I. L. P. model : (p-median)

$$\text{Min} \sum_{i \in I} \sum_{j \in J} h_i \cdot d_{ij} \cdot y_{ij}$$

Subject to :

$$\sum_{j \in J} x_j = p$$

$$\sum_{j \in J} y_{ij} = 1 \quad \forall i \in I$$

$$y_{ij} \leq x_j \quad \forall i \in I, j \in J$$

$$x_j \in \{0, 1\} \quad \forall j \in J$$

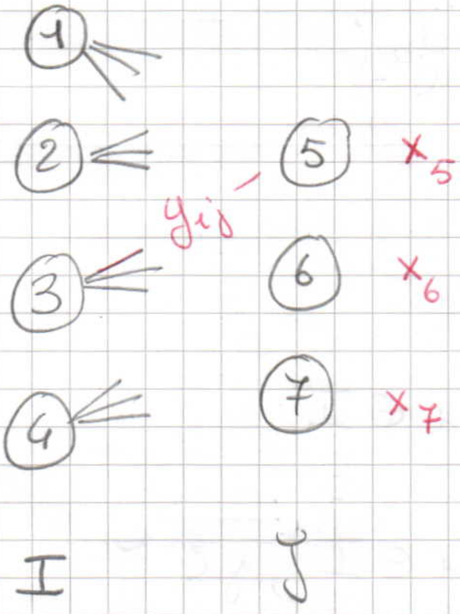
$$y_{ij} \in \{0, 1\} \quad \forall i \in I, j \in J$$

Obs 1 : (location and allocation) variables as well as constraints are common to formulation (p-center) : only the objective function is different

Obs 2 : the formulation also minimizes the average travel distance (just divide the objective function by $|I|$)

Time complexity : NP-Hard for variable values of p

Example (p-median)



< complete graph >

	5	6	7
1	20	25	60
2	30	15	10
3	30	15	10
4	30	15	10

d_{ij}

let $p = 2$ and $h_i = 1$, $i = 1, 2, 3, 4$

$$\begin{aligned} \text{Min } & 20\, y_{15} + 25\, y_{16} + 60\, y_{17} + 30\, y_{25} + \\ & + 15\, y_{26} + 10\, y_{27} + 30\, y_{35} + 15\, y_{36} + \\ & + 10\, y_{37} + 30\, y_{45} + 15\, y_{46} + 10\, y_{47} \end{aligned}$$

Sub. to:

$$x_5 + x_6 + x_7 = 2$$

$$\left.\begin{aligned} y_{15} + y_{16} + y_{17} &= 1 \\ y_{25} + y_{26} + y_{27} &= 1 \\ &\vdots \end{aligned}\right\} \text{allocation constraints}$$

< the same for nodes 3 and 4 >

$$\left.\begin{array}{l} y_{15} \le x_5 \\ y_{16} \le x_6 \\ y_{17} \le x_7 \end{array}\right\}$$ linking between location and allocation

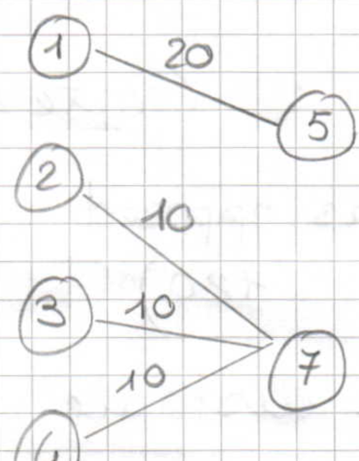
$\vdots$

< the same for the other links >

$$x_5, x_6, x_7 \in \{0, 1\}$$

$$y_{15}, y_{16}, y_{17} \ldots \in \{0, 1\}$$

What is an optimal solution of the toy example? by inspection:



