Logistics

Lecture notes[*]

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Chapter 2

Introduction to modeling: Linear Programming

Our approach to address and solve logistics decision problems is the following:

1. to formulate a suitable mathematical model to represent the decision problem;

2. to implement and solve the model.

See Figure 2.1 for a graphical representation of the considered problem solving approach. Precisely, we move within *Management Science*, and specifically *Operations Research*, field of study which uses computer science, mathematics and statistics to solve decision problems.

Today, electronic spreadsheets provide a simple and useful way for business people to implement a model and analyze decision alternatives, although many other, more sophisticated and powerful solvers exist: *spreadsheet models* (i.e. models implemented via a spreadsheet) will be used hereafter.
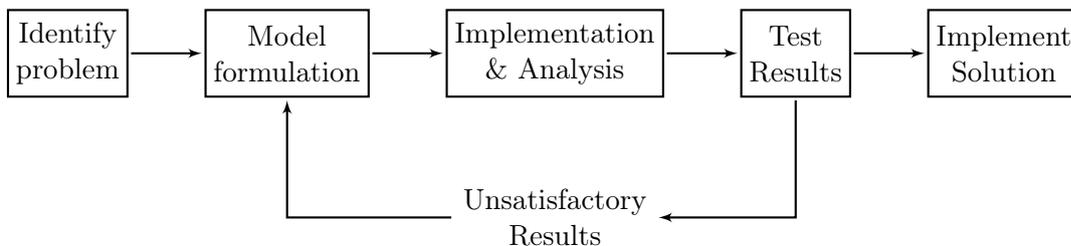
Figure 2.1: The problem solving approach

2.1   Optimization problems

The addressed problems consist in deciding how to use the limited resources available in an efficient way. Typically, this must be accomplished by maximizing profits or minimizing costs: i.e. the addressed problems are optimization problems.

Mathematical Programming (MP) is the area of Operations Research aiming at modeling and solving optimization problems. Its applications include, among the others, manufacturing, financial planning and logistics.

In any case, an optimization problem involves:

- *decisions* to be taken (e.g. how much of each product should be produced, shipped, etc.);

- restrictions, or *constraints*, to be placed on the alternatives available to the decision maker (e.g. limited amount of raw materials and labor when producing);

- the goal, or *objective*, to be considered by the decision maker when deciding (e.g. to choose the mix of products that maximizes profits).

2.1.1   Expressing optimization problems via mathematical models

How can we mathematically represent decisions, constraints and objective?

- *decisions* are represented by decision variables: $x_1, x_2, \ldots x_n$;

- *constraints* are formulated by (in)equalities:
$$f(x_1, x_2, \ldots x_n) \leq b \quad \text{or}$$
$$f(x_1, x_2, \ldots x_n) \geq b \quad \text{or}$$
$$f(x_1, x_2, \ldots x_n) = b$$

- the *objective* is modelled by an objective function to be maximized or minimized: $\max f(x_1, x_2, \ldots x_n)$ or $\min f(x_1, x_2, \ldots x_n)$.

Therefore, the general mathematical model for an optimization problem is:

$$\max / \min \ f_0(x_1, x_2, \ldots x_n) \quad \text{subject to}$$
$$\begin{cases} f_1(x_1, x_2, \ldots x_n) \leq b_1 \\ \vdots \\ f_k(x_1, x_2, \ldots x_n) \geq b_k \\ \vdots \\ f_m(x_1, x_2, \ldots x_n) = b_m \end{cases}$$

2.2 Linear Programming

Linear Programming can be used to model and solve very complicated business problems, saving companies millions of dollars. A Linear Programming (LP) problem is a special case of the mentioned optimization problem where constraints and objective function are linear, i.e. $f_0, f_1, \ldots f_m$ are weighted sums (linear combinations) of the decision variables.

An LP problem has, therefore, the following general form:

$$
\max / \min \ c_1 x_1 + c_2 x_2 + \cdots + c_n x_n
$$
$$
\begin{cases}
a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \leq b_1 \\
\vdots \\
a_{k1} x_1 + a_{k2} x_2 + \cdots + a_{kn} x_k \geq b_k \\
\vdots \\
a_{m1} x_1 + a_{m2} x_2 + \ldots a_{mn} x_n = b_m
\end{cases}
\tag{2.1}
$$

where:

- $c_1, \ldots c_n$ are the objective function (cost) coefficients; they represent the marginal profits (or costs) associated with the variables;

- a_{ij} denotes the coefficient in constraint i for variable x_j, $i = 1 \ldots m, j = 1 \ldots n$.

