An Introduction to Mixed Integer Linear Problems (MILP)

(Ragsdale : Chap. 6)

Many business problems need integer solutions (e.g. how many employees to assign to each shift, how many vehicles to purchase...)

→ (Mixed) Integer Linear Programming: certain decision variables must assume only integer values

An example: Blue Ridge Hot Tubs

$$\text{Max } 350\,x_1 + 300\,x_2$$

$$x_1 + x_2 \leq 200$$

$$9\,x_1 + 6\,x_2 \leq 1566 \quad 1520$$

$$12\,x_1 + 16\,x_2 \leq 2880 \quad 2650$$

$$x_1\,,\ x_2 \geq 0$$

integrality constraints $\quad x_1\,,\ x_2$ integer

- The optimal solution was integer ($x_1^* = 122$, $x_2^* = 78$)
- However, by changing the two LHS we would get $x_1^* = 116,9444$ and $x_2^* = 77,9167$: so, integrality constraints have to be added to the model (from LP to (M)ILP)

<u>How can we address ILP ?</u>

1) Solving its <u>Linear Relaxation</u>, i.e. eliminating the integrality constraints, and then rounding the obtained solution

e.g. $\tilde{x}_1 = 116$ $\hat{x}_2 = 77$ (profit: 63.700)

<u>Obs</u>: this is not necessarily an optimum solution to the ILP

<u>Obs</u>: the optimum LP value, 64.306, is <u>an upper bound</u> to the optimum ILP value (for a maximization problem)

2) Solving directly the ILP model (e.g. via the EXCEL solver)

< Read Section 6.7 to implement the Integer Blue Ridge Hot Tubs as a spreadsheet model: EXCEL file available>

Other examples

1) A fixed-charge problem
(Ragsdale: Sect. 6.12, 6.13)

- Remington Manufacturing is planning its next production cycle

- 3 products (say Product 1, Product 2 and Product 3), each of which must undergo machining, grinding and assembley operations to be completed

• Table summarizing the hours required by each unit of product, and the total hours available for each operation

	Prod 1	Prod 2	Prod 3	Total Hours
Machining	2	3	6	600
Grinding	6	3	4	300
Assembly	5	6	2	400
Unitary profit	48	55	50	

However: manufacturing units of Product 1 requires a setup operation on the production line that costs 1000 ; similarly for Product 2 (800) and Product 3 (900) (fixed-charge costs)

The optimization problem : determine the most profitable mix of products to produce

Defining the decision variables

x_i : amount of PRODUCT i to be produced, $i = 1, 2, 3$

$$y_i = \begin{cases} 1 & \text{if } x_i > 0 \\ 0 & \text{otherwise} \end{cases} \quad i = 1, 2, 3$$

these are auxiliary binary variables, which will be used to express logical conditions

ILP model

$$\text{Max } 48x_1 + 55x_2 + 50x_3 - 1000y_1 - 800y_2 - 900y_3$$

$$\left.\begin{aligned} 2x_1 + 3x_2 + 6x_3 &\le 600 \\ 6x_1 + 3x_2 + 4x_3 &\le 300 \\ 5x_1 + 6x_2 + 2x_3 &\le 400 \end{aligned}\right\} \text{constraints on total hours availability}$$

M

$$\left.\begin{aligned} x_1 &\le 50\, y_1 \\ x_2 &\le 67\, y_2 \\ x_3 &\le 75\, y_3 \end{aligned}\right\} \text{linking constraints}$$

$$x_i \ge 0 \quad i = 1, 2, 3$$

$$y_i \in \{0, 1\} \quad i = 1, 2, 3$$

- M is an upper bound on the optimal value of the x_i (Big M)

- The models are much easier to solve if M is kept as small as possible: for example we can set $M = 60$ in the first linking constraint (no more than 60 units of PRODUCT 1 can be produced), and to 67 and 75 in the remaining cases

< Read Section 6.13.5 for implementing as a spreadsheet model: EXCEL file available >

2) <u>A Transportation problem</u>

(Ragsdale: Section 6.16)

- <u>B&G Construction</u>: commercial building company that signed contracts to construct 4 buildings

- Each building project requires a large amount of cement to be delivered to the building sites
- 3 cement companies have submitted bids for supplying the cement

Table summarizing the prices for delivered ton of cement and the maximum amount of cement the 3 companies may deliver

