Routing problems

(Toth - Vigo : Chap 1)

- Management of the provision of goods and services in distribution systems
- The use of optimization procedures allows substantial savings (from 5% to 20%) in the global transportation costs

Here we address Vehicle Routing Problems (VRP) or Vehicle Scheduling Problems : concern the distribution of goods from depots to final users (customers)

Typical applications : solid waste collection, school bus routing, dial-a-ride systems ... but also, for example, home care applications

General VRP formulation: given a set of customers, a fleet of vehicles located in one or more depots, and given a road network, determine a set of routes, each performed by a single vehicle that starts and ends at its own depot, in such a way as all the customer requirements are fulfilled, all the operational constraints (if present) are satisfied, by minimizing the global transportation cost.

- Let us describe some typical VRP characteristics by considering the main components, the different operational constraints that can be imposed, and the possible objectives to be achieved.

Typical VRP characteristics

Main components:

1. **Road network**: it is described by means of a graph, where nodes correspond to road junctions, and to depot and customer locations, and where arcs represent streets.

- Arcs can be
  - directed (e.g. to model one-way streets)
  - undirected (traversal in both directions)

- Each arc is associated with:
  - a cost, which generally represents its length
  - travel time, which may depend on the vehicle type or considered period

② Customers

Typical characteristics:

- node of the road network where the customer is located
- amount of goods (demand), possibly of different types, that must be delivered or collected
- period of time (time window) during which the customer can be served
- times required to deliver or collect goods at the customer location (unloading and loading times, resp.)
- subset of vehicles that can be used to serve the customer

If some customers can not be fully satisfied, then different priorities, or penalties associated with (partial) lack of service, can be assigned to customers

3 Routes

- Each route starts and ends at a depot (usually the same)
- Each depot is characterized by number and types of vehicles associated with it, and by the total amount of goods it can deal with

4 Fleet of vehicles

Main characteristics:

- home depot (vehicles may end the service at a depot other than home depot)
- capacity ≡ maximum weight or volume or number of units the vehicle can load
- possible subdivision into compartments
- possible devices available for loading and unloading
- subset of arcs which can be traversed by the vehicle

- costs associated with vehicle utilization (per distance unit, per time unit...)

(5) Drivers ≡ vehicles in our context

(6) Operational constraints

Typical constraints:

- current load can not exceed vehicle capacity
- customers served in a route can require only delivery, only collection or both
- customers served within their time windows
- precedence constraints on the order in which customers in a route are served:
  - pickup and delivery problems: route can perform both collection and delivery of goods, and goods collected by pickup customers must be carried to the

corresponding delivery customers

- VRP with Backhauls: all deliveries must be performed before the collections

- Evaluation of global cost of the routes, and the check of the operational constraints, require to know the travel time and the travel cost between each pair of customers and between depots and customers

- To this end, the original (sparse) road network is usually modelled as a complete graph G such as:
  - the nodes in G correspond to customers and depots
  - $\forall (i,j)$ in G:
    - c_{ij} is the cost of the shortest path from i to j in the road network
    - t_{ij} : total time of such a shortest path in the road network

G can be directed or undirected depending on the property of the corresponding cost and travel time matrices to be asymmetric or symmetric, resp.

< The following models will refer to G >

7. Optimization objectives (often contrasting objectives)

- minimization of the global transportation cost: depends on the global distance travelled and on the fixed costs associated with the used vehicles
- minimization of the number of vehicles used to satisfy all customers
- balancing of the routes (for travel time, vehicle load...)
- minimization of the penalties for partial service of customers

< often a weighted combination of these objectives is addressed >

Relevant variants of VRP

- Stochastic VRP: demands and/or travel times are random variables
- Dynamic (time-dependent) VRP: demands, costs and/or travel times are time dependent
- Arc Routing Problems: customers are located along the arcs of the road network (e.g. in postal delivery services)

A particular case of VRP: Traveling Salesman Problem (TSP): VRP with a single depot, a single (uncapacitated) vehicle and no operational constraints!

