PSC 2023/24 (375AA, 9CFU)

Principles for Software Composition

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13a - Cartesian domains

HOFL
Towards a denotational semantics

Imp

three syntactic categories (types) $Aexp, Bexp, Com$

one interpretation function each

$$\mathcal{A}[\![\cdot]\!] : Aexp \to \Sigma \to \mathbb{Z}$$

$$\mathcal{B}[\![\cdot]\!] : Bexp \to \Sigma \to \mathbb{B}$$

$$\mathcal{C}[\![\cdot]\!] : Com \to \Sigma \to \Sigma_{\perp}$$

$$\Sigma \triangleq Ide \to \mathbb{Z}$$

environment

semantic values

meaning of identifiers

syntax

one semantic domain each

HOFL

one syntactic category for pre-terms T

infinitely many types $\tau \ ::= \ int \ | \ \tau_0 * \tau_1 \ | \ \tau_0 \to \tau_1$

infinitely many categories for typeable terms T_τ

one semantic domain each D_τ

one parametric interpretation function $[\![\cdot]\!]$

variables also have different types $x : \tau$

the environment must be type-sensitive ρ

Requirements

$t : \tau$ $\quad\quad$ $[\![t]\!]\rho \in D_\tau$ — a domain for each type!

environment $\rho : Var \to \bigcup_{\tau \in \mathcal{T}} D_\tau$

type consistent assignment of values to variables $x : \tau \Rightarrow \rho(x) \in D_\tau$

t may diverge (e.g. $\mathbf{rec}\ x.\ x$) $\Rightarrow$ D_τ must include a bottom element $\bot_{D_\tau}$

Requirements

$t : \tau$ $\qquad \llbracket t \rrbracket \rho \in D_\tau$ — a domain for each type!

environment $\quad \rho : Var \to \bigcup_{\tau \in \mathcal{T}} D_\tau$

type consistent assignment of values to variables $\quad x : \tau \Rightarrow \rho(x) \in D_\tau$

$$\llbracket \mathbf{rec}\ x.\ t \rrbracket \rho = \llbracket t \rrbracket \rho[{}^{\llbracket \mathbf{rec}\ x.\ t \rrbracket \rho}/_{x}]$$

$$\Gamma_{x,t} \triangleq \lambda d.\ \llbracket t \rrbracket \rho[{}^{d}/_{x}]$$

$$\llbracket \mathbf{rec}\ x.\ t \rrbracket \rho = \Gamma_{x,t}\ (\llbracket \mathbf{rec}\ x.\ t \rrbracket \rho)$$

$$\llbracket \mathbf{rec}\ x.\ t \rrbracket = \mathit{fix}\ \Gamma_{x,t}$$

to solve recursive equations:

D_τ must be a $\mathrm{CPO}_\perp$

$\Gamma_{x,t}$ must be continuous

Requirements

a domain for each type!

$t : \tau$ $\quad$ $[\![t]\!]\rho \in D_\tau$

environment $\quad \rho : Var \to \bigcup_{\tau \in \mathcal{T}} D_\tau$

type consistent assignment of values to variables $\quad x : \tau \Rightarrow \rho(x) \in D_\tau$

$$\tau \ ::= \ int \ \mid \ \tau_0 * \tau_1 \ \mid \ \tau_0 \to \tau_1$$

we must be able to combine $\mathrm{CPO}_\perp$

using cartesian product and function spaces

Requirements

a domain for each type!

$t : \tau \qquad [\![t]\!]\rho \in D_\tau$

environment $\rho : Var \to \bigcup_{\tau \in \mathcal{T}} D_\tau$

type consistent assignment of values to variables $x : \tau \Rightarrow \rho(x) \in D_\tau$