2.2.1 A simple LP example

"Blue Ridge Hot Tubs" produces and sells two types of hot tubs: "Aqua-Spa" and "Hydro-Lux". The manager buys hot tub shells from a local supplier, and then adds pumps and tubing to the shells to create hot tubs.

The resources available in the next production cycle are:

- 200 pumps;

- 2,880 feet of tubing;

- 1,566 production labor hours.

The operating requisites are:

- each Aqua-Spa requires 12 feet of tubing and 9 hours of labor;

- each Hydro-Lux requires 16 feet of tubing and 6 hours of labor.

The profit for the manager is 350 on each Aqua-Spa he sells and 300 on each Hydro-Lux he sells (the manager is confident to sell each hot tub he/she produces).

The decision problem of Blue Ridge Hot Tubs can be stated in the following way: how many Aqua-Spa and Hydro-Lux are to be produced, taking into account the limited

resources and the operating requisites, so as to maximize the profit during the next production cycle?

In order to express the problem via an LP model, we follow these steps:

- identify the decision variables: what are the fundamental decisions that must be made to solve the problem?

  - x_1: number of Aqua-Spa hot tubs to produce;

  - x_2: number of Hydro-Lux hot tubs to produce;

- state the constraints as linear combinations of the decision variables:

  - only 200 pumps are available, and each hot tub requires one pump:

  $$x_1 + x_2 \leq 200;$$

  - only 1,566 labor hours are available, and each Aqua-Spa requires 9 labor hours while each Hydro-Lux requires 6 labor hours:

  $$9x_1 + 6x_2 \leq 1{,}566;$$

  - only 288 feet of tubing is available, and each Aqua-Spa requires 12 feet while each Hydro-Lux requires 16 feet:

  $$12x_1 + 16x_2 \leq 2{,}880;$$

- state the objective function as a linear combination of the decision variables: the manager has a profit of 350 on each Acqua-Spa he/she sells, and of 300 on each Hydro-Lux he/she sells; therefore the total profit, to be maximized, is

$$\max\ 350x_1 + 300x_2;$$

- nonnegativity constraints: we cannot produce a negative number of hot tubs:

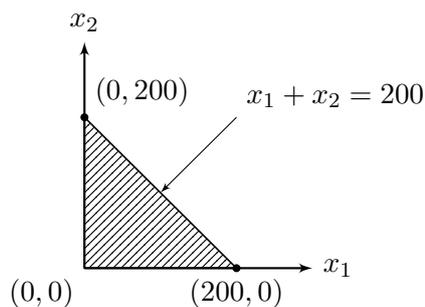$$x_1 \geq 0,\ x_2 \geq 0.$$

The overall LP model is therefore:

$$
\begin{aligned}
\max\ 350x_1 + 300x_2 \\
x_1 +\quad x_2 &\leq 200 \\
9x_1 +\quad 6x_2 &\leq 1{,}566 \\
12x_1 +\quad 16x_2 &\leq 2{,}880 \\
x_1 \qquad &\geq 0 \\
x_2 &\geq 0
\end{aligned}
$$

### 2.2.2 Solving LP problems: an intuitive approach

How can we compute the best (i.e. optimal) solution? In the case of only two decision variables, we can use a graphical approach (see G. Bigi, A. Frangioni, G. Gallo, and M. Scutellà ((2014)) for more rigorous mathematical approaches, based on the LP Duality Theory):

1. Plot the constraints and identify the LP *feasible region*; this is done by plotting the "boundary lines" of the constraints and identifying the points (x_1, x_2) which satisfy all the constraints.

