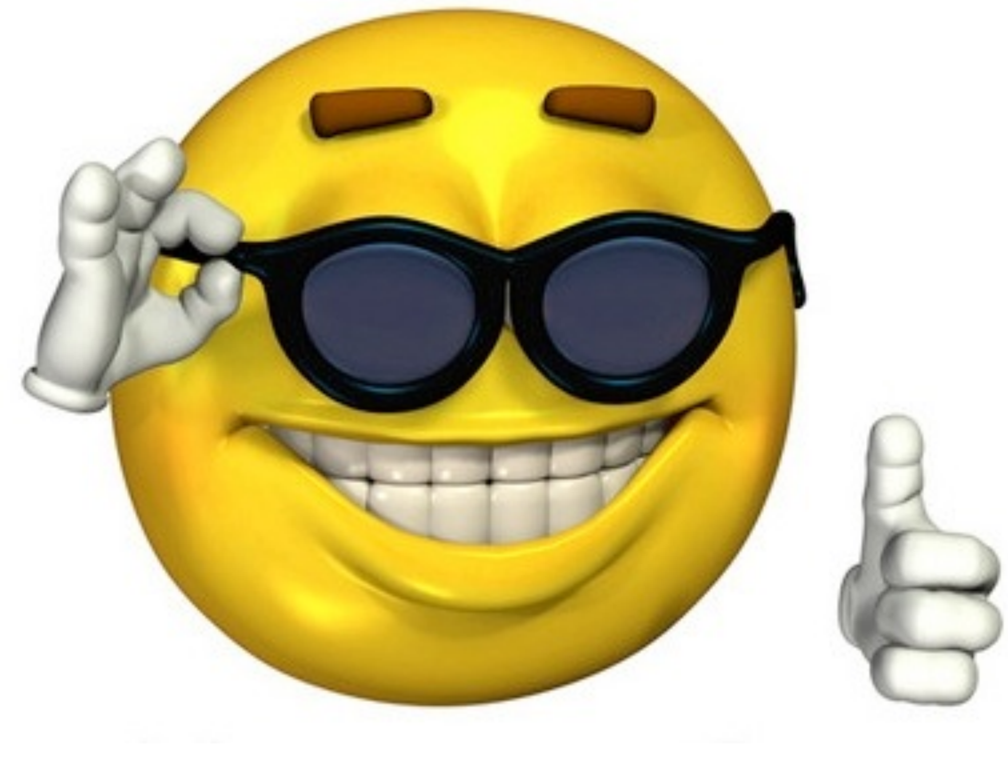
while $(\exists p \in Q,\ \exists t \in \bullet p,\ t \notin Q \bullet)$

$Q := Q \setminus \{p\}$

return Q If Q is empty then M_0 marks every proper siphon

Main consequence

The problem to decide
if a free-choice system is live and bounded
can be solved in polynomial time
(thanks to the Rank Theorem)



Recap: free-choice nets

f.c. net: place liveness <**=>** liveness

f.c. net: non-live => exists a proper siphon R and $M \in [M_0\rangle$
such that $M(R)=0$