$$\tau \ ::= \ int \ \mid \ \tau_0 * \tau_1 \ \mid \ \tau_0 \to \tau_1$$

choose D_{int}

given D_{τ_0}, D_{τ_1} build $D_{\tau_0 * \tau_1}$ $D_{\tau_0 \to \tau_1}$

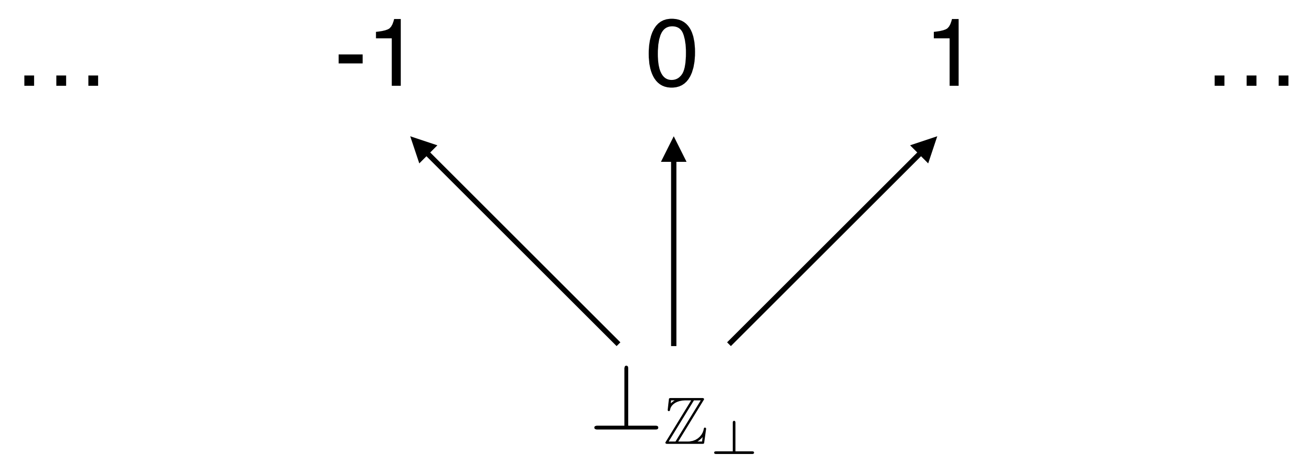
Flat domain of Integers

Flat domain of Integers

$\mathbb{Z}$ ... -1 0 1 ...

Flat domain of Integers

$\mathbb{Z}_\perp$



PO: flat order

bottom: any flat order has bottom

completeness: any flat order is complete
(only finite chains are possible)

Strict extensions

$\mathsf{op} \in \{+, - \times\}$

$\mathsf{op} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$

$\underline{\mathsf{op}}_{\perp} : \mathbb{Z}_{\perp} \times \mathbb{Z}_{\perp} \to \mathbb{Z}_{\perp}$

$1 \underline{+}_{\perp} \perp = \perp$

$\perp \underline{\times}_{\perp} 5 = \perp$

$\perp \underline{-}_{\perp} \perp = \perp$

$$v_1 \ \underline{\mathsf{op}}_{\perp} \ v_2 \triangleq \begin{cases} v_1 \ \underline{\mathsf{op}} \ v_2 & \text{if } v_1, v_2 \in \mathbb{Z} \\ \perp_{\mathbb{Z}_{\perp}} & \text{otherwise } (v_1 = \perp_{\mathbb{Z}_{\perp}} \text{ or } v_2 = \perp_{\mathbb{Z}_{\perp}}) \end{cases}$$

called *strict* extension

to prove: $\underline{\mathsf{op}}_{\perp}$ is monotone and continuous

is $\mathbb{Z}_{\perp} \times \mathbb{Z}_{\perp}$ a $\mathrm{CPO}_{\perp}$?

Cartesian product of domains

Cartesian product

$\mathcal{D} = (D, \sqsubseteq_D)$

$\mathcal{E} = (E, \sqsubseteq_E)$

$$\text{CPO}_\perp \quad \Rightarrow \quad \mathcal{D} \times \mathcal{E} = (\ D \times E\ ,\ \sqsubseteq_{D \times E}\ )$$

how to order pairs?

$$(d_0, e_0) \sqsubseteq_{D \times E} (d_1, e_1) \quad \text{iff} \quad d_0 \sqsubseteq_D d_1 \wedge e_0 \sqsubseteq_E e_1$$

example $\mathbb{Z}_\perp \times \mathbb{Z}_\perp$

$$(0, 1) \overset{?}{\sqsubseteq}_{\mathbb{Z}_\perp \times \mathbb{Z}_\perp} (1, 2)$$ ✗

$$(\perp_{\mathbb{Z}_\perp}, 1) \overset{?}{\sqsubseteq}_{\mathbb{Z}_\perp \times \mathbb{Z}_\perp} (1, 1)$$ ✓

$$(2, \perp_{\mathbb{Z}_\perp}) \overset{?}{\sqsubseteq}_{\mathbb{Z}_\perp \times \mathbb{Z}_\perp} (2, 0)$$ ✓

$$(0, \perp_{\mathbb{Z}_\perp}) \overset{?}{\sqsubseteq}_{\mathbb{Z}_\perp \times \mathbb{Z}_\perp} (\perp_{\mathbb{Z}_\perp}, 0)$$ ✗

Cartesian CPO

$$\mathcal{D} \times \mathcal{E} = (\ D \times E\ ,\ \sqsubseteq_{D \times E}\ )$$

is it a partial order?

reflexivity, antisymmetry, transitivity of $\sqsubseteq_{D \times E}$
follow immediately from those of $\sqsubseteq_D$ $\sqsubseteq_E$

is there a bottom element?

let $\bot_{D \times E} = (\bot_D, \bot_E)$

take any pair $(d, e) \in D \times E$

since $\bot_D \sqsubseteq_D d$ and $\bot_E \sqsubseteq_E e$ then $\bot_{D \times E} = (\bot_D, \bot_E) \sqsubseteq_{D \times E} (d, e)$

Cartesian CPO (ctd)

$$\mathcal{D} \times \mathcal{E} = (\ D \times E\ ,\ \sqsubseteq_{D \times E}\ )$$

is it complete?