$$x_5^* = x_7^* = 1 \qquad x_6^* = 0$$

$$y_{15}^* = y_{27}^* = y_{37}^* = y_{47}^* = 1$$

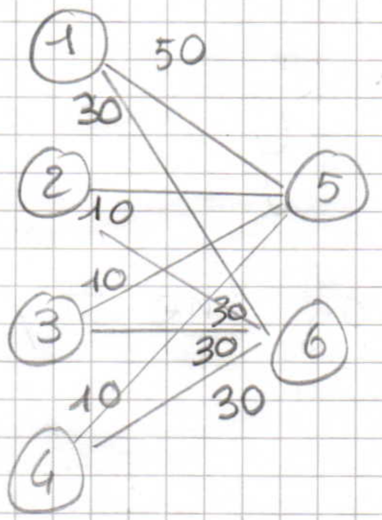
$$(y_{ij}^* = 0 \text{ other.})$$

with optimal value $\equiv$ minimum total distance $= 50$

Observe that this solution is also optimal for the p-center problem defined on the same instance: the minimum maximum distance is 20!

Howex, (p-center) and (p-median) are not equivalent!

Counterexample (an optimal solution to p-median is not necessarily optimal to p-center)



$p = 1$

$h_i = 1 \quad , i = 1, 2, 3, 4$

Optimal solution to p-median: locate at 5 : total distance : 80 (as opposed to 120)

- the corresponding maximum distance is 50

Optimal solution to p-center: locate at 6

maximum distance : 30 (as opposed to 50)

- the corresponding total distance is 120

(6) Fixed charge location problem

Location problems such as p-median makes three assumptions:

1) each candidate site has the same fixed cost for locating a facility
2) the facilities are uncapacitated, i.e. there is no upper limit on the demand they can serve
3) the number of facilities to be opened, i.e. p, is known a priori

- These assumptions are not appropriate in some cases; relax them:

The problem: determine the optimal number and locations of facilities, which are now capacitated, and determine the assignment of each demand node to an open facilities, in order to minimize the total cost (= fixed costs for opening facilities + transportation costs)

Obs: since facilities have capacities, demand may not be assigned to its closest open facility (as in previous cases)

additional input data:

- f_j fixed cost of locating a facility at j, $\forall j \in J$
- C_j capacity of a facility at j, $\forall j \in J$
- α cost per unit demand per unit distance

I. L. P. model

(FCLP) $$\text{Min} \sum_{j \in J} f_j \cdot x_j + \alpha \sum_{i \in I} \sum_{j \in J} h_i \cdot d_{ij} \cdot y_{ij}$$

Subject to : $$\sum_{j \in J} y_{ij} = 1 \quad \forall i \in I$$ assignment constraints

$$y_{ij} \leq x_j \quad \forall i \in I, j \in J$$ linking between location and allocation

"new" constraints $$\sum_{i \in I} h_i \cdot y_{ij} \leq C_j \cdot x_j \quad \forall j \in J$$

$$x_j \in \{0, 1\} \quad \forall j \in J$$

$$y_{ij} \in \{0, 1\} \quad \forall i \in I, j \in J$$

<u>meaning of:</u>

$$\sum_{i \in I} h_i y_{ij} \leq C_j \cdot x_j \quad \forall j \in J$$

- if $x_j = 1$ (facility at j) $\Rightarrow \underbrace{\sum_{i \in I} h_i y_{ij}}_{\text{total demand served by } j} \leq C_j$

  <u>capacity constraint</u>

- if $x_j = 0$ (no facility at j) $\Rightarrow \sum_{i \in I} h_i y_{ij} \leq 0$

  <u>that</u> is $y_{ij} = 0 \quad \forall i \in I$

  no client assigned to j

  <u>linking between location and allocation</u>

<u>Therefore</u> $y_{ij} \leq x_j \quad \forall i \in I, j \in J$ are <u>redundant</u>, and so can be eliminated (however they enhance the LP relaxation)

Variants of the model:

1) $\sum_{j \in J} y_{ij} = 1 \quad \forall i \in I$

$y_{ij} \in \{0,1\} \quad \forall i \in I, j \in J$

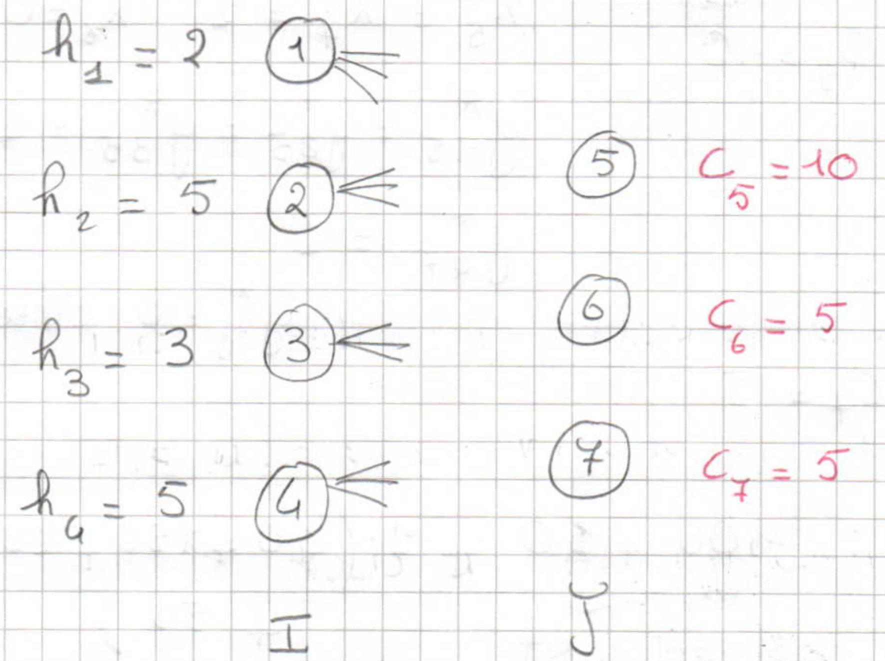
single - source allocation ≡ each client is served by a unique facility

By replacing (relaxing) $y_{ij} \in \{0,1\}$ with $0 \leq y_{ij} \leq 1 \quad \forall i \in I, j \in J$ we allow node i to be served (partially) by multiple facilities

2) if we remove the capacity constraints then we get the Uncapacitated fixed charge location problem (UFCLP):
in this case each client will be served by its closest (open) facility : allocation is "easy"!

example (FCLP)

Consider now the same instance of (p-median), but different h_i:



	5	6	7
1	20	25	60
2	30	15	10
3	30	15	10
4	30	15	10

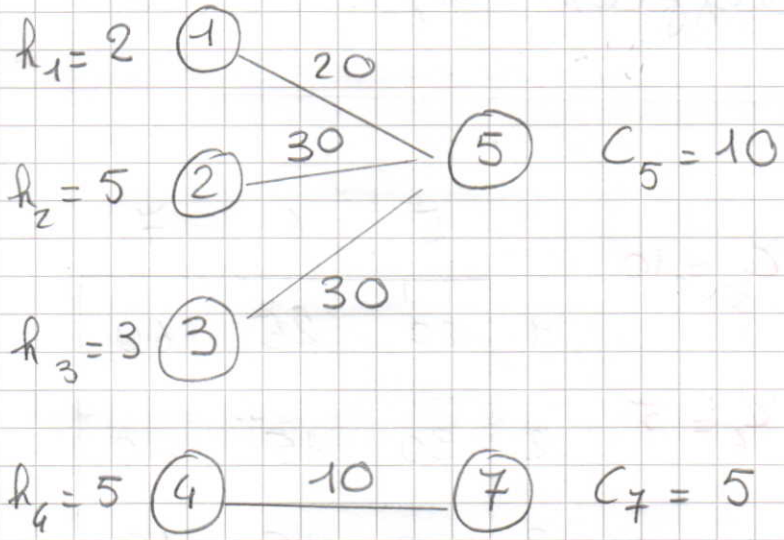
d_{ij}

Differently than (p-median), we have:

- facility capacities (in the figure)
- fixed costs: $f_5 = 50$, $f_6 = f_7 = 100$
- unit cost per demand per distance: $\alpha = 1$

Obs: since the total demand is 15, we need to open at least two facilities (due to their capacities)

An optimal solution for the toy example (by inspection):



$x_5^* = x_7^* = 1 \quad x_6^* = 0$

$y_{15}^* = y_{25}^* = y_{35}^* = 1$

$y_{47}^* = 1$

($y_{ij}^* = 0$ otherw.)