The Capacitated VRP (CVRP)

Assumptions:

- customers of the same kind (e.g. delivery), with deterministic and unsplittable demand
- homogeneous fleet of vehicles located in a unique home depot
- operational constraints: vehicle capacity

Let $G = (V, A)$ be a complete graph:

- $V = \{0, 1, \dots n\}$ set of nodes

  0 denotes the depot

  $\{1, \dots n\}$ set of customers

- $c_{ij} \geq 0 \ \forall (i,j) \in A$ traveling cost from i to j

If G is directed, then $c_{ij} \neq c_{ji}$ is possible:

Asymmetric CVRP (ACVRP)

If G is undirected, and so $c_{ij} = c_{ji}$:

Symmetric CVRP (SCVRP)

- Let $d_i \geqslant 0$ be the demand of customer i, $i = 1, \dots n$ ($d_0 = 0$)
- Assume to have K identical vehicles available at the depot, with capacity C (assume $d_i \leq C$, $i = 1, \dots n$)

Assumption $K \geqslant K_{min}$, where K_{min} is the minimum number of vehicles to serve all customers ($K_{min} \geqslant \underbrace{\left\lceil \frac{\sum_{i=1}^{n} d_i}{C} \right\rceil}_{\text{lower bound!}}$)

The CVRP problem: determine the tours of the K vehicles (i.e. K directed cycles in G) such as:

(i) each tour includes the depot (i.e. node 0)

(ii) each customer belongs to exactly one tour (i.e. he is visited by exactly one vehicle)

(iii) the sum of the demands of the customers belonging to the same tour does not exceed C (the capacity of the vehicle),

by minimizing the total cost.

Interesting variants of CVRP

- There can be unused vehicles (if $K > K_{min}$); therefore it is possible to have fixed costs for using the vehicles
- Vehicles may have different capacities C_k, $k = 1, \dots K$
- Tours formed by a single customer can be forbidden

Observe that CVRP generalizes TSP:

- TSP is the special case where $K = 1$ and $C \geq \sum_{i=1}^{n} d_i$

Therefore : CVRP is NP-Hard (in a strong sense)

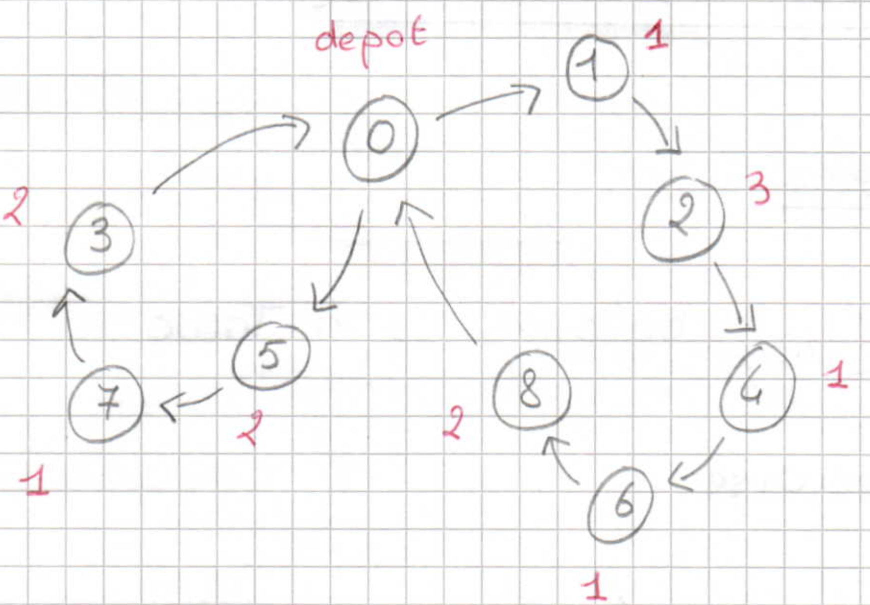
An example

$n = 8$

d	1	2	3	4	5	6	7	8
	1	3	2	1	2	1	1	2

$K = 2 \qquad C = 8$

Obs $\quad K = K_{min} = \left\lceil \dfrac{\sum_{i=1}^{8} d_i}{8} \right\rceil = \left\lceil \dfrac{13}{8} \right\rceil = 2$



A feasible CVRP solution

If all arcs in the figure cost 1, then the cost of the feasible solution to CVRP is 10

Basic models to CVRP

- Models for the asymmetric version (ACVRP) (can be easily adopted to the symmetric one)

1 Basic two-index model

Decision variables

$$x_{ij} = \begin{cases} 1 & \text{if } (i,j) \text{ belongs to a tour} \\ 0 & \text{otherwise} \end{cases}$$

$$\forall (i,j) \in A$$

- $O(n^2)$ variables

(VRP 1)

$$\min \sum_{(i,j)\in A} c_{ij}\, x_{ij}$$

(1) $\displaystyle\sum_{(i,j)\in BS(j)} x_{ij} = 1 \qquad \forall j \in V\setminus\{0\}$