take a chain $\{(d_i, e_i)\}_{i \in \mathbb{N}}$ we need to find its lub

we prove its lub is $\left(\bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right)$

1. it is an upper bound of the chain
2. it is smaller than or equal to any other upper bound

Cartesian CPO (ctd)

$\mathcal{D} \times \mathcal{E} = (D \times E , \sqsubseteq_{D \times E})$ take a chain $\{(d_i, e_i)\}_{i \in \mathbb{N}}$

1. $\left(\bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right)$ is an upper bound of the chain

take a generic element of the chain (d_j, e_j)

we have $d_j \sqsubseteq_D \bigsqcup_{i \in \mathbb{N}} d_i$ $e_j \sqsubseteq_E \bigsqcup_{i \in \mathbb{N}} e_i$ thus $(d_j, e_j) \sqsubseteq_{D \times E} \left(\bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right)$

Cartesian CPO (ctd)

$\mathcal{D} \times \mathcal{E} = (D \times E , \sqsubseteq_{D \times E})$ take a chain $\{(d_i, e_i)\}_{i \in \mathbb{N}}$

2. $\left(\bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right)$ is the least among upper bounds

take a generic upper bound (d, e): $\forall i \in \mathbb{N}.\ (d_i, e_i) \sqsubseteq_{D \times E} (d, e)$

by def $\forall i \in \mathbb{N}.\ d_i \sqsubseteq_D d \ \wedge \ \forall i \in \mathbb{N}.\ e_i \sqsubseteq_E e$

i.e., d is an upper bound of $\{d_i\}_{i \in \mathbb{N}}$

e is an upper bound of $\{e_i\}_{i \in \mathbb{N}}$

$\Rightarrow$ $\bigsqcup_{i \in \mathbb{N}} d_i \sqsubseteq_D d$, $\bigsqcup_{i \in \mathbb{N}} e_i \sqsubseteq_E e$

hence $\left(\bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right) \sqsubseteq_{D \times E} (d, e)$

Cartesian CPO: recap

$$\mathcal{D} \times \mathcal{E} = (\ D \times E\ ,\ \sqsubseteq_{D \times E}\ )$$

$$(d_0, e_0) \sqsubseteq_{D \times E} (d_1, e_1) \quad \text{iff} \quad d_0 \sqsubseteq_D d_1 \wedge e_0 \sqsubseteq_E e_1$$

$$\bot_{D \times E} \triangleq (\bot_D, \bot_E)$$

$$\bigsqcup_{i \in \mathbb{N}} (d_i, e_i) \triangleq \left(\bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right)$$

is $\mathbb{Z}_\bot \times \mathbb{Z}_\bot$ a $\mathrm{CPO}_\bot$? ✓

Projections

$\pi_1 : D \times E \to D$

$\pi_1(d, e) = d$

$\pi_2 : D \times E \to E$

$\pi_2(d, e) = e$

TH. projections are monotone

proof. take $(d_0, e_0) \sqsubseteq_{D \times E} (d_1, e_1)$

we want to prove $\pi_1(d_0, e_0) \sqsubseteq_D \pi_1(d_1, e_1)$

$\pi_2(d_0, e_0) \sqsubseteq_E \pi_2(d_1, e_1)$

$$\pi_1(d_0, e_0) = d_0 \sqsubseteq_D d_1 = \pi_1(d_1, e_1)$$

$$\Uparrow$$

$$(d_0, e_0) \sqsubseteq_{D \times E} (d_1, e_1)$$

the case of π_2 is analogous

Projections (ctd)

$\pi_1 : D \times E \to D$ $\qquad$ $\pi_2 : D \times E \to E$

$\pi_1(d, e) = d$ $\qquad$ $\pi_2(d, e) = e$

TH. projections are continuous

proof. take $\{(d_i, e_i)\}_{i \in \mathbb{N}}$

we want to prove $\pi_1 \left(\bigsqcup_{i \in \mathbb{N}} (d_i, e_i) \right) = \bigsqcup_{i \in \mathbb{N}} \pi_1(d_i, e_i)$

$$\pi_1 \left(\bigsqcup_{i \in \mathbb{N}} (d_i, e_i) \right) = \pi_1 \left(\bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right) = \bigsqcup_{i \in \mathbb{N}} d_i = \bigsqcup_{i \in \mathbb{N}} \pi_1(d_i, e_i)$$

by def of lub $\qquad$ by def of π_1 $\qquad$ by def of π_1

the case of π_2 is analogous

Exercise: smashed prod

$$\mathcal{D} = (D, \sqsubseteq_D) \qquad \mathcal{E} = (E, \sqsubseteq_E) \qquad \text{CPO}_\perp \quad \Rightarrow \quad \mathcal{D} \otimes \mathcal{E} = (\ D \otimes E\ ,\ \sqsubseteq_{D \otimes E}\ )$$

$$D \otimes E \triangleq \{(d, e) \mid (d, e) \in D \times E, d = \perp_D \Leftrightarrow e = \perp_E\}$$

how to order pairs?

bottom element?

complete order?