Minimum cost: $\underbrace{100 + 50}_{\text{fixed cost}} + 1 \cdot \underbrace{(40 + 150 + 90 + 50)}_{\text{transportation cost}}$

$= 480$

- Observe that nodes 2 and 3 are not served by their closest facility (as it would be happen in (p-median)), due to capacity restrictions

(FCLP)

$$\text{Min } \overbrace{50 \cdot x_5 + 100 \cdot x_6 + 100 \cdot x_7}^{\text{fixed costs}} +$$

$$+ \underbrace{2 \cdot 20 \cdot y_{15} + 2 \cdot 25 \cdot y_{16} + \dots}_{\text{transportation costs}}$$

Subject to:

$$\left.\begin{aligned} y_{15} + y_{16} + y_{17} &= 1 \\ y_{25} + y_{26} + y_{27} &= 1 \\ y_{35} + y_{36} + y_{37} &= 1 \\ y_{45} + y_{46} + y_{47} &= 1 \end{aligned}\right\} \text{allocation constraints}$$

$$2 \cdot y_{15} + 5 \cdot y_{25} + 3 \cdot y_{35} + 5 \cdot y_{45} \leq 10 \cdot x_5$$

$$2 \cdot y_{16} + 5 \cdot y_{26} + 3 \cdot y_{36} + 5 \cdot y_{46} \leq 5 \cdot x_6$$

$$2 \cdot y_{17} + 5 \cdot y_{27} + 3 \cdot y_{37} + 5 \cdot y_{47} \leq 5 \cdot x_7$$

$$x_5, x_6, x_7 \in \{0, 1\}$$

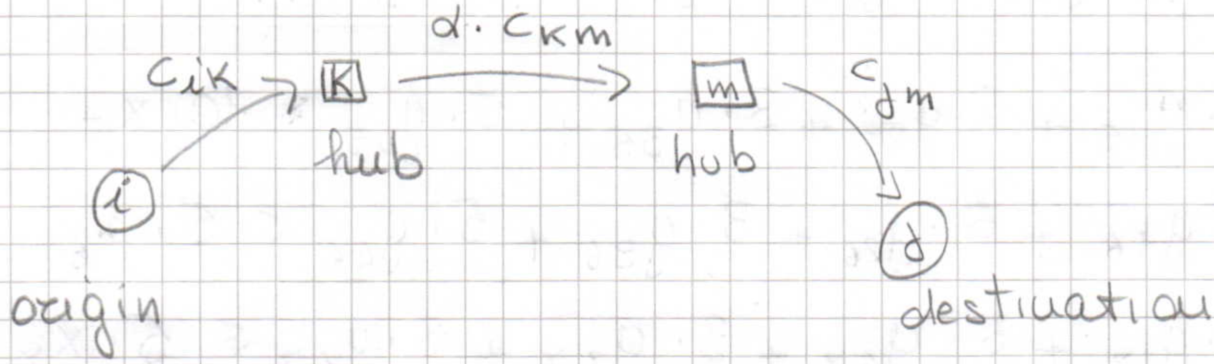
$$y_{15}, y_{16}, \dots \in \{0, 1\}$$

capacity constraints + allocation-location linking

7 Hub location problems

Logistics systems such as airline networks use <u>hub and spoke systems</u>, in order to use larger capacity or faster vehicles during the delivery from an origin to a destination, so reducing the overall transportation cost:

<u>example</u> for an origin-destination pair (i,j):



where α is a <u>discount factor for transportation between hubs</u>

<u>Basic p-hub location problem</u> : locate p hubs and assign each demand node to exactly a hub, so as to minimize the total transportation cost between the origin-destination pairs

<u>Obs</u>: The set of demand nodes I determines a set of origin - destination pairs (i,j), and we know the <u>amount of flow</u> h_{ij} that the origin i must send to j along the network (whereas the previous models assume to know the request of each demand nodes versus a facility, and not versus other demand nodes)