(2) $\displaystyle\sum_{(j,i)\in FS(j)} x_{ji} = 1 \qquad \forall j \in V\setminus\{0\}$

(3) $\displaystyle\sum_{(i,0)\in A} x_{i0} = K$

(4) $\displaystyle\sum_{(0,i)\in A} x_{0i} = K$

(5) $\displaystyle\sum_{i\notin S}\sum_{j\in S} x_{ij} \geq r(S) \qquad \forall S \subseteq V\setminus\{0\},\ S\neq\emptyset$

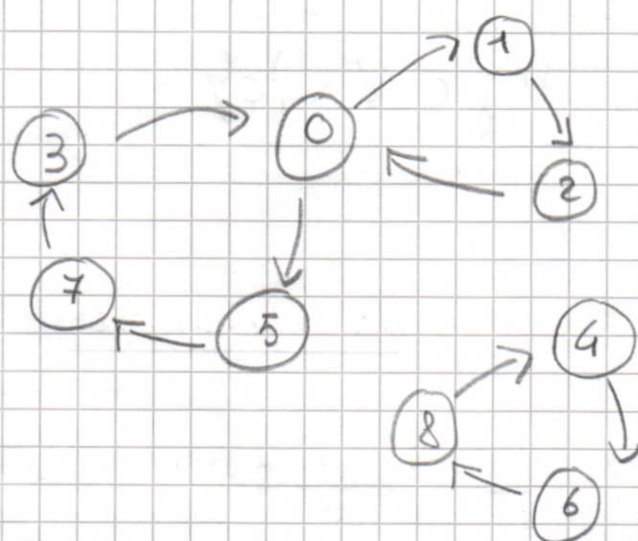
(6) $x_{ij}\in\{0,1\} \qquad \forall (i,j)\in A$

Obs : (1), (2), (3) and (4) are linearly dependent (one constraint could be removed)

Are (1), (2), (3), (4) sufficient to model ACVRP solutions?

<u>a counterexample</u>

consider the example in 140 ; then



<u>subtour</u> (tour which does not include depot)

satisfies (1), (2), (3) and (4), but this is not feasible to ACVRP (why?)

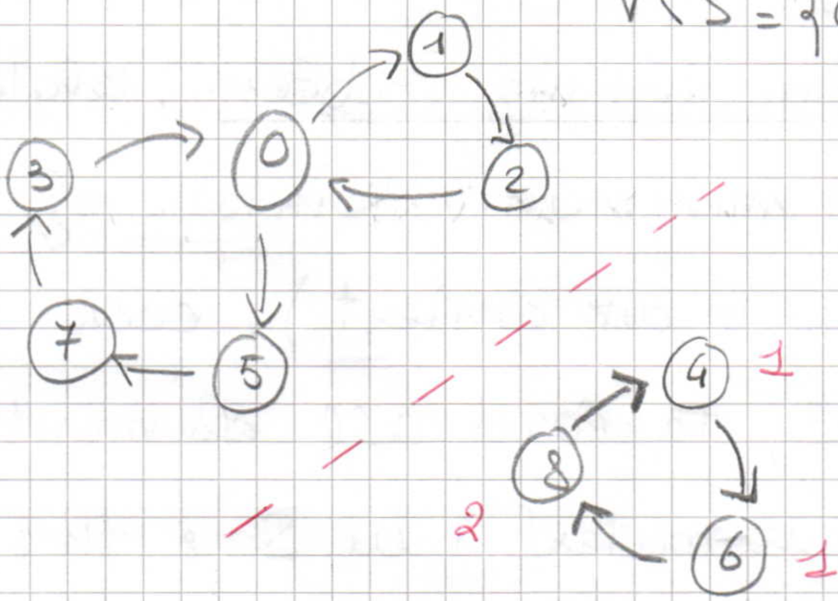
- The <u>Cut Capacity Constraints</u> (<u>CCC</u>) (5) are introduced to <u>guarantee the connection of each tour to the depot</u>, **i.e.** to avoid the situation above ; in (5), $r(S)$ is the minimum number of vehicles to serve all customers in S ($r(S)$ are assumed to be input data)

