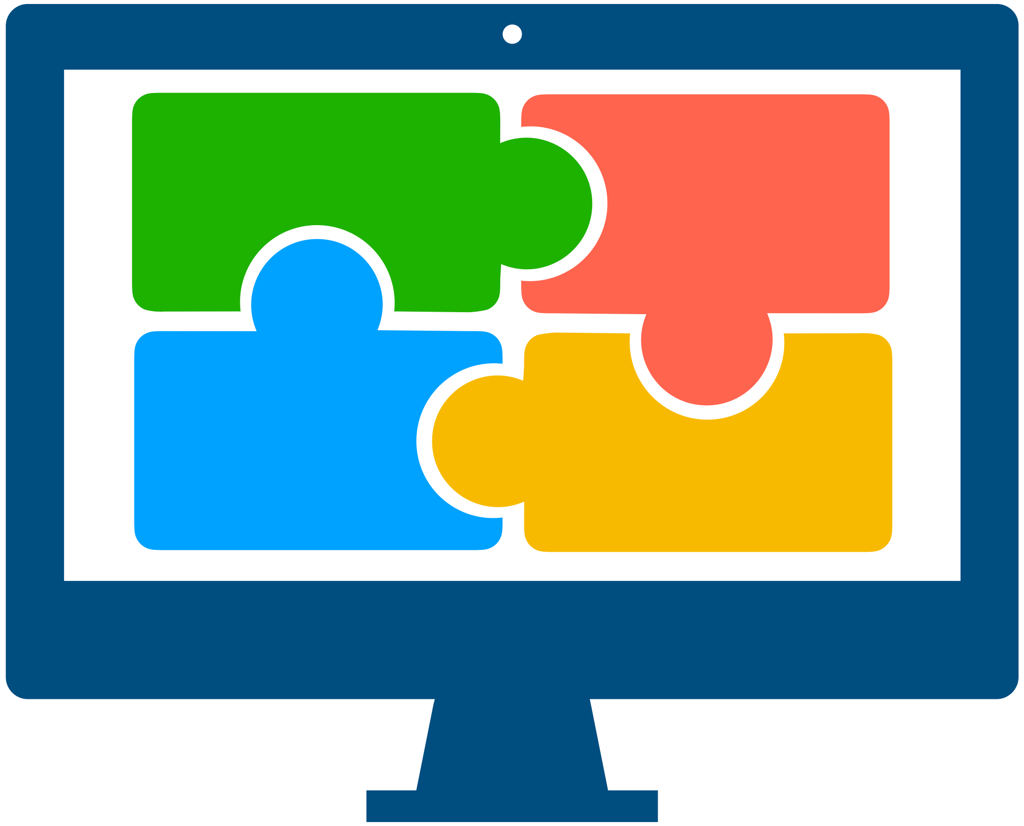


http://didawiki.di.unipi.it/doku.php/magistraleinformatica/psc/

PSC 2020/21 (375AA, 9CFU)

Principles for Software Composition

Roberto Bruni
http://www.di.unipi.it/~bruni/

13b - Functional domains

Switch lemma

Switch Lemma

$\mathcal{E} = (E, \sqsubseteq)$ CPO

a set of elements (not a chain) $\{e_{n,m}\}_{n,m \in \mathbb{N}}$

such that $e_{n,m} \sqsubseteq e_{n',m'}$ if $n \leq n' \wedge m \leq m'$

$e_{0,0}$

Switch Lemma

$\mathcal{E} = (E, \sqsubseteq)$ CPO

a set of elements (not a chain) $\{e_{n,m}\}_{n,m \in \mathbb{N}}$

such that $e_{n,m} \sqsubseteq e_{n',m'}$ if $n \leq n' \wedge m \leq m'$

$e_{0,0} \sqsubseteq e_{0,1}$

Switch Lemma

$\mathcal{E} = (E, \sqsubseteq)$ CPO

a set of elements (not a chain) $\{e_{n,m}\}_{n,m \in \mathbb{N}}$

such that $e_{n,m} \sqsubseteq e_{n',m'}$ if $n \leq n' \wedge m \leq m'$

$e_{0,0} \sqsubseteq e_{0,1} \sqsubseteq e_{0,2} \sqsubseteq \cdots$

Switch Lemma

$\mathcal{E} = (E, \sqsubseteq)$ CPO

a set of elements (not a chain) $\{e_{n,m}\}_{n,m \in \mathbb{N}}$

such that $e_{n,m} \sqsubseteq e_{n',m'}$ if $n \leq n' \wedge m \leq m'$

$$e_{0,0} \sqsubseteq e_{0,1} \sqsubseteq e_{0,2} \sqsubseteq \cdots \sqsubseteq e_{0,m} \sqsubseteq \cdots$$