<u>Input data</u>:

- h_{ij} : units of product (flow) to be sent from i to j, $\forall i, j \in I$ (note that $h_{ij} = 0$ if no sending is required)
- c_{ij} : unit transportation cost from i to j, $\forall i, j$ in the network
- α : discount factor for inter-hub sending

Decision variables:

$$x_j = \begin{cases} 1 & \text{if we locate a hub at } j \\ 0 & \text{otherwise} \end{cases} \quad \forall j \in J$$

$$y_{ij} = \begin{cases} 1 & \text{if node } i \text{ is assigned to a hub at } j \\ 0 & \text{otherwise} \end{cases} \quad \forall i \in I,\ j \in J$$

A mathematical model:

(p-hub location)

$$\text{Min} \sum_{i \in I} \sum_{j \in I} h_{ij} \Big(\sum_{k \in J} c_{ik} \cdot y_{ik} + \sum_{m \in J} c_{jm} \cdot y_{jm} + \alpha \cdot \sum_{k \in J} \sum_{m \in J} c_{km} \cdot \underbrace{y_{ik} \cdot y_{jm}}_{\text{nonlinear terms}} \Big)$$

Subject to:

$$\sum_{j \in J} x_j = p \qquad \text{exactly p-hubs}$$

$$\sum_{j \in J} y_{ij} = 1 \quad \forall i \in I \qquad \text{allocation constraints}$$

$$y_{ij} \le x_j \quad \forall i \in I,\ j \in J \qquad \text{linking between location and allocation}$$

$$x_j \in \{0,1\} \quad \forall j \in J$$

$$y_{ij} \in \{0,1\} \quad \forall i \in I,\ j \in J$$

Assumptions

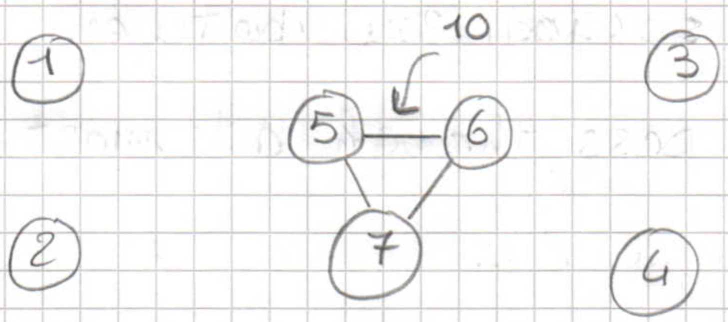
- the hub portion of the logistics network is a complete graph: so the flow between any pair (i,j) will pass through at most two hubs
- each demand node is assigned to exactly one hub (we can relax $\sum_{j \in J} y_{ij} = 1$, $y_{ij} \in \{0,1\}$ $i \in I, j \in J$ to allow that (large) nodes be served by more hubs)

Model characteristics:

- node-to-node flows rather than simply demands at nodes (versus facilities)
- the model is not I.L.P. since the objective function is not linear (it is quadratic)
- in the optimal solution, nodes are not necessarily assigned to the nearest hub (the cost is measured in terms of origin-destination flows)

Example (p-hub location)

$I = \{1, 2, 3, 4\}$ $\quad J = \{5, 6, 7\}$



$c_{ij} = 100 \ \forall i, j \in J, \ i \neq j$

$\alpha = \frac{1}{10}$

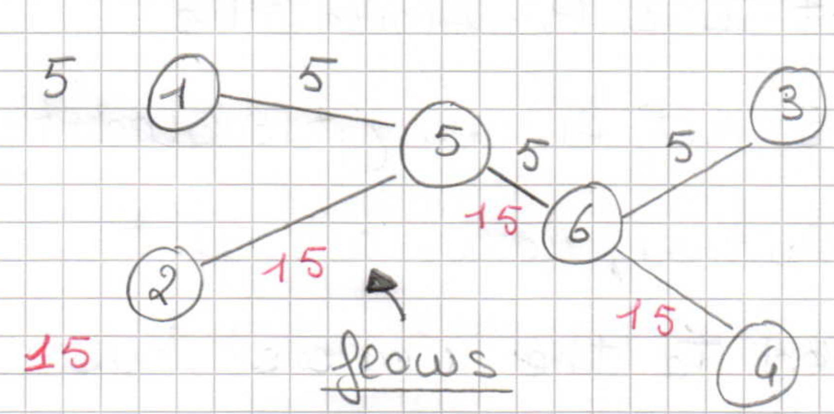
Costs between demand nodes and candidate sites:

	5	6	7
1	10	1000	10
2	10	1000	10
3	1000	10	10
4	1000	10	10

$c_{ij}, i \in I, j \in J$

Let $h_{13} = 5$,
$h_{24} = 15$ and
$h_{ij} = 0$ otherwise

If $p = 2$, a feasible solution is:



$x^*_5 = x^*_6 = 1$

$x^*_7 = 0$

$y^*_{15} = y^*_{25} = 1$

$y^*_{36} = y^*_{46} = 1$

$y^*_{ij} = 0$ otherwise

The <u>corresponding transportation cost</u> is:

$$5\left(10 + 10 + 100 \cdot \overset{\alpha\,=}{\frac{1}{10}}\right) + \quad \text{cost for pair } (1,3)$$

$$+\,15\left(10 + 10 + 100 \cdot \frac{1}{10}\right) = \quad \text{cost for pair } (2,4)$$

$$= 5 \cdot 30 + 15 \cdot 30 = 600$$

<u>The model</u>:

$$\text{Min } 5\,(10 \cdot y_{15} + 1000 \cdot y_{16} + 10\, y_{17} +$$
$$+\,1000\, y_{35} + 10\, y_{36} + 10\, y_{37} +$$
$$+\,\frac{1}{10}\,(\underbrace{0}_{c_{55}} \cdot y_{15} \cdot y_{35} + 100 \cdot y_{15}\, y_{36} \cdots\ ) +$$
$$15\,(10 \cdot y_{25} + 1000\, y_{26} + 10\, y_{27} +$$
$$+\,1000\, y_{45} + 10\, y_{46} + 10\, y_{47} +$$
$$+\,\frac{1}{10}\,(0 \cdot y_{25} \cdot y_{45} + 100\, y_{25} \cdot y_{46} \cdots\ )$$

<u>Subject to</u>:

$$x_5 + x_6 + x_7 = 2$$

$$\left.\begin{array}{l} y_{15} + y_{16} + y_{17} = 1 \\ y_{25} + y_{26} + y_{27} = 1 \\ y_{35} + y_{36} + y_{37} = 1 \\ y_{45} + y_{46} + y_{47} = 1 \end{array}\right\} \text{allocation}$$

$$\left.\begin{array}{l} y_{15} \leq x_5 \\ y_{16} \leq x_6 \\ y_{17} \leq x_7 \\ \vdots \end{array}\right\} \text{linking between location and allocation}$$

$$x_5, x_6, x_7 \in \{0, 1\}$$

$$y_{15}, y_{16} \ldots \in \{0, 1\}$$

8 Maxisum location problem

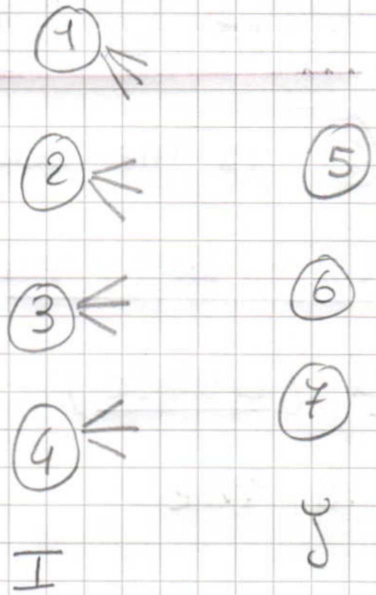
(in Drezner - Hamacher, page 94 : only problem description, no model there)

- The previous location problems involving both demand nodes and facilities assume that it is desirable to locate facilities as close as possible to demand nodes

- This is not true in case of undesirable facilities (e.g. prisons, power plants, waste repositories..) : we would like to locate facilities far from demand nodes!

