Multicommodity flows

By generalizing the previous examples, we can define the minimum cost flow problem as follows:

- given a directed graph $G = (N, A)$
- b_i balance of node i, $\forall i \in N$
- u_{ij} upper capacity of link (i,j), $\forall (i,j) \in A$
- c_{ij} unit transportation cost along (i,j), $\forall (i,j) \in A$

introduce the following decision variables: (flow variables)

x_{ij} : amount (flow) to be pushed along (i,j), $\forall (i,j) \in A$

then we can state the following Linear Programming (LP) model:

Mincost flow model:

$$\text{Min} \sum_{(i,j)\in A} c_{ij} \cdot x_{ij}$$

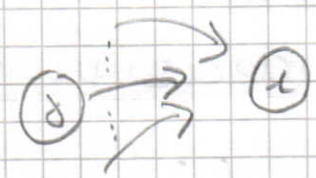
Subject to:

$$\sum_{(j,i)\in BS(i)} x_{ji} - \sum_{(i,j)\in FS(i)} x_{ij} = b_i, \ \forall i \in N \qquad (1)$$

$$0 \leq x_{ij} \leq u_{ij} \qquad , \forall (i,j)\in A \qquad (2)$$

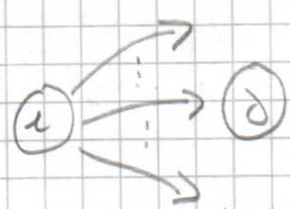
(1) are the flow conservation constraints:

- BS(i) backward star of i



BS(i)

- FS(i) forward star of i



FS(i)

- $b_i \begin{cases} < 0 & \text{if } i \text{ is a supply mode} \\ > 0 & \text{if } i \text{ is a demand mode} \\ 0 & \text{if } i \text{ is a Transshipment mode} \end{cases}$

(2) are the capacity constraints

Feasibility necessary conditions: $\sum_{i \in N} b_i = 0$

Integrality property: if b_i are integer $\forall i \in N$, and u_{ij} are integer $\forall (i,j) \in A$, then there exists an integer minimum cost flow (solving the LP mincost flow model we get an optimum integer solution)

Special cases (already shown)

1. The shortest path problem from $s \in N$ to $t \in N$ (e.g. ACA)

specialize the mincost flow model by setting:

- $b_i = \begin{cases} -1 & \text{if } i = s \\ +1 & \text{if } i = t \\ 0 & \text{otherwise} \end{cases}$

- $u_{ij} = +\infty \quad \forall (i,j) \in A$ uncapacitated problem

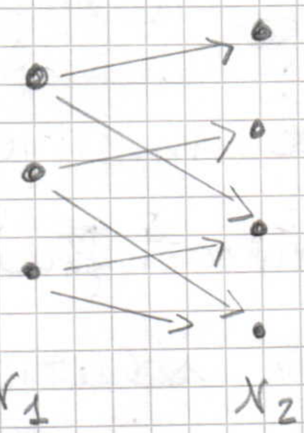
(2) The transportation problem

(e.g. Tropicsun)

special minimum cost flow case where:

- G is a bipartite graph, i.e. $G = (\underbrace{N_1, N_2}_{N}, A)$

e.g.



- $b_i < 0 \quad \forall i \in N_1$ (all nodes in N_1 are supply nodes)

- $b_i > 0 \quad \forall i \in N_2$ (all nodes in N_2 are demand nodes)

<u>Notice</u>: real minimum cost flow problems are solved by means of "ad hoc", very efficient, network flow algorithms (see <u>Ricerca Operativa</u> and <u>MOR</u> (Network Optimization Methods) <u>courses</u> for more info)

- The studied problems are <u>single commodity flow problems</u>, i.e. they address the transportation along a network, from some origins to some destinations, of a unique type of product or <u>commodity</u> (e.g. cars in (BMC) and (ACA), fruit in Tropicsun)

- However, in several applications many products must be sent along the same network (<u>multicommodity scenario</u>):