<u>Obs</u>: $\forall S$, $r(S)$ can be computed by solving a "<u>Bin Packing Problem</u>" related to S: find the minimum number of bins ($\equiv$ vehicles, in our context), each having capacity C, to load all elements ($\equiv$ customers, in our context) in S: this is NP-Hard, but solvable in an efficient way

<u>Explanation of constraints (ccc)</u>:

for each <u>cut</u> $(V \setminus S, S)$ in the logistics network, which separates the customers in S from the depot, at least $r(S)$ arcs going from $V \setminus S$ to S must be present in any feasible solution, i.e. a sufficient ($r(S)$) number of vehicles must travel from $V \setminus S$ (where the depot is located) to S to serve the customers in S

example (cont.)



$V \setminus S = \{0, 1, 2, 3, 5, 7\}$

$S = \{4, 6, 8\}$

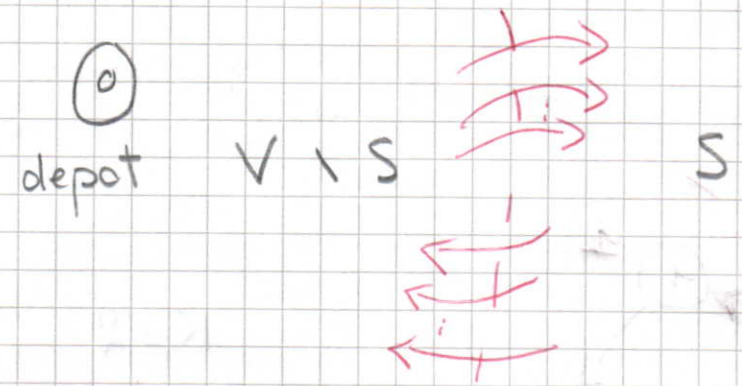
S, and so the cut $(V \setminus S, S)$ violates the CCC constraints: in fact, since $C = 8$, then $r(S) = 1$, and so the CCC constraint related to $S = \{4, 6, 8\}$ is

$$\sum_{i \in \underbrace{\{0,1,2,3,5,7\}}_{V \setminus S}} \sum_{j \in \underbrace{\{4,6,8\}}_{S}} x_{ij} \geq \underbrace{1}_{r(S)}$$

and it is not satisfied. Therefore, the solution above is not feasible to model (VRP1) (in fact, it is not feasible for CVRP)

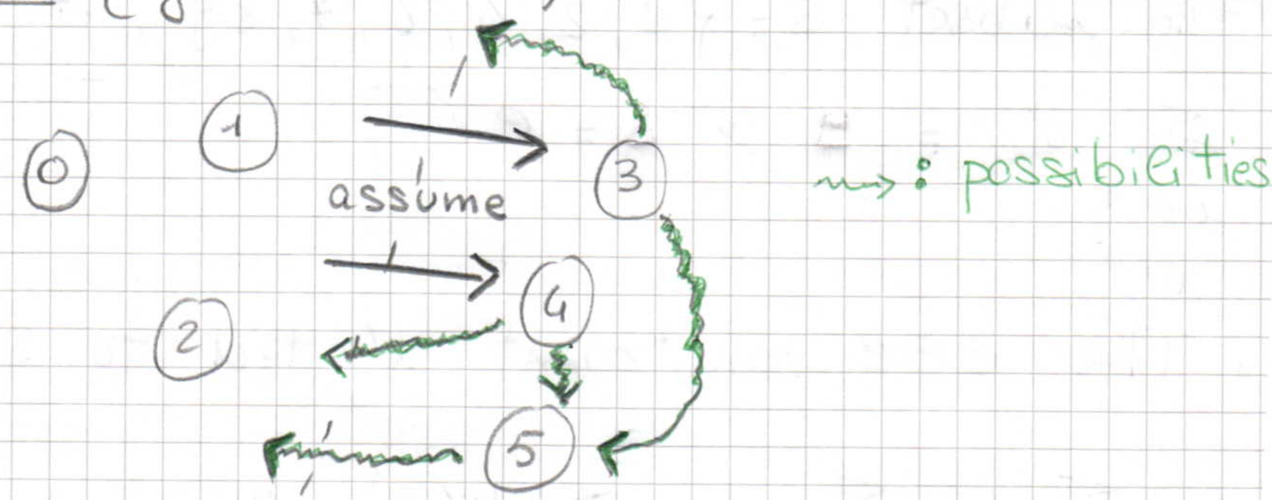
<u>Note that</u> (1)-(4) imply:

$$\sum_{i \in V \setminus S} \sum_{j \in S} x_{ij} = \sum_{i \in S} \sum_{j \in V \setminus S} x_{ij} \quad \forall S \neq \emptyset, \; S \subseteq V \setminus \{0\}$$