<u>The problem</u> : determine the locations of p facilities (p is given) in order to <u>maximize</u> the total weighted distance between demand nodes and facilities

<u>An example</u> (same instance of (p-median))



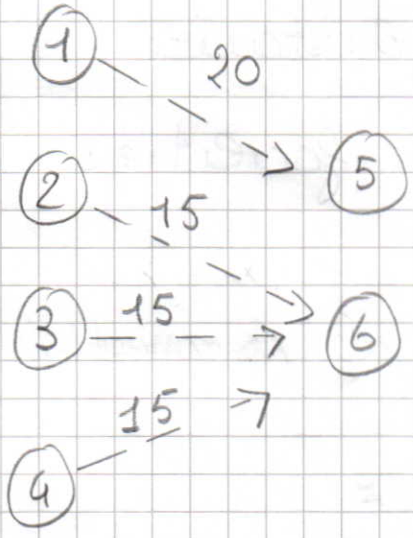
	5	6	7
1	20	25	60
2	30	15	10
3	30	15	10
4	30	15	10

d_{ij}

<complete graph>

- $p = 2$
- $h_i = 1 \quad , i = 1, 2, 3, 4$

What happens if we open the two facilities in 5 and 6 ?



In this scenario, the most critical facility for 1 is facility in 5, since it is closer than 6.

Instead, the most critical (= closer) facility to 2, 3 and 4 is the one in 6. Therefore the total distance between demand nodes and their critical (closer) open facilities is 65 : this is the objective function value

- Opening in 5 and 7 : obj. f. value =

  20 + 10 + 10 + 10 = 50

- Opening in 6 and 7 : obj. f. value =

  25 + 10 + 10 + 10 = 55

Since we want to "maximize", opening in 5 and in 6 gives the optimal solution

How can we model the Maxisum location problem?

I.L.P. model (not in the text book)

Decision variables:

$$x_j = \begin{cases} 1 & \text{if we open at } j \\ 0 & \text{otherwise} \end{cases} \qquad \forall j \in J$$

z_i : distance between i and the corresponding critical ($\equiv$ closer) facility, $\forall i \in I$

(maxisum) $\quad \text{Max} \sum_{i \in I} z_i$

Subject to: $\sum_{j \in J} x_j = P$

$$(d_{ij} - M)\, x_j + M \geq z_i \quad \forall i \in I,\ j \in J$$

$$x_j \in \{0, 1\} \quad \forall j \in J$$

- M denotes the usual very large value

Constraint explanation:

consider $i \in I$:

$\forall j \in J$:

if $x_j = 1$: $d_{ij} \cancel{- M} \cancel{+ M} \geq z_i$

i.e. $z_i \leq d_{ij}$

i.e. z_i is a lower bound to the distance between i and the opened facility in j

if $x_j = 0$: $z_i \leq M$ always satisfied

i.e. in this case the constraint "does not constrain", since in j there is no facility

Therefore : $z_i \leq d_{ij}$ $\forall j \in J$ s.t. there is a facility

and so z_i is a lower bound to the minimum distance (which is the critical one!) between i and the opened facilities, $\forall i \in I$

By maximizing $\sum_{i \in I} z_i$, the solver forces each z_i to be as large as possible, so maximizing the total maxisum distance, i.e. the total distance between each demand mode and its closer facility (in the optimal solution, z_i^* gives such a minimum distance $\forall i \in I$)

example (cont.)

$$\text{Max} \quad z_1 + z_2 + z_3 + z_4$$

Subject to $\quad x_5 + x_6 + x_7 = 2$

$$\left.\begin{aligned} (20 - M)\, x_5 + M &\geq z_1 \\ (25 - M)\, x_6 + M &\geq z_1 \\ (60 - M)\, x_7 + M &\geq z_1 \end{aligned}\right\} \text{demand node } 1$$

$$\left.\begin{aligned} (30 - M)\, x_5 + M &\geq z_2 \\ (15 - M)\, x_6 + M &\geq z_2 \\ (10 - M)\, x_7 + M &\geq z_2 \end{aligned}\right\} \text{demand node } 2$$