(1) if the products (commodities) are independent, then we can decompose the problem into several independent single commodity mincost problems, one per commodity

(2) if the products share the network resources (e.g. link capacities), then we need to model and solve a

multicommodity flow problem, which generalizes the minimum cost flow problem (which is single commodity)

e.g. in airline scheduling the commodities are the different fligths, each one having its own departure node (origin) and destination node, in the logistics network associated with the airline company

Minimum cost multicommodity flow problem

Let:

- $G = (N, A)$ directed graph (logistics network)
- K : number of commodities
- b_i^k : balance of i for commodity k, $\forall i \in N$, $k = 1, \dots K$
- u_{ij}^k : capacity of (i,j) for commodity k, $\forall (i,j) \in A$, $k = 1, \dots K$

- u_{ij} : global capacity of (i,j), $\forall (i,j) \in A$
- c_{ij}^{k} : unit transportation cost along (i,j) for commodity k, $\forall (i,j) \in A$, $k = 1, .. \mathbf{K}$

Define the following <u>multicommodity flow variables</u>:

x_{ij}^{k} : amount of commodity k to be pushed along (i,j), $\forall (i,j) \in A$, $k = 1, .. \mathbf{K}$

<u>Min cost multicommodity flow model</u>:

$$\text{Min} \sum_{(i,j)\in A} \sum_{k=1}^{\mathbf{K}} c_{ij}^{k} \cdot x_{ij}^{k}$$

<u>Subject to</u>

$$\sum_{(j,i)\in BS(i)} x_{ji}^{k} - \sum_{(i,j)\in FS(i)} x_{ij}^{k} = b_i^k \quad \forall i \in N, \; k = 1, \dots \mathbf{K}$$

$$0 \le x_{ij}^{k} \le u_{ij}^{k} \quad \forall (i,j) \in A, \; k = 1, \dots \mathbf{K}$$

$$\boxed{\sum_{k=1}^{\mathbf{K}} x_{ij}^{k} \le u_{ij}} \quad \forall (i,j) \in A$$

These are the constraints linking the K commodities (without these linking constraints, the model could be solved by solving K independent minimum cost flow problems

Obs: now the integrality property does not hold

Assume now that, in addition to organize the transportation of the commodities along the network, you have also to design the network: you can send some products along (i,j) only if you first decide to build (activate, install...) (i,j), with a related fixed cost f_{ij}:

additional input data

f_{ij} : fixed cost for activating (i,j), $\forall (i,j) \in A$

Additional decision variables (design variables):

$$y_{ij} = \begin{cases} 1 & \text{if } (i,j) \text{ is activated} \\ 0 & \text{otherwise} \end{cases} \qquad \forall (i,j) \in A$$

Fixed-charge network design problem

I.L.P. model

$$\text{Min} \underbrace{\sum_{(i,j)\in A} f_{ij} \cdot y_{ij}}_{\text{fixed cost}} + \underbrace{\sum_{(i,j)\in A} \sum_{k=1}^{K} c_{ij}^k \cdot x_{ij}^k}_{\text{transportation cost}}$$

Subject to:

$$\sum_{(j,i)\in BS(i)} x_{ji}^k - \sum_{(i,j)\in FS(i)} x_{ij}^k = b_i^k \qquad \forall i \in N,\ k = 1, \dots K$$

$$0 \leq x_{ij}^k \leq u_{ij}^k \qquad \forall (i,j) \in A,\ k = 1, \dots K$$

$$\boxed{\sum_{k=1}^{K} x_{ij}^k \leq u_{ij} \cdot y_{ij}} \qquad \forall (i,j) \in A$$

$$y_{ij} \in \{0, 1\} \qquad \forall (i,j) \in A$$

these are both capacity constraints and constraints linking the transportation decisions ($\{x_{ij}^k\}$) with the design decisions ($\{y_{ij}\}$): linking constraints

Time Complexity : NP-Hard

"Ad hoc" exact and heuristic algorithms:

< MOR course >

(no algorithmic approaches in this course)