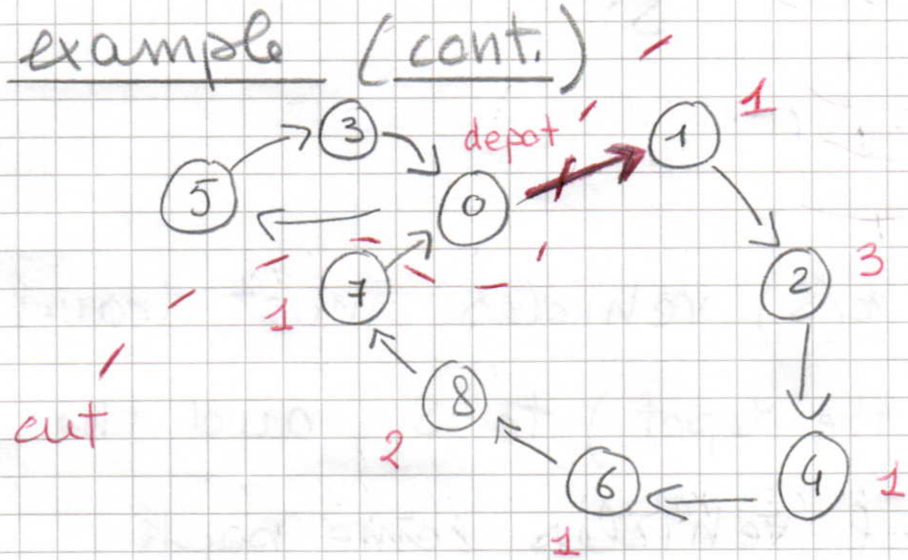
i.e., at least r(S) vehicles must travel from V \ S (i.e. the depot) to S, and the same number of vehicles come back from S to V \ S (i.e. the depot)

<u>example</u> (just a hint)



The ccc constraints guarantee not only the connection of each tour to the depot, but also the <u>vehicle capacity satisfaction</u>:

<u>example</u> (<u>cont.</u>)



$K = 2$

- This solution is composed of two tours, but it is <u>infeasible</u>: in fact, for the subset $S = \{1, 2, 4, 6, 7, 8\}$, it is

$$\sum_{i \in S} d_i = 9 > 8 = C$$

- The ccc constraint related to S is:

$$\sum_{\underbrace{i \in V \setminus S}_{\{0,3,5\}}} \sum_{\underbrace{j \in S}_{\{1,2,4,6,7,8\}}} x_{ij} \geq r(S) = 2$$

This number takes into account capacity C

This constraint is violated by the solution above (only one arc crosses the cut), which therefore (correctly) is not feasible to the ILP model (VRP1)

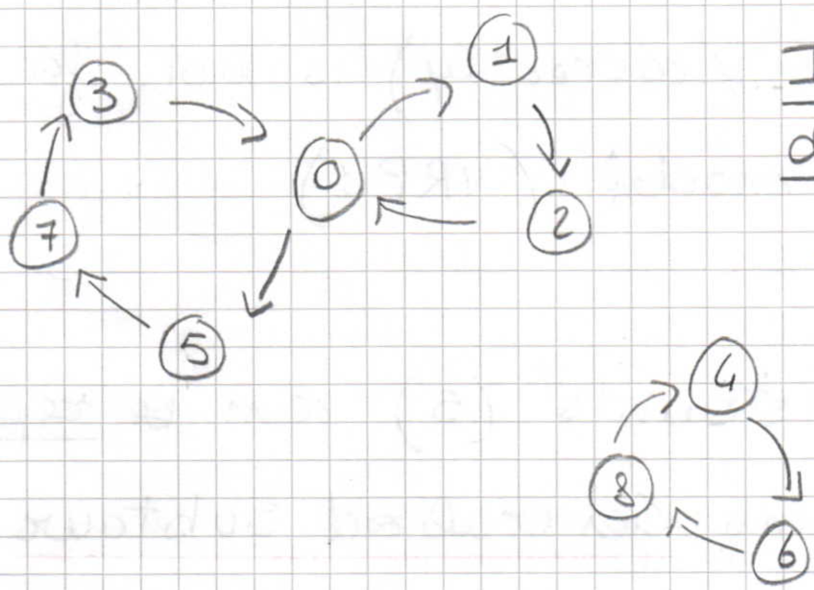
The CCC constraints (5) can be replaced by the following Generalized Subtour Elimination Constraints (GSEC):

$$(7) \quad \sum_{i \in S} \sum_{j \in S} x_{ij} \leq |S| - r(S), \quad \forall S \subseteq V \setminus \{0\}, \; S \neq \emptyset$$