Switch Lemma

$\mathcal{E} = (E, \sqsubseteq)$ CPO

a set of elements (not a chain) $\{e_{n,m}\}_{n,m \in \mathbb{N}}$

such that $e_{n,m} \sqsubseteq e_{n',m'}$ if $n \leq n' \wedge m \leq m'$

$$
\begin{array}{ccccccccccc}
e_{1,0} & \sqsubseteq & e_{1,1} & \sqsubseteq & e_{1,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{1,m} & \sqsubseteq & \cdots \\
\sqcup\!| & & \sqcup\!| & & \sqcup\!| & & & & \sqcup\!| & & \\
e_{0,0} & \sqsubseteq & e_{0,1} & \sqsubseteq & e_{0,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{0,m} & \sqsubseteq & \cdots
\end{array}
$$

Switch Lemma

$\mathcal{E} = (E, \sqsubseteq)$ CPO

a set of elements (not a chain) $\{e_{n,m}\}_{n,m \in \mathbb{N}}$

such that $e_{n,m} \sqsubseteq e_{n',m'}$ if $n \leq n' \wedge m \leq m'$

$$
\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & \iddots & & \vdots & & \\
\sqcup\!| & & \sqcup\!| & & \sqcup\!| & & & & \sqcup\!| & & \\
e_{2,0} & \sqsubseteq & e_{2,1} & \sqsubseteq & e_{2,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots \\
\sqcup\!| & & \sqcup\!| & & \sqcup\!| & & & & \sqcup\!| & & \\
e_{1,0} & \sqsubseteq & e_{1,1} & \sqsubseteq & e_{1,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{1,m} & \sqsubseteq & \cdots \\
\sqcup\!| & & \sqcup\!| & & \sqcup\!| & & & & \sqcup\!| & & \\
e_{0,0} & \sqsubseteq & e_{0,1} & \sqsubseteq & e_{0,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{0,m} & \sqsubseteq & \cdots
\end{array}
$$

Switch Lemma

$\mathcal{E} = (E, \sqsubseteq)$ CPO

a set of elements (not a chain) $\{e_{n,m}\}_{n,m \in \mathbb{N}}$

such that $e_{n,m} \sqsubseteq e_{n',m'}$ if $n \leq n' \wedge m \leq m'$

$$
\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & & & \vdots & & \iddots \\
\sqcup\!\!\mid & & \sqcup\!\!\mid & & \sqcup\!\!\mid & & & & \sqcup\!\!\mid & & \\
e_{n,0} & \sqsubseteq & e_{n,1} & \sqsubseteq & e_{n,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{n,m} & \sqsubseteq & \cdots \\
\sqcup\!\!\mid & & \sqcup\!\!\mid & & \sqcup\!\!\mid & & & & \sqcup\!\!\mid & & \\
\vdots & & \vdots & & \vdots & & \iddots & & \vdots & & \\
\sqcup\!\!\mid & & \sqcup\!\!\mid & & \sqcup\!\!\mid & & & & \sqcup\!\!\mid & & \\
e_{2,0} & \sqsubseteq & e_{2,1} & \sqsubseteq & e_{2,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots \\
\sqcup\!\!\mid & & \sqcup\!\!\mid & & \sqcup\!\!\mid & & & & \sqcup\!\!\mid & & \\
e_{1,0} & \sqsubseteq & e_{1,1} & \sqsubseteq & e_{1,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{1,m} & \sqsubseteq & \cdots \\
\sqcup\!\!\mid & & \sqcup\!\!\mid & & \sqcup\!\!\mid & & & & \sqcup\!\!\mid & & \\
e_{0,0} & \sqsubseteq & e_{0,1} & \sqsubseteq & e_{0,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{0,m} & \sqsubseteq & \cdots
\end{array}
$$

Switch Lemma

$\mathcal{E} = (E, \sqsubseteq)$ CPO

a set of elements (not a chain) $\{e_{n,m}\}_{n,m \in \mathbb{N}}$

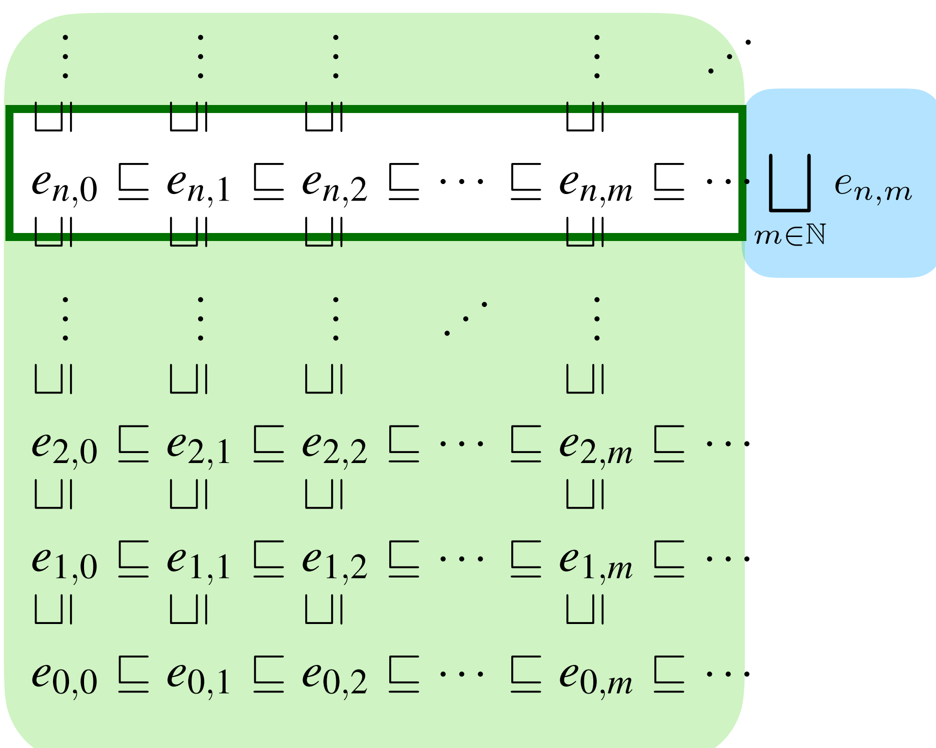
such that $e_{n,m} \sqsubseteq e_{n',m'}$ if $n \leq n' \land m \leq m'$

