Logistics

Lecture notes*

Maria Grazia Scutellà

Dipartimento di Informatica
Università di Pisa

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Chapter 3

Introduction to Mixed Integer Linear Problems

Many business problems need integer solutions (e.g. we need to decide how many employees to assign to each shift, how many vehicles to purchase...), so it is useful to introduce a new type of problem called *(Mixed) Integer Linear Programming ((M)ILP)*: it is a linear programming problem where certain decision variables must assume only integer values. Hereafter we will often use ILP also to denote mixed integer linear problems.

For example, let's resume the Blue Ridge Hot Tubs problem from the previous chapter:

$$\begin{aligned} \max\ 350x_1 &+ 300x_2 \\ x_1 + \quad & x_2 \leq 200 \\ 9x_1 + \quad & 6x_2 \leq 1{,}566\ (1{,}520) \\ 12x_1 + \quad & 16x_2 \leq 2{,}880\ (2{,}650) \\ x_1\ , \quad & x_2 \geq 0 \\ x_1\ , \quad & x_2 \text{ integer} \end{aligned}$$

whose optimal solution was already integer ($x_1^* = 122, x_2^* = 78$). However, by changing the two right hand sides (RHS) as in parentheses, we would get $x_1^* = 116.9444, x_2^* = 77.9167$: so, integrality constraints have to be added to the model (from LP to ILP).

How can we address ILP?

1. Solving its *Linear Relaxation*, i.e. eliminating the integrality constraints, and then rounding the obtained solution, e.g. $\tilde{x}_1 = 116,\ \tilde{x}_2 = 77$ (with profit = 63,700). However, this is not necessarily an optimum solution to the ILP. Notice that the optimum LP value, that is 64,306, is an upper bound to the optimum ILP value (for a maximization problem).

2. Solving directly the ILP model (e.g. via the Excel solver).

3.1 Other examples

3.1.1 A fixed-charge problem

Remington Manufacturing is planning its next production cycle. They produce 3 products (say Product 1, Product 2 and Product 3), each of which must undergo machining, grinding and assembly operations to be completed. Table 3.1 summarizes the hours required by each unit of product, and the total hours available for each operation.

However, manufacturing units of Product 1 requires a setup operation on the production line that costs 1,000; similarly for Product 2 (800) and Product 3 (900) (*fixed-charge costs*). In order to solve the optimization problem we must determine the most profitable mix of products to produce.

Define the following decision variables:

- x_i: amount of Product i to be produced, $i = 1, 2, 3$;
- $y_i = \begin{cases} 1 & \text{if } x_i > 0 \\ 0 & \text{otherwise} \end{cases}, i = 1, 2, 3.$

  The y_i are auxiliary binary variables, which will be used to express logical conditions.

Using these variables, an ILP model for Remington Manufacturing is:

$$
\begin{array}{lll}
\max\ 48x_1 + 55x_2 + 50x_3 - 1000y_1 - 800y_2 - 900y_3 & & \\
2x_1 + 3x_2 + 6x_3 \le 600 & & \\
6x_1 + 3x_2 + 4x_3 \le 300 & & \\
5x_1 + 6x_2 + 2x_3 \le 400 & & \\
x_1 \le \textcircled{50}y_1 & & \text{linking} \\
x_2 \le \textcircled{67}y_2 & & \text{constraints} \\
x_3 \le \textcircled{75}y_3 & & \\
x_i \ge 0, & & i = 1, 2, 3 \\
y_i \in \{0, 1\}, & & i = 1, 2, 3
\end{array}
$$

	Prod 1	Prod 2	Prod 3	total hours
machining	2	3	6	600
grinding	6	3	4	300
assembly	5	6	2	400
unitary profit	48	55	50	

Table 3.1: Production requirements and availability for Remington Manufacturing

In the linking constraints, the circled values (let us call them M) are upper bounds on the optimal values of the x_i. The models are much easier to solve if M is kept as small as possible: therefore we can set $M = 60$ in the first linking constraint (no more than 60 units of Product 1 can be produced), and to 67 and 75 in the remaining cases.

In order to implement the Remington Manufacturing model as a spreadsheet model, we refer the interested reader to C. Ragsdale (2004): Section 6.13.5 and to C. Ragsdale (2004): Section 6.12, 6.13.

3.1.2 A transportation problem

B&G Construction is a commercial building company that signed contracts to construct four buildings. Each building project requires a large amount of cement to be delivered to the building sites, and three cement companies have submitted bids for supplying the cement. Table 3.2 summarizes the prices for delivered ton of cement and the maximum amount of cement each of the three companies may deliver.

	cost per ton				**max supply**
	Proj. 1	Proj. 2	Proj. 3	Proj. 4	
Company 1	120	115	130	125	525
Company 2	100	150	110	105	450
Company 3	140	95	145	165	550
tons needed	450	275	300	350	

Table 3.2: Cement requirements and costs for B&G Construction

However constraints get more complicated, because each cement company placed special conditions on its bid:

1. Company 1 will not supply orders of less than 150 tons for any project;
2. Company 2 can supply more than 200 tons to no more than one of the projects;
3. Company 3 will accept only orders that total 200, 400 or 550 tons (or 0, i.e. no order).

The optimization problem is to determine the amounts of cement to purchase from each supplier to meet the demands for each project, by satisfying the complicating constraints, at a minimum cost. This is a *transportation problem* with additional constraints.