f.c. net: every siphon contains a marked trap <=> live

f.c. net: bounded and live <=> 6 conditions in Rank Theorem

$\bullet R \subseteq R \bullet$

siphon

empty siphons remain empty

Exercise

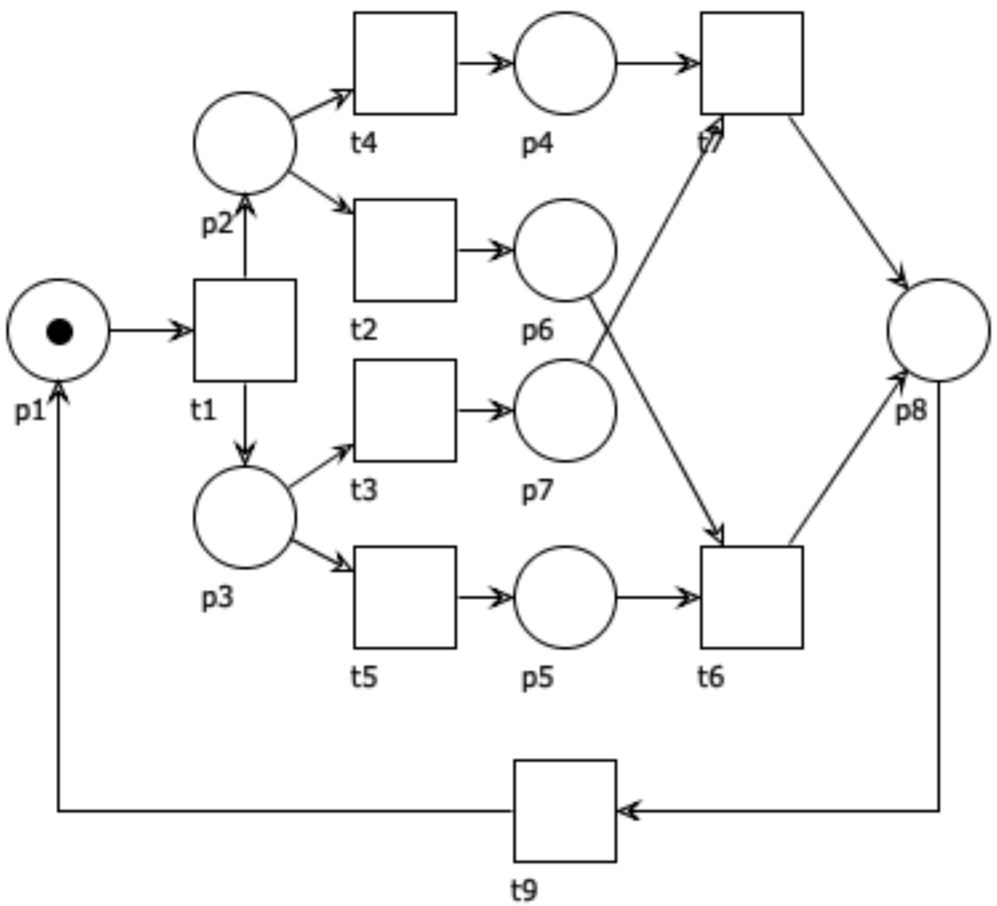
$\bullet Q \supseteq Q \bullet$

trap

marked traps remain marked

The system below is free-choice and non-live:
find a proper siphon that does not include a marked trap

Hint: take
$R=\{p_1,p_2,p_3,p_4,p_5,p_8\}$
and show that:
it is a siphon and
it contains no trap



$\bullet R \subseteq R \bullet$

siphon

empty siphons remain empty

Exercise

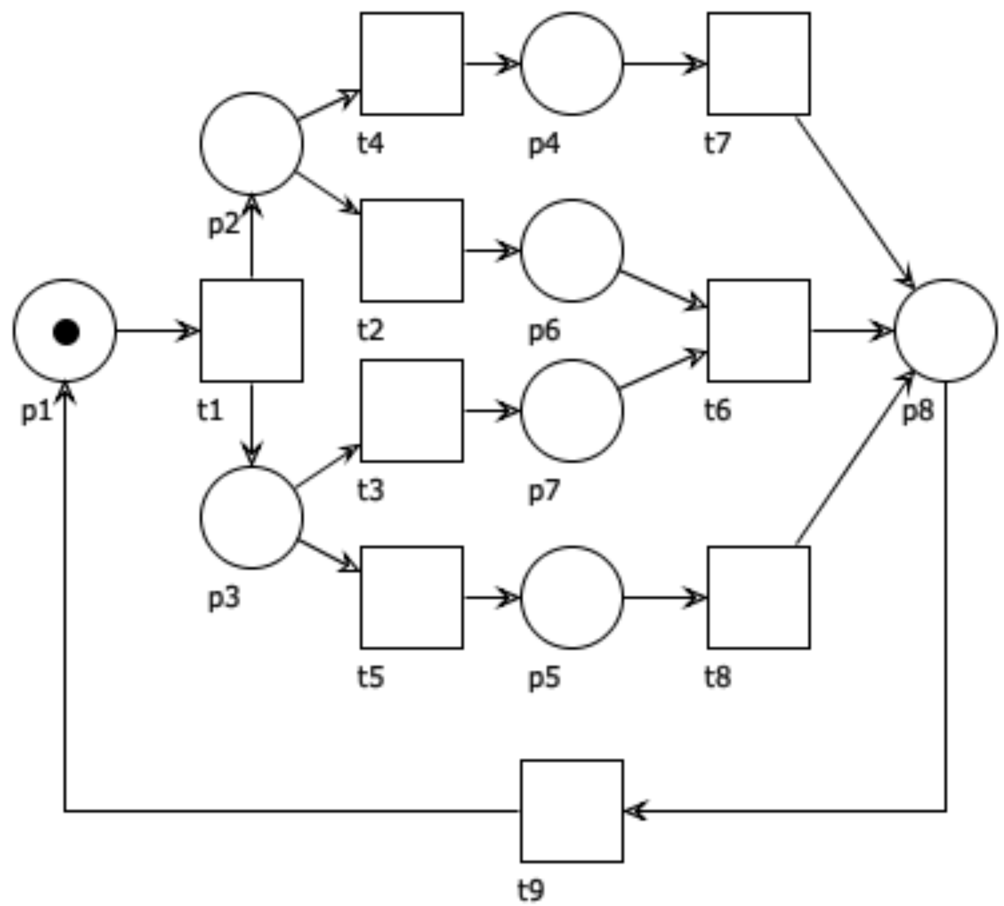
$\bullet Q \supseteq Q \bullet$

trap

marked traps remain marked

The system below is free-choice and live:
show that every proper siphon includes a marked trap

Hint: the only proper siphons are $R_1=\{p_1,p_2,p_3,p_4,p_5,p_7,p_8\}$ and $R_2=\{p_1,p_2,p_3,p_4,p_5,p_6,p_8\}$



Exercise

Draw the net corresponding to the formula

$$x_2 \wedge (x_1 \vee \overline{x}_3 \vee \overline{x}_4) \wedge (x_1 \vee \overline{x}_2) \wedge (\overline{x}_1 \vee x_4) \wedge (\overline{x}_2 \vee \overline{x}_4)$$

Is it satisfiable?