   For example, the boundary of the first constraint in the Blue Ridge Hot Tubs model is the straight line defined by $x_1 + x_2 = 200$:


   The shaded area is the region satisfying the first constraint $(x_1 + x_2 \leq 200)$ and $x_1 \geq 0$, $x_2 \geq 0$. However, feasible points must also satisfy the other constraints. So, plot the boundary line for the second constraint, i.e. $9x_1 + 6x_2 = 1{,}566$:
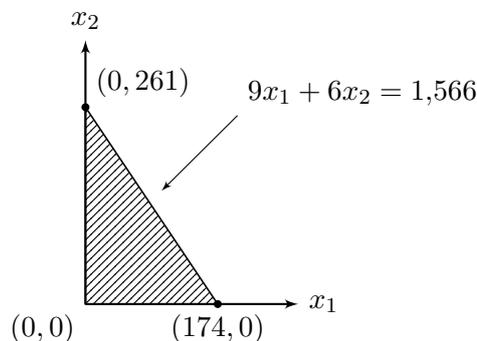
   Since we have to satisfy both constraints:
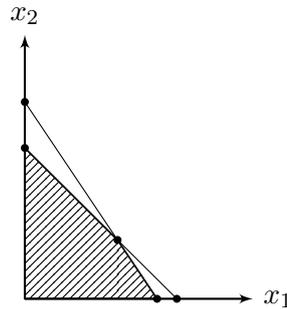
By intersecting the two regions we obtain the region of the graph satisfying the first two constraints. By plotting the last constraint (we already addressed the nonnegativity ones), we get:
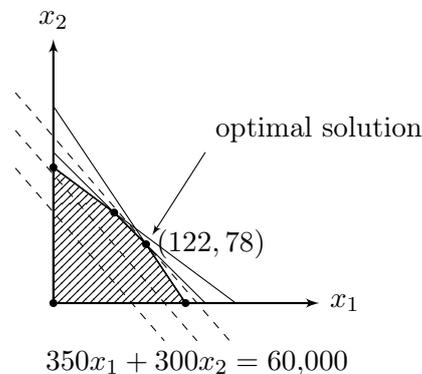
As it is possible to observe, there is an infinite number of feasible solutions: how can we find the best one? Fortunately, the following property holds: *if an LP problem has an optimal solution, there exists an optimal solution occurring at a* corner point *or* extreme point *of the feasible region* (black dots on the graph). Since the number of corner points is finite, we can find an optimal solution by comparing the objective function value of such corner points.

2. How can we discover such an optimal solution? Consider the objective function, and fix a level of profit, e.g. 60,000:

$$350x_1 + 300x_2 = 60{,}000.$$

This equation defines a straight line, which we can plot on the graph.

If we iterate this process of drawing lines with larger and larger value of the objective function (profit), we obtain a set of parallel lines (*level curves*) shifting away from the origin: the very last level curve which still intersects the feasible region is an optimal solution (extreme point).

In our example, $(x_1^*, x_2^*) = (122, 78)$ is an optimal solution, with maximum profit 66,100.
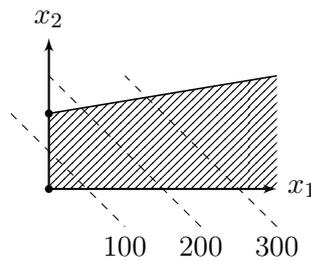
2.2.3 Other possible outcomes in solving LPs

Other possible outcomes in LP solving are:

1. Alternate optimal solutions; e.g. if we want to maximize x_1 (the dashed line in the figure represents a corresponding level curve) over the feasible region depicted below:
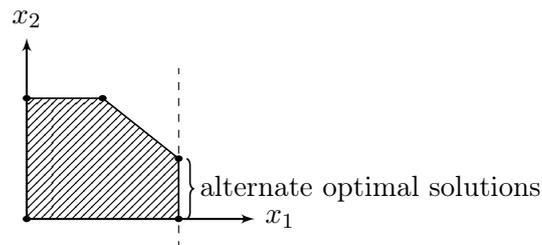
2. Unbounded optimal solutions; e.g. if we want to maximize $x_1 + x_2$ over the feasible region depicted below:


   In this case, the objective function value can be made infinitely large (for a maximization problem). In practice, this usually indicates that there is something wrong with the problem formulation.