fixed n the set $\{e_{n,m}\}_{m \in \mathbb{N}}$

forms a chain (a row in the picture)

and thus has a lub (E is a CPO)

$$\bigsqcup_{m \in \mathbb{N}} e_{n,m}$$

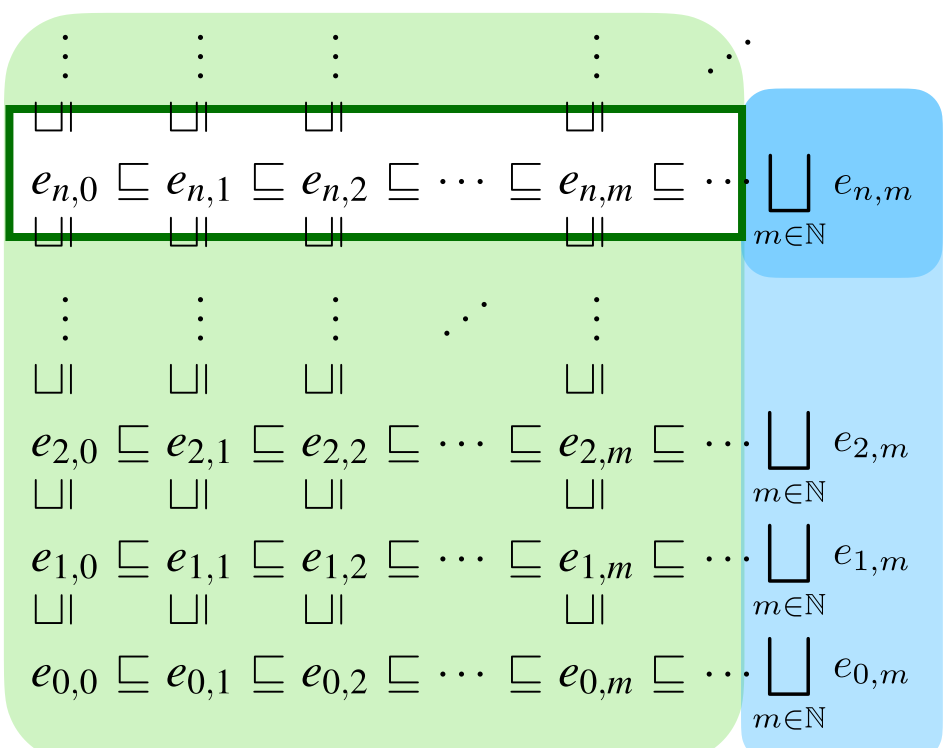


Switch Lemma

$\mathcal{E} = (E, \sqsubseteq)$ CPO

a set of elements (not a chain) $\{e_{n,m}\}_{n,m\in\mathbb{N}}$

such that $e_{n,m} \sqsubseteq e_{n',m'}$ if $n \leq n' \wedge m \leq m'$



fixed n the set $\{e_{n,m}\}_{m\in\mathbb{N}}$

forms a chain (a row in the picture)

and thus has a lub (E is a CPO)

$$\bigsqcup_{m\in\mathbb{N}} e_{n,m}$$

we form the chain of all row-lubs

$$\left\{\bigsqcup_{m\in\mathbb{N}} e_{n,m}\right\}_{n\in\mathbb{N}}$$

Switch Lemma

$\mathcal{E} = (E, \sqsubseteq)$ CPO

a set of elements (not a chain) $\{e_{n,m}\}_{n,m \in \mathbb{N}}$

such that $e_{n,m} \sqsubseteq e_{n',m'}$ if $n \leq n' \wedge m \leq m'$

fixed m the set $\{e_{n,m}\}_{n \in \mathbb{N}}$

forms a chain (a column in the picture)

and thus has a lub (E is a CPO)

