

# Multicommodity flows

118

By generalizing the previous examples, we can define the minimum cost flow problem as follows:

- given a directed graph  $G = (N, A)$
- $b_i$ : balance of node  $i$ ,  $\forall i \in N$
- $u_{ij}$ : upper capacity of link  $(i, j)$ ,  $\forall (i, j) \in A$
- $c_{ij}$ : unit transportation cost along  $(i, j)$ ,  $\forall (i, j) \in A$

introduce the following decision variables:  
(flow variables)

$x_{ij}$ : amount (flow) to be pushed along  $(i, j)$ ,  $\forall (i, j) \in A$

then we can state the following  
Linear Programming (LP) model:

# Mincost flow model:

119

$$\text{Min } \sum_{(i,j) \in EA} c_{ij} \cdot x_{ij}$$

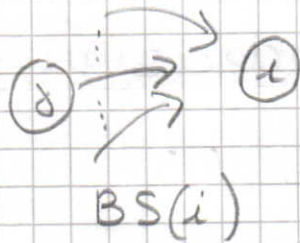
Subject to:

$$\sum_{(j,i) \in BS(i)} x_{ji} - \sum_{(i,j) \in FS(i)} x_{ij} = b_i, \forall i \in N \quad (1)$$

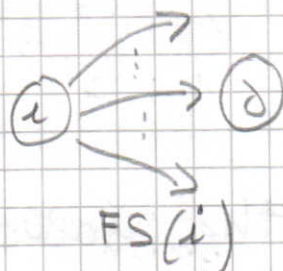
$$0 \leq x_{ij} \leq u_{ij}, \forall (i,j) \in EA \quad (2)$$

(1) are the flow conservation constraints:

- $BS(i)$  backward star of  $i$



- $FS(i)$  forward star of  $i$



- $b_i$   $\begin{cases} < 0 & \text{if } i \text{ is a } \underline{\text{supply mode}} \\ > 0 & \text{if } i \text{ is a } \underline{\text{demand mode}} \\ 0 & \text{if } i \text{ is a } \underline{\text{transshipment mode}} \end{cases}$

(z) are the capacity constraints

120

Feasibility necessary conditions :  $\sum_{i \in N} b_i = 0$

Integrality property : if  $b_i$  are integer  $\forall i \in N$ , and  $u_{ij}$  are integer  $\forall (i,j) \in A$ , then there exists an integer minimum cost flow (solving the LP mincost flow model we get an optimum integer solution)

Special cases (already shown)

① The shortest path problem from  $s \in N$  to  $t \in N$  (e.g. AEA)

specialize the mincost flow model by

setting :

$$b_i = \begin{cases} -1 & \text{if } i = s \\ +1 & \text{if } i = t \\ 0 & \text{otherwise} \end{cases}$$

•  $u_{ij} = +\infty \quad \forall (i,j) \in A$  uncapacitated problem