3. Infeasibility; for example:

$$\begin{aligned}
\max\ &x_1 + x_2 \\
&x_1 + x_2 \le 150 \\
&x_1 + x_2 \ge 200 \\
&x_1 \qquad\ \ge 0 \\
&\qquad x_2 \ge 0
\end{aligned}$$

   This may be due to an error in the problem formulation; some constraints have to be eliminated or loosened to get feasible solutions.

Finally, observe that *redundant constraints* may be present. A constraint is redundant if it has no role in determining the feasible region, therefore we can remove it from the model.
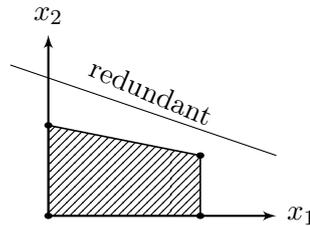
References   C. Ragsdale (2004): Chapter 2

2.3   Solving LP models via spreadsheets

Solving LP problems is an easy task by using spreadsheet packages, for example Excel: just formulate the problem correctly and implement it in an accurate way. The same concepts and techniques apply to spreadsheet packages other than Excel. Furthermore, specialized, more powerful optimization packages do exist: Cplex, Lindo, Coin...

In order to implement the Blue Ridge Hot Tubs model as a spreadsheet model, we refer the interested reader to C. Ragsdale (2004): Sections 3.3 to 3.6.

References   C. Ragsdale (2004): Chapter 3

### 2.3.1   A production and inventory planning problem

"Upton Corporation" produces air compressors. The manager wants to plan its production and inventory levels for the next 6 months. Table 2.1 summarizes the monthly production costs, demands and production capacities that are expected over the next 6 months.

	1	**2**	**3**	**4**	**5**	**6**
unit production cost	240	250	265	285	280	260
units demanded	1,000	4,500	6,000	5,500	3,500	4,000
maximum production	4,000	3,500	4,000	4,500	4,000	3,500

Table 2.1: Production costs, demands and capacities in the next six months for Upton Corporation

The following constraints must be satisfied:

- size of the warehouse: a maximum of 6,000 units can be held in inventory at the end of each month;

- safety stock: at least 1,500 units must be held in inventory at the end of each month, to meet unexpected demand;
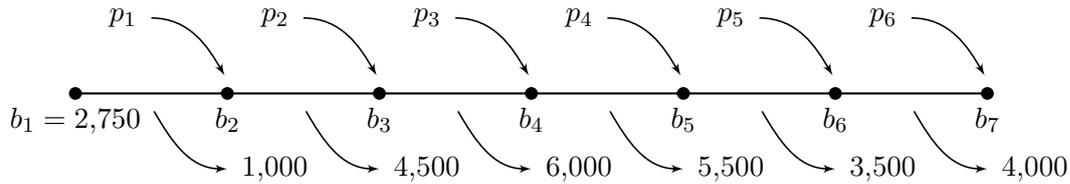
Figure 2.2: Time horizon planning

- current inventory: 2,750 units;

- stable workforce: the production must be greater than or equal to a half of the monthly production capacity;

- inventory costs: the company estimates that the cost of carrying in inventory a unit in any month is about 1.5 % of the unit production cost in the same month;

- Upton estimates the number of units carried in inventory each month as the average of the initial and final inventory level in the corresponding month.

The problem of Upton Corporation is to determine the levels of production and inventory for the next 6 months so as to satisfy the monthly demands by minimizing the overall (production + inventory) costs. This is an example of a simple lot-sizing model.

In order to state the problem, let us define the following decision variables:

- p_i: number of units produced in month i, $i = 1 \ldots 6$;

- b_i: initial inventory level for month i, $i = 1 \ldots 6$ (while b_7 denotes the final inventory level).

Figure 2.2 shows a schematic representation of the problem of Upton Corporation. For each month i, b_i represents its starting inventory level, the incoming arrow represents the production level in month i and the outgoing arrow represents the required monthly demand in month i.