$$\bigsqcup_{n \in \mathbb{N}} e_{n,m}$$

$$\bigsqcup_{n \in \mathbb{N}} e_{n,m}$$

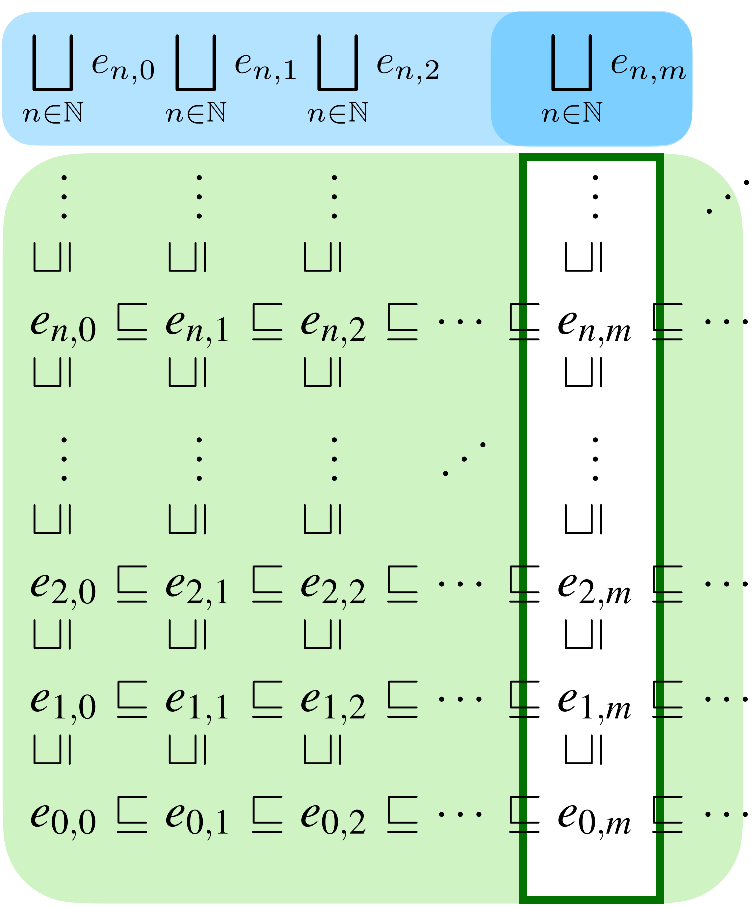
$$\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & & & \vdots & & \iddots \\
\sqsubseteq & & \sqsubseteq & & \sqsubseteq & & & & \sqsubseteq & & \\
e_{n,0} & \sqsubseteq & e_{n,1} & \sqsubseteq & e_{n,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{n,m} & \sqsubseteq & \cdots \\
\sqsubseteq & & \sqsubseteq & & \sqsubseteq & & & & \sqsubseteq & & \\
\vdots & & \vdots & & \vdots & & \iddots & & \vdots & & \\
\sqsubseteq & & \sqsubseteq & & \sqsubseteq & & & & \sqsubseteq & & \\
e_{2,0} & \sqsubseteq & e_{2,1} & \sqsubseteq & e_{2,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots \\
\sqsubseteq & & \sqsubseteq & & \sqsubseteq & & & & \sqsubseteq & & \\
e_{1,0} & \sqsubseteq & e_{1,1} & \sqsubseteq & e_{1,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{1,m} & \sqsubseteq & \cdots \\
\sqsubseteq & & \sqsubseteq & & \sqsubseteq & & & & \sqsubseteq & & \\
e_{0,0} & \sqsubseteq & e_{0,1} & \sqsubseteq & e_{0,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{0,m} & \sqsubseteq & \cdots
\end{array}$$

Switch Lemma

$\mathcal{E} = (E, \sqsubseteq)$ CPO

a set of elements (not a chain) $\{e_{n,m}\}_{n,m \in \mathbb{N}}$

such that $e_{n,m} \sqsubseteq e_{n',m'}$ if $n \leq n' \wedge m \leq m'$



fixed m the set $\{e_{n,m}\}_{n \in \mathbb{N}}$

forms a chain (a column in the picture)

and thus has a lub (E is a CPO)