$$\left.\begin{aligned}(30-M)\,x_5+M &\geq z_3\\(15-M)\,x_6+M &\geq z_3\\(10-M)\,x_7+M &\geq z_3\end{aligned}\right\}\ \text{demand node } 3$$

$$\left.\begin{aligned}(30-M)\,x_5+M &\geq z_4\\(15-M)\,x_6+M &\geq z_4\\(10-M)\,x_7+M &\geq z_4\end{aligned}\right\}\ \text{demand node } 4$$

$$x_5,\ x_6,\ x_7 \in \{0,1\}$$

The optimal solution:

$$x_5^* = x_6^* = 1 \qquad x_7^* = 0$$

$$z_1^* = 20$$

$$z_2^* = z_3^* = z_4^* = \mathbf{15}$$

optimal obj. function value : **65**

Location problems in the Public Sector

(Drezner - Hamacher : Section 4.2.3 (until page 126))

e.g. emergency services (ambulances, fire stations, police units), school system, postal facilities

- optimization criteria: social cost minimization, universality of service, efficiency and equity (as opposed as profit maximization)

- since these objectives are difficult to measure, often they are "surrogated" by minimization of costs needed for full coverage of the service (i.e. (SCLP)), or search for maximal coverage given a budget (i.e. (MCLP) and its variants); the basic models in this sector are in fact (SCLP) and (MCLP) (also (p-center) and (p-median))

Let us consider a more complex example arising in Mobile Emergency Services:

<u>FLEET</u> (≡ Facility Location and Equipment Emplacement Technique)

- generalization of (MCLP)
- simultaneous location of more types of facilities

<u>The problem</u>: decide where to locate two different types of fire-fighting servers (1) pump or engine brigades 2) ladder or truck brigades), as well as the depots housing them, so as to cover the maximum number of people by <u>both</u> an engine company sited within a standard distance E (given) <u>and</u> a truck company sited within another standard distance T (also given)

Input data

- $N_i^E = \{ j \in J : d_{ij} \leq E \} \quad \forall i \in I$
- $N_i^T = \{ j \in J : d_{ij} \leq T \} \quad \forall i \in I$
- p^S : number of fire stations (depots) to open
- p^{E+T} : number of (engine or truck) companies to open
- a_i : inhabitants in i, $\forall i \in I$

Decision variables

$$x_j^E = \begin{cases} 1 & \text{if an engine company is positioned in a fire station at } j \\ 0 & \text{otherwise} \end{cases} \quad \forall j \in J$$

$$x_j^T = \begin{cases} 1 & \text{if a truck company is positioned in a fire station at } j \\ 0 & \text{otherwise} \end{cases} \quad \forall j \in J$$

$$x_j^S = \begin{cases} 1 & \text{if a fire station (depot) is established at site } j \\ 0 & \text{otherwise} \end{cases} \quad \forall j \in J$$

<u>I.L.P. model</u> (used for locating in Baltimore):

$$\text{Max} \sum_{i \in I} a_i z_i$$

"covered"

$$z_i \leq \sum_{j \in N_i^E} x_j^E \quad \forall i \in I$$

$$z_i \leq \sum_{j \in N_i^T} x_j^T \quad \forall i \in I$$

* z_i can be 1 only if at least one engine is within E and one truck is within T *

$$x_j^E \leq x_j^S \quad \forall j \in J$$

$$x_j^T \leq x_j^S \quad \forall j \in J$$

* allow housing of companies only at nodes with a depot *

$$\sum_{j \in J} x_j^S = p^S$$

$$\sum_{j \in J} x_j^E + \sum_{j \in J} x_j^T = p^{E+T}$$

$$x_j^S, x_j^E, x_j^T \in \{0,1\} \quad \forall j \in J$$

$$z_i \in \{0,1\} \quad \forall i \in I$$

<u>where</u> $z_i = \begin{cases} 1 & \text{if } i \text{ is covered} \\ 0 & \text{otherwise} \end{cases} \quad \forall i \in I$

(as in (MCLP))