That is: at least $r(S)$ arcs (= vehicles) must go out of S (and so, enter S from outside)

By considering the previous (infeasible) solutions:

I)



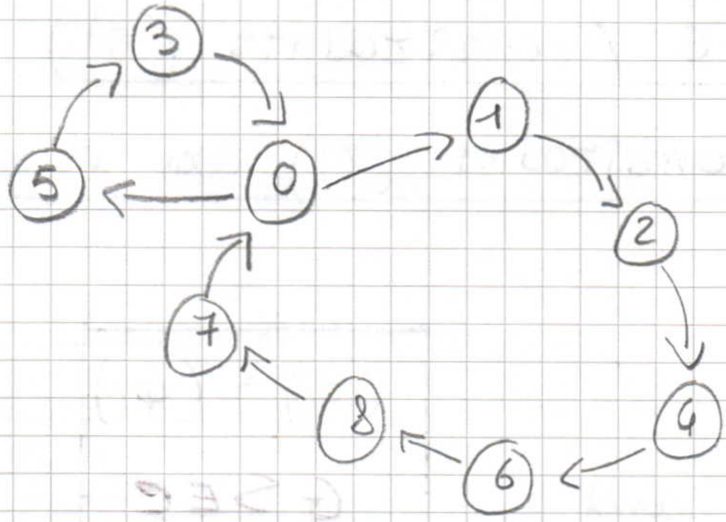
Infeasibility due to a subtour

By considering $S = \{4, 6, 8\}$, the related GSEC constraint is:

$$\sum_{i \in S} \sum_{j \in S} \underbrace{x_{ij}}_{\text{internal selected arcs}} = 3 \leq \underbrace{|S|}_{=3} - \underbrace{r(S)}_{=1} = 2$$

it is violated, so the solution is not feasible to the ILP model (as it must be!)

II)



Infeasibility due to capacity violation

By considering $S = \{1, 2, 4, 6, 7, 8\}$, the related GSEC constraint is:

$$\sum_{i \in S} \sum_{j \in S} x_{ij} = 5 \leq \underbrace{|S|}_{=6} - \underbrace{r(S)}_{=2} = 4$$

it is violated (due to $r(S) = 2$, caused by capacity considerations); so, also this solution is not feasible to our ILP model (as it must be!)

Indeed, CCC (constraints (5))

and GSEC (constraints (7)) are equivalent,

so

(1) - (4)
CCC
(6)

and

(1) - (4)
GSEC
(6)

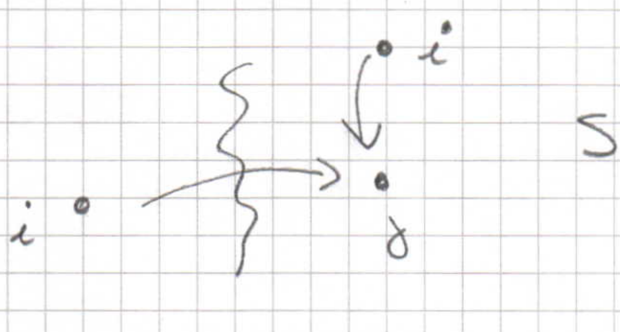
are equivalent formulations to CVRP

Why? for each subset of customers S

$$\sum_{j \in S} \underbrace{\sum_{(i,j) \in BS(j)}}_{= 1} x_{ij} = |S|$$

This is equivalent to:

$$\sum_{j\ S} \left(\sum_{i \in S} x_{ij} + \sum_{i \in V \setminus S} x_{ij} \right) = |S|$$



That is:

$$\sum_{i \in S} \sum_{j \in S} x_{ij} + \sum_{i \in V \setminus S} \sum_{j \in S} x_{ij} = |S| \quad \text{for each } S$$