$$\bigsqcup_{n \in \mathbb{N}} e_{n,m}$$

we form the chain of all column-lubs

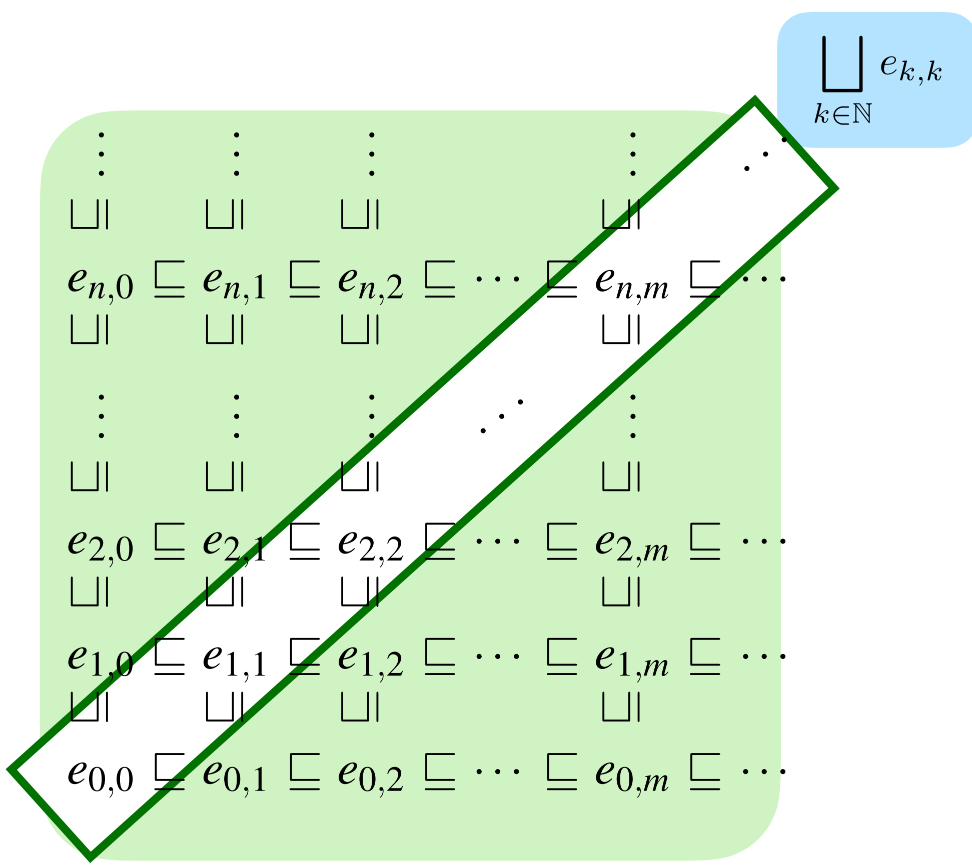
$$\left\{ \bigsqcup_{n \in \mathbb{N}} e_{n,m} \right\}_{m \in \mathbb{N}}$$

Switch Lemma

$\mathcal{E} = (E, \sqsubseteq)$ CPO

a set of elements (not a chain) $\{e_{n,m}\}_{n,m \in \mathbb{N}}$

such that $e_{n,m} \sqsubseteq e_{n',m'}$ if $n \leq n' \wedge m \leq m'$



the diagonal elements $\{e_{k,k}\}_{k \in \mathbb{N}}$

also forms a chain

and thus has a lub (E is a CPO)

$$\bigsqcup_{k \in \mathbb{N}} e_{k,k}$$

Switch Lemma

$\mathcal{E} = (E, \sqsubseteq)$ CPO

a set of elements (not a chain) $\{e_{n,m}\}_{n,m \in \mathbb{N}}$

such that $e_{n,m} \sqsubseteq e_{n',m'}$ if $n \leq n' \wedge m \leq m'$

$$
\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & & & \vdots & & \iddots \\
\sqcup\!| & & \sqcup\!| & & \sqcup\!| & & & & \sqcup\!| & & \\
e_{n,0} & \sqsubseteq & e_{n,1} & \sqsubseteq & e_{n,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{n,m} & \sqsubseteq & \cdots \\
\sqcup\!| & & \sqcup\!| & & \sqcup\!| & & & & \sqcup\!| & & \\
\vdots & & \vdots & & \vdots & & \iddots & & \vdots & & \\
\sqcup\!| & & \sqcup\!| & & \sqcup\!| & & & & \sqcup\!| & & \\
e_{2,0} & \sqsubseteq & e_{2,1} & \sqsubseteq & e_{2,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots \\
\sqcup\!| & & \sqcup\!| & & \sqcup\!| & & & & \sqcup\!| & & \\
e_{1,0} & \sqsubseteq & e_{1,1} & \sqsubseteq & e_{1,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{1,m} & \sqsubseteq & \cdots \\
\sqcup\!| & & \sqcup\!| & & \sqcup\!| & & & & \sqcup\!| & & \\
e_{0,0} & \sqsubseteq & e_{0,1} & \sqsubseteq & e_{0,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{0,m} & \sqsubseteq & \cdots
\end{array}
$$

we prove that
all sets we have seen
have the same
set of upper bounds
and thus the same
least upper bound

$$\bigsqcup_{n \in \mathbb{N}} \bigsqcup_{m \in \mathbb{N}} e_{n,m} = \bigsqcup_{k \in \mathbb{N}} e_{k,k} = \bigsqcup_{m \in \mathbb{N}} \bigsqcup_{n \in \mathbb{N}} e_{n,m}$$

$$\{e_{n,m}\}_{n,m \in \mathbb{N}}$$

$$\left\{ \bigsqcup_{m \in \mathbb{N}} e_{n,m} \right\}_{n \in \mathbb{N}}$$

$$\left\{ \bigsqcup_{n \in \mathbb{N}} e_{n,m} \right\}_{m \in \mathbb{N}}$$

$$\{e_{k,k}\}_{k \in \mathbb{N}}$$

(i)