Based on these decision variables, we can state the Upton Corporation problem via the following linear constraints:

- the production level of each month must not exceed the production capacity and must be greater than or equal to a half of it:

$$2{,}000 \leq p_1 \leq 4{,}000$$
$$1{,}750 \leq p_2 \leq 3{,}500$$
$$2{,}000 \leq p_3 \leq 4{,}000$$
$$2{,}250 \leq p_4 \leq 4{,}500$$
$$2{,}000 \leq p_5 \leq 4{,}000$$
$$1{,}750 \leq p_6 \leq 3{,}500;$$

- the inventory levels at the end of each month must lay between 1,500 and 6,000:

$$1{,}500 \leq b_2 \leq 6{,}000$$
$$1{,}500 \leq b_3 \leq 6{,}000$$
$$1{,}500 \leq b_4 \leq 6{,}000$$
$$1{,}500 \leq b_5 \leq 6{,}000$$
$$1{,}500 \leq b_6 \leq 6{,}000$$
$$1{,}500 \leq b_7 \leq 6{,}000;$$

- the relationship among production variables, inventory variables and monthly demands can be expressed as follows:

$$b_2 = b_1 + p_1 - 1{,}000$$
$$b_3 = b_2 + p_2 - 4{,}500$$
$$b_4 = b_3 + p_3 - 6{,}000$$
$$b_5 = b_4 + p_4 - 5{,}500$$
$$b_6 = b_5 + p_5 - 3{,}500$$
$$b_7 = b_6 + p_6 - 4{,}000;$$

- the initial inventory level is 2,750:

$$b_1 = 2{,}750.$$

The goal is to minimize the total cost. There are two kinds of cost: production costs and inventory costs. The total production cost is obtained simply by multiplying the unit production costs by the number of units produced:

$$240p_1 + 250p_2 + 265p_3 + 285p_4 + 280p_5 + 260p_6.$$

Inventory costs in each month are estimated as the $1.5\,\%$ of the unit production cost multiplied by the average number of units in inventory in that month:

$$3.6\frac{b_1 + b_2}{2} + 3.75\frac{b_2 + b_3}{2} + 3.98\frac{b_3 + b_4}{2} + 4.28\frac{b_4 + b_5}{2} + 4.20\frac{b_5 + b_6}{2} + 3.9\frac{b_6 + b_7}{2}.$$

Putting all together, the overall LP model is the one presented in Model 2.1. In order to implement the Upton Corporation model as a spreadsheet model, we refer the interested reader to C. Ragsdale (2004): Section 3.12.

$$
\min \overbrace{240}^{\text{production cost}} p_1 + 250p_2 + 265p_3 + 285p_4 + 280p_5 + 260p_6 +
$$
$$
+ 3.6(b_1 + b_2)/2 + 3.75(b_2 + b_3)/2 + 3.98(b_3 + b_4)/2 +
$$
$$
+ 4.28(b_4 + b_5)/2 + 4.20(b_5 + b_6)/2 + \underbrace{3.9}_{\text{inventory cost (1.5\% of 260)}} (b_6 + b_7)/2
$$

$2{,}000 \le p_1 \le 4{,}000$

$1{,}750 \le p_2 \le 3{,}500$ constraints on

$2{,}000 \le p_3 \le 4{,}000$ monthly production

$2{,}250 \le p_4 \le 4{,}500$ levels

$2{,}000 \le p_5 \le 4{,}000$

$1{,}750 \le p_6 \le 3{,}500$

$1{,}500 \le b_2 \le 6{,}000$

$1{,}500 \le b_3 \le 6{,}000$ constraints on

$1{,}500 \le b_4 \le 6{,}000$ monthly final

$1{,}500 \le b_5 \le 6{,}000$ inventory levels

$1{,}500 \le b_6 \le 6{,}000$

$1{,}500 \le b_7 \le 6{,}000$

$b_2 = b_1 + p_1 - 1{,}000$

$b_3 = b_2 + p_2 - 4{,}500$ relationship

$b_4 = b_3 + p_3 - 6{,}000$ between initial

$b_5 = b_4 + p_4 - 5{,}500$ and final inventory

$b_6 = b_5 + p_5 - 3{,}500$ levels + demand

$b_7 = b_6 + p_6 - 4{,}000$ satisfaction

$b_1 = 2{,}750$

Model 2.1: The Upton Corporation problem

Textbooks

Bigi, G., A. Frangioni, G. Gallo, and M.G. Scutellà (2014). *Appunti di Ricerca Operativa*. Italian. URL: http://didawiki.di.unipi.it/doku.php/ingegneria/ricercaoperativa1/start#testi_di_riferimento.

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