Therefore:

$$\sum_{i \in V \setminus S} \sum_{j \in S} x_{ij} \geq r(S) \qquad \text{CCC constraints}$$

if and only if

$$\sum_{i \in S} \sum_{j \in S} x_{ij} \leq |S| - r(S) \quad \text{GSEC constraints}$$

- Inconvenience for both CCC and GSEC: their cardinality can be exponential with respect to the input size (i.e. n)

- Possible approaches:
  1) Cutting Plane approach (not discussed here)
  2) Alternative constraints, of polynomial cardinality

Miller, Tucker and Zemlin (MTZ) constraints

- Alternative to CCC and to GSEC

MTZ constraints:

- $u_i \geq 0 \quad \forall i \in V \setminus \{0\}$ additional auxiliary variables

(i) $u_i - u_j + C x_{ij} \leq C - d_j \quad \forall i, j \in V \setminus \{0\}$ such that $d_i + d_j \leq C$

(ii) $d_i \leq u_i \leq C \quad \forall i \in V \setminus \{0\}$

Explanation of MTZ constraints:

- u_i represents the load of the vehicle serving customer i, after the loading operation at i, $\forall i \in V \setminus \{0\}$

Therefore:

(ii) $d_i \leq u_i \leq C$, since the load must be at least d_i but no more than C

vehicle capacity constraints

Meaning of constraints (1):

Two cases are possible $\forall\, i,j \in V - \{0\}$:

- $\boxed{x_{ij} = 0}$, i.e. no vehicle moves from i to j along (i,j)

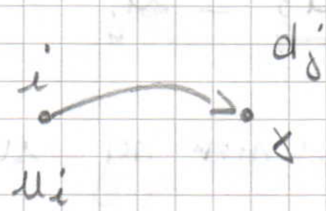
$$u_i - u_j \le C - d_j$$ always true since $u_i \le C$ and $u_j \ge d_j$

that is, if $x_{ij} = 0$ the constraint is redundant

- $\boxed{x_{ij} = 1}$, i.e. a vehicle moves from i to j along (i,j)

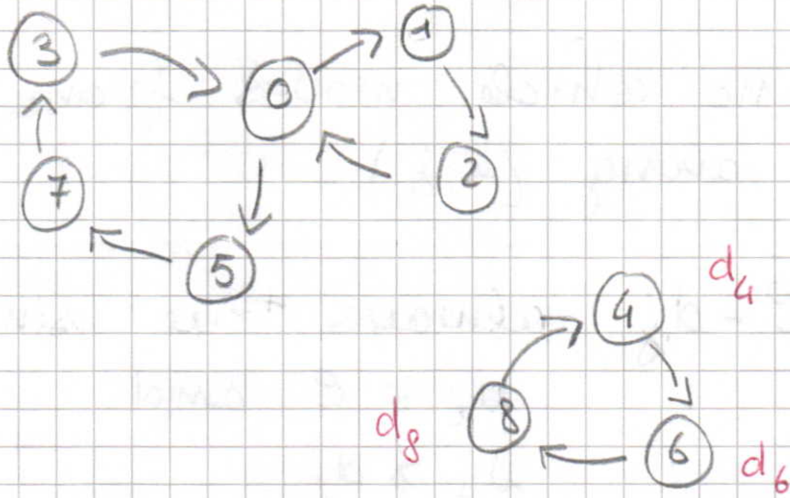
$$u_i - u_j + \cancel{C} \le \cancel{C} - d_j$$ i.e.

$$u_j \ge u_i + d_j$$



in fact, there is a vehicle serving j after i, and so its load after visiting j, i.e. u_j, is $\ge u_i + d_j$

why MTZ constraints exclude solutions containing subtours, such as ?



since MTZ impose:

$$u_4 - u_6 + \cancel{C} \underset{=1}{x_{46}} \le \cancel{C} - d_6$$

$$u_6 - u_8 + \cancel{C} \underset{=1}{x_{68}} \le \cancel{C} - d_8$$

$$u_8 - u_4 + \cancel{C} \underset{=1}{x_{84}} \le \cancel{C} - d_4$$

By summing up the inequalities, they imply:

$$0 \le -d_4 - d_6 - d_8 \quad \text{i.e.}$$

$$d_4 + d_6 + d_8 \geq 0$$

which is true if and only if $d_4 = d_6 = d_8 = 0$ (but then we can disregard customers 4, 6 and 8)