$\{e_{n,m}\}_{n,m\in\mathbb{N}}$ same u.b. as $\{e_n\}_{n\in\mathbb{N}}$

where $e_n = \bigsqcup_{m\in\mathbb{N}} e_{n,m}$

$$
\begin{array}{ccccccccccccc}
\vdots & & \vdots & & \vdots & & & & \vdots & & & & \vdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} \\
e_{n,0} & \sqsubseteq & e_{n,1} & \sqsubseteq & e_{n,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{n,m} & \sqsubseteq & \cdots & \sqsubseteq & e_n = \bigsqcup_{m\in\mathbb{N}} e_{n,m} \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} \\
\vdots & & \vdots & & \vdots & & & & \vdots & & & & \vdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} \\
e_{2,0} & \sqsubseteq & e_{2,1} & \sqsubseteq & e_{2,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots & \sqsubseteq & e_2 = \bigsqcup_{m\in\mathbb{N}} e_{2,m} \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} \\
e_{1,0} & \sqsubseteq & e_{1,1} & \sqsubseteq & e_{1,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots & \sqsubseteq & e_1 = \bigsqcup_{m\in\mathbb{N}} e_{1,m} \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} \\
e_{0,0} & \sqsubseteq & e_{0,1} & \sqsubseteq & e_{0,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots & \sqsubseteq & e_0 = \bigsqcup_{m\in\mathbb{N}} e_{0,m}
\end{array}
$$

(i) $\{e_{n,m}\}_{n,m\in\mathbb{N}}$ same u.b. as $\{e_n\}_{n\in\mathbb{N}}$

where $e_n = \bigsqcup_{m\in\mathbb{N}} e_{n,m}$

1. take an upper bound e of $\{e_{n,m}\}_{n,m\in\mathbb{N}}$

   we want to prove it is an upper bound for $\{e_n\}_{n\in\mathbb{N}}$

e

$$
\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & & & \vdots & & & \vdots \\
\sqsupseteq & & \sqsupseteq & & \sqsupseteq & & & & \sqsupseteq & & & \sqsupseteq \\
e_{n,0} & \sqsubseteq & e_{n,1} & \sqsubseteq & e_{n,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{n,m} & \sqsubseteq & \cdots \sqsubseteq & e_n = \bigsqcup_{m\in\mathbb{N}} e_{n,m} \\
\sqsupseteq & & \sqsupseteq & & \sqsupseteq & & & & \sqsupseteq & & & \sqsupseteq \\
\vdots & & \vdots & & \vdots & & & & \vdots & & & \vdots \\
\sqsupseteq & & \sqsupseteq & & \sqsupseteq & & & & \sqsupseteq & & & \sqsupseteq \\
e_{2,0} & \sqsubseteq & e_{2,1} & \sqsubseteq & e_{2,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots \sqsubseteq & e_2 = \bigsqcup_{m\in\mathbb{N}} e_{2,m} \\
\sqsupseteq & & \sqsupseteq & & \sqsupseteq & & & & \sqsupseteq & & & \sqsupseteq \\
e_{1,0} & \sqsubseteq & e_{1,1} & \sqsubseteq & e_{1,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots \sqsubseteq & e_1 = \bigsqcup_{m\in\mathbb{N}} e_{1,m} \\
\sqsupseteq & & \sqsupseteq & & \sqsupseteq & & & & \sqsupseteq & & & \sqsupseteq \\
e_{0,0} & \sqsubseteq & e_{0,1} & \sqsubseteq & e_{0,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots \sqsubseteq & e_0 = \bigsqcup_{m\in\mathbb{N}} e_{0,m}
\end{array}
$$

take any (row) index n

we prove $e_n \sqsubseteq e$

$\{e_{n,m}\}_{m\in\mathbb{N}} \subseteq \{e_{n,m}\}_{n,m\in\mathbb{N}}$

a row — the matrix

e is an u.b. of $\{e_{n,m}\}_{m\in\mathbb{N}}$

$e_n = \bigsqcup_{m\in\mathbb{N}} e_{n,m}$ is the lub

therefore $e_n \sqsubseteq e$

(i)

$\{e_{n,m}\}_{n,m\in\mathbb{N}}$ same u.b. as $\{e_n\}_{n\in\mathbb{N}}$

where $e_n = \bigsqcup_{m\in\mathbb{N}} e_{n,m}$

2. take an upper bound e of $\{e_n\}_{n\in\mathbb{N}}$

we want to prove it is an upper bound for $\{e_{n,m}\}_{n,m\in\mathbb{N}}$

$$
\begin{array}{ccccccccccccc}
 & & & & & & & & & & & & e \\
\vdots & & \vdots & & \vdots & & & & \vdots & & & & \vdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} \\
e_{n,0} & \sqsubseteq & e_{n,1} & \sqsubseteq & e_{n,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{n,m} & \sqsubseteq & \cdots & \sqsubseteq & e_n = \bigsqcup_{m\in\mathbb{N}} e_{n,m} \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} \\
\vdots & & \vdots & & \vdots & & & & \vdots & & & & \vdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} \\
e_{2,0} & \sqsubseteq & e_{2,1} & \sqsubseteq & e_{2,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots & \sqsubseteq & e_2 = \bigsqcup_{m\in\mathbb{N}} e_{2,m} \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} \\
e_{1,0} & \sqsubseteq & e_{1,1} & \sqsubseteq & e_{1,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots & \sqsubseteq & e_1 = \bigsqcup_{m\in\mathbb{N}} e_{1,m} \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} \\
e_{0,0} & \sqsubseteq & e_{0,1} & \sqsubseteq & e_{0,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots & \sqsubseteq & e_0 = \bigsqcup_{m\in\mathbb{N}} e_{0,m}
\end{array}
$$