	Cost per ton				
	Proj 1	Proj 2	Proj 3	Proj 4	Max Supply
Company 1	120	115	130	125	525
Company 2	100	150	110	105	450
Company 3	140	95	145	165	550
Total tons needed for each project	450	275	300	350	

Complicating constraints: each cement company placed special conditions on its bid:

1) <u>company 1</u>: it will not supply orders of < 150 tons for any project

2) <u>company 2</u>: it can supply more than 200 tons to no more than one of the projects

3) <u>company 3</u>: it will accept only orders that total 200 tons, 400 tons or 550 tons (or 0, i.e. no order)

<u>The optimization problem</u>: determine the amounts of cement to purchase from each supplier to meet the demands for each project, by satisfying the complicating constraints, at a minimum cost

(<u>transportation problem with additional constraints</u>)

Defining the decision variables:

x_{ij} : tons of cement from company i for project j, $i = 1, \dots 3$, $j = 1, \dots 4$

+ auxiliary binary variables to formulate the complicating (side) constraints :

$y_{1j} \in \{0, 1\}$, $j = 1, \dots 4$ for side constraint Comp. 1

$y_{2j} \in \{0, 1\}$, $j = 1, \dots 4$ " " " Comp. 2

$y_{3k} \in \{0, 1\}$, $k = 1, 2, 3$ " " Comp. 3

The ILP model

$$\text{Min } 120\, x_{11} + 115\, x_{12} + 130\, x_{13} + 125\, x_{14} +$$
$$+ 100\, x_{21} + 150\, x_{22} + 110\, x_{23} + 105\, x_{24} +$$
$$+ 140\, x_{31} + 95\, x_{32} + 145\, x_{33} + 165\, x_{34}$$

Subject to :

$$x_{11} + x_{12} + x_{13} + x_{14} \leq 525$$
$$x_{21} + x_{22} + x_{23} + x_{24} \leq 450$$
$$x_{31} + x_{32} + x_{33} + x_{34} \leq 550$$

Constraints on max. supply for each company

$$x_{11} + x_{21} + x_{31} = 450$$
$$x_{12} + x_{22} + x_{32} = 275$$
$$x_{13} + x_{23} + x_{33} = 300$$
$$x_{14} + x_{24} + x_{34} = 350$$

demand for cement satisfaction

Comp. 1

$$x_{11} \leq 525\, y_{11}$$
$$x_{12} \leq 525\, y_{12}$$
$$x_{13} \leq 525\, y_{13}$$
$$x_{14} \leq 525\, y_{14}$$

Logical constraints: "if $x_{11}, x_{12} \ldots > 0$, then the associated binary variable must be 1"

$$x_{11} \geq 150\, y_{11}$$
$$x_{12} \geq 150\, y_{12}$$
$$x_{13} \geq 150\, y_{13}$$
$$x_{14} \geq 150\, y_{14}$$

"If $x_{11}, x_{12} \ldots > 0$ (and so the associated binary variable is 1), then it must be ≥ 150"

Comp 2

$$x_{21} \leq 200 + 250\, y_{21}$$
$$x_{22} \leq 200 + 250\, y_{22}$$
$$x_{23} \leq 200 + 250\, y_{23}$$
$$x_{24} \leq 200 + 250\, y_{24}$$
$$y_{21} + y_{22} + y_{23} + y_{24} \leq 1$$

Express "more than 200 tons to no more than one project"

Comp. 3
$$\begin{cases} x_{31} + x_{32} + x_{33} + x_{34} = \\ \quad = 200\, y_{31} + 400\, y_{32} + 550\, y_{33} \\ y_{31} + y_{32} + y_{33} \leq 1 \end{cases}$$

Expressing that the total amount ordered from Comp. 3 can be only
200 (if $y_{31} = 1$) or
400 (if $y_{32} = 1$) or
550 (if $y_{33} = 1$) or
0 (if $y_{31} = y_{32} = y_{33} = 0$)
i.e. only 4 distinct values

$x_{ij} \geq 0 \quad i = 1, 2, 3$
$\qquad\qquad j = 1, 2, 3, 4$

$y_{1j} \in \{0, 1\} \quad j = 1, 2, 3, 4$

$y_{2j} \in \{0, 1\} \quad j = 1, 2, 3, 4$

$y_{3k} \in \{0, 1\} \quad k = 1, 2, 3$

< Read Section 6.16 for implementing the B&G Construction model as a spreadsheet model: EXCEL file available>