Define the decision variables as follows:

- x_{ij}: tons of cement supplied from company i for project j;
- $y_{1j} \in \{0, 1\},\ j = 1, \dots 4$ for the additional constraint of Company 1;
- $y_{2j} \in \{0, 1\},\ j = 1, \dots 4$ for the additional constraint of Company 2;

- $y_{3k} \in \{0, 1\},\ k = 1, 2, 3$ for the additional constraint of Company 3.

The objective function, to be minimized, is the sum of the amounts of cement supplied by each company for each project, multiplied by their associated unitary costs, to be minimized:

$$\begin{aligned} \min\ & 120x_{11} + 115x_{12} + 130x_{13} + 125x_{14} + \\ & + 100x_{21} + 150x_{22} + 110x_{23} + 105x_{24} + \\ & + 140x_{31} + 95x_{32} + 145x_{33} + 165x_{34}. \end{aligned}$$

Each company may supply a total maximum amount of cement:

$$\begin{aligned} x_{11} + x_{12} + x_{13} + x_{14} &\le 525 \\ x_{21} + x_{22} + x_{23} + x_{24} &\le 450 \\ x_{31} + x_{32} + x_{33} + x_{34} &\le 550. \end{aligned}$$

Each project must be supplied with a specific total amount of cement from any of the companies:

$$\begin{aligned} x_{11} + x_{21} + x_{31} &= 450 \\ x_{12} + x_{22} + x_{32} &= 275 \\ x_{13} + x_{23} + x_{33} &= 300 \\ x_{14} + x_{24} + x_{34} &= 350. \end{aligned}$$

The additional constraint placed by Company 1 on its bid can be expressed as follows: "if $x_{1j} > 0$, then the associated boolean variable y_{1j} must be 1":

$$\begin{aligned} x_{11} &\le 525y_{11} \\ x_{12} &\le 252y_{12} \\ x_{13} &\le 525y_{13} \\ x_{14} &\le 525y_{14}; \end{aligned}$$

and also "if $y_{1j} = 1$ then x_{1j} must be greater than or equal to 150":

$$\begin{aligned} x_{11} &\ge 150y_{11} \\ x_{12} &\ge 150y_{12} \\ x_{13} &\ge 150y_{13} \\ x_{14} &\ge 150y_{14}. \end{aligned}$$

The additional constraint placed by Company 2 on its bid ("more than 200 tons to no more than one project") can be expressed as follows:

$$
\begin{aligned}
x_{21} \qquad\qquad\qquad\qquad & \leq 200 + 250y_{21} \\
x_{22} \qquad\qquad & \leq 200 + 250y_{22} \\
x_{23} \qquad & \leq 200 + 250y_{23} \\
x_{24} & \leq 200 + 250y_{24} \\
y_{21} + y_{22} + y_{23} + y_{24} & \leq 1.
\end{aligned}
$$

The total amount of cement ordered from Company 3 can be only 200 (if $y_{31} = 1$), 400 (if $y_{32} = 1$), 550 (if $y_{33} = 1$) or 0 (if $y_{31} = y_{32} = y_{33} = 0$):

$$
\begin{aligned}
x_{31} + x_{32} + x_{33} + x_{34} &= 200y_{31} + 400y_{32} + 550y_{33} \\
y_{31} + y_{32} + y_{33} &\leq 1.
\end{aligned}
$$

The overall ILP model is presented below. In order to implement the B&G Construction model as a spreadsheet model, we refer the interested reader to C. Ragsdale (2004): Section 6.16.

References C. Ragsdale (2004): Chapter 6

$$
\begin{aligned}
\min\ & 120x_{11} + 115x_{12} + 130x_{13} + 125x_{14} + \\
& + 100x_{21} + 150x_{22} + 110x_{23} + 105x_{24} + \\
& + 140x_{31} + 95x_{32} + 145x_{33} + 165x_{34} \\
& x_{11} + x_{12} + x_{13} + x_{14} \leq 525 \\
& x_{21} + x_{22} + x_{23} + x_{24} \leq 450 \\
& x_{31} + x_{32} + x_{33} + x_{34} \leq 550 \\
& x_{11} + x_{21} + x_{31} = 450 \\
& x_{12} + x_{22} + x_{32} = 275 \\
& x_{13} + x_{23} + x_{33} = 300 \\
& x_{14} + x_{24} + x_{34} = 350 \\
& x_{11} \leq 525y_{11} \\
& x_{12} \leq 525y_{12} \\
& x_{13} \leq 525y_{13} \\
& x_{14} \leq 525y_{14} \\
& x_{11} \geq 150y_{11} \\
& x_{12} \geq 150y_{12} \\
& x_{13} \geq 150y_{13} \\
& x_{14} \geq 150y_{14} \\
& x_{21} \leq 200 + 250y_{21} \\
& x_{22} \leq 200 + 250y_{22} \\
& x_{23} \leq 200 + 250y_{23} \\
& x_{24} \leq 200 + 250y_{24} \\
& y_{21} + y_{22} + y_{23} + y_{24} \leq 1 \\
& x_{31} + x_{32} + x_{33} + x_{34} = 200y_{31} + 400y_{32} + 550y_{33} \\
& y_{31} + y_{32} + y_{33} \leq 1 \\
& x_{ij} \geq 0 \quad i = 1,2,3,\ j = 1,2,3,4 \\
& y_{1j} \in \{0,1\} \quad j = 1,2,3,4 \\
& y_{2j} \in \{0,1\} \quad j = 1,2,3,4 \\
& y_{3k} \in \{0,1\} \quad k = 1,2,3
\end{aligned}
$$

Model 3.1: The B&G problem

Textbooks

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