take any indices n, m

we prove $e_{n,m} \sqsubseteq e$

$$e_{n,m} \sqsubseteq \bigsqcup_{m\in\mathbb{N}} e_{n,m} = e_n \sqsubseteq e$$

one element of a row

the lub of that row

(ii) $\{e_{n,m}\}_{n,m\in\mathbb{N}}$ same u.b. as $\left\{\bigsqcup_{n\in\mathbb{N}} e_{n,m}\right\}_{m\in\mathbb{N}}$

the proof is analogous to the previous case
(reason by columns, not by rows)

$$\bigsqcup_{n\in\mathbb{N}} e_{n,0} \quad \bigsqcup_{n\in\mathbb{N}} e_{n,1} \quad \bigsqcup_{n\in\mathbb{N}} e_{n,2} \qquad \bigsqcup_{n\in\mathbb{N}} e_{n,m}$$

$$\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & & & \vdots & & \\
\sqsupseteq & & \sqsupseteq & & \sqsupseteq & & & & \sqsupseteq & & \\
e_{n,0} & \sqsubseteq & e_{n,1} & \sqsubseteq & e_{n,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{n,m} & \sqsubseteq & \cdots \\
\sqsupseteq & & \sqsupseteq & & \sqsupseteq & & & & \sqsupseteq & & \\
\vdots & & \vdots & & \vdots & & & & \vdots & & \\
\sqsupseteq & & \sqsupseteq & & \sqsupseteq & & & & \sqsupseteq & & \\
e_{2,0} & \sqsubseteq & e_{2,1} & \sqsubseteq & e_{2,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots \\
\sqsupseteq & & \sqsupseteq & & \sqsupseteq & & & & \sqsupseteq & & \\
e_{1,0} & \sqsubseteq & e_{1,1} & \sqsubseteq & e_{1,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots \\
\sqsupseteq & & \sqsupseteq & & \sqsupseteq & & & & \sqsupseteq & & \\
e_{0,0} & \sqsubseteq & e_{0,1} & \sqsubseteq & e_{0,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots
\end{array}$$

(iii) $\{e_{n,m}\}_{n,m\in\mathbb{N}}$ same u.b. as $\{e_{k,k}\}_{k\in\mathbb{N}}$

1. take an upper bound e of $\{e_{n,m}\}_{n,m\in\mathbb{N}}$

   we want to prove it is an upper bound for $\{e_{k,k}\}_{k\in\mathbb{N}}$

$$e$$

$$\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & & & \vdots & & \\
\sqcup\!\!| & & \sqcup\!\!| & & \sqcup\!\!| & & & & \sqcup\!\!| & & \\
e_{n,0} & \sqsubseteq & e_{n,1} & \sqsubseteq & e_{n,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{n,m} & \sqsubseteq & \cdots \\
\sqcup\!\!| & & \sqcup\!\!| & & \sqcup\!\!| & & & & \sqcup\!\!| & & \\
\vdots & & \vdots & & \vdots & & & & \vdots & & \\
\sqcup\!\!| & & \sqcup\!\!| & & \sqcup\!\!| & & & & \sqcup\!\!| & & \\
e_{2,0} & \sqsubseteq & e_{2,1} & \sqsubseteq & e_{2,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots \\
\sqcup\!\!| & & \sqcup\!\!| & & \sqcup\!\!| & & & & \sqcup\!\!| & & \\
e_{1,0} & \sqsubseteq & e_{1,1} & \sqsubseteq & e_{1,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots \\
\sqcup\!\!| & & \sqcup\!\!| & & \sqcup\!\!| & & & & \sqcup\!\!| & & \\
e_{0,0} & \sqsubseteq & e_{0,1} & \sqsubseteq & e_{0,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots
\end{array}$$

but this is immediate, because

$$\{e_{k,k}\}_{k\in\mathbb{N}} \subseteq \{e_{n,m}\}_{n,m\in\mathbb{N}}$$

the diagonal   the whole matrix

(iii) $\{e_{n,m}\}_{n,m\in\mathbb{N}}$ same u.b. as $\{e_{k,k}\}_{k\in\mathbb{N}}$

2. take an upper bound e of $\{e_{k,k}\}_{k\in\mathbb{N}}$

we want to prove it is an upper bound for $\{e_{n,m}\}_{n,m\in\mathbb{N}}$

e

$$
\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & & & \vdots & & \\
\sqsupseteq & & \sqsupseteq & & \sqsupseteq & & & & \sqsupseteq & & \\
e_{n,0} & \sqsubseteq & e_{n,1} & \sqsubseteq & e_{n,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{n,m} & \sqsubseteq & \cdots \\
\sqsupseteq & & \sqsupseteq & & \sqsupseteq & & & & \sqsupseteq & & \\
\vdots & & \vdots & & \vdots & & & & \vdots & & \\
\sqsupseteq & & \sqsupseteq & & \sqsupseteq & & & & \sqsupseteq & & \\
e_{2,0} & \sqsubseteq & e_{2,1} & \sqsubseteq & e_{2,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots \\
\sqsupseteq & & \sqsupseteq & & \sqsupseteq & & & & \sqsupseteq & & \\
e_{1,0} & \sqsubseteq & e_{1,1} & \sqsubseteq & e_{1,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots \\
\sqsupseteq & & \sqsupseteq & & \sqsupseteq & & & & \sqsupseteq & & \\
e_{0,0} & \sqsubseteq & e_{0,1} & \sqsubseteq & e_{0,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots
\end{array}
$$

take any indices n, m

we prove $e_{n,m} \sqsubseteq e$

let $k = \max\{n, m\}$

$$e_{n,m} \sqsubseteq e_{k,k} \sqsubseteq e$$

$n \leq k \wedge m \leq k$

e is an u.b. of $\{e_{k,k}\}_{k\in\mathbb{N}}$

Switch Lemma: recap

$$\{e_{n,m}\}_{n,m\in\mathbb{N}}$$

$e_{n,m} \sqsubseteq e_{n',m'}$ if $n \leq n' \wedge m \leq m'$

same set of upper bounds as

$$\left\{\bigsqcup_{m\in\mathbb{N}} e_{n,m}\right\}_{n\in\mathbb{N}} \qquad \{e_{k,k}\}_{k\in\mathbb{N}} \qquad \left\{\bigsqcup_{n\in\mathbb{N}} e_{n,m}\right\}_{m\in\mathbb{N}}$$

$$\bigsqcup_{n\in\mathbb{N}} \bigsqcup_{m\in\mathbb{N}} e_{n,m} = \bigsqcup_{k\in\mathbb{N}} e_{k,k} = \bigsqcup_{m\in\mathbb{N}} \bigsqcup_{n\in\mathbb{N}} e_{n,m}$$

Functional domains

Function space

$$\mathcal{D} = (D, \sqsubseteq_D)$$

$$\mathcal{E} = (E, \sqsubseteq_E)$$

$$\text{CPO}_\perp \;\Rightarrow\; [\mathcal{D} \to \mathcal{E}] = (\ [D \to E]\ ,\ \sqsubseteq_{[D \to E]}\ )$$

$$D \to E \triangleq \{\ f \ \mid\ f : D \to E\ \}$$

$$[D \to E] \triangleq \{\ f \ \mid\ f : D \to E\ ,\ f \text{ continuous}\ \}$$

how to order functions?

$$f \sqsubseteq_{[D \to E]} g \quad \text{iff} \quad \forall d \in D.\ f(d) \sqsubseteq_E g(d)$$

Example

$$f \sqsubseteq_{[D \to E]} g \quad \text{iff} \quad \forall d \in D.\ f(d) \sqsubseteq_E g(d)$$

$$f, g : \mathbb{Z}_\perp \to \mathbb{Z}_\perp$$

$$f(x) \triangleq 0 \qquad f \overset{?}{\sqsubseteq}_{[\mathbb{Z}_\perp \to \mathbb{Z}_\perp]} g \qquad g(x) \triangleq \begin{cases} 1 & x \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

$$f(1) = 0 \ \not\sqsubseteq_{\mathbb{Z}_\perp} 1 = g(1)$$

total functions on $\mathbb{Z}_\perp$ are not comparable

(unless they are equal)

any total function is maximal in $\mathbb{Z}_\perp \to \mathbb{Z}_\perp$

Example

$$f \sqsubseteq_{[D \to E]} g \quad \text{iff} \quad \forall d \in D.\ f(d) \sqsubseteq_E g(d)$$

$$f, g : \mathbb{Z}_\perp \to \mathbb{Z}_\perp$$

$$f(x) \triangleq \begin{cases} 1 & x \text{ odd} \\ \perp_{\mathbb{Z}_\perp} & \text{otherwise} \end{cases} \qquad g(x) \triangleq \begin{cases} 1 & x \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

$$f \overset{?}{\sqsubseteq}_{[\mathbb{Z}_\perp \to \mathbb{Z}_\perp]} g$$

Example

$$f \sqsubseteq_{[D \to E]} g \quad \text{iff} \quad \forall d \in D.\ f(d) \sqsubseteq_E g(d)$$

$$f, g : \mathbb{Z}_\perp \to \mathbb{Z}_\perp$$

$$f(x) \triangleq \begin{cases} 1 & x \text{ odd} \\ \perp_{\mathbb{Z}_\perp} & \text{otherwise} \end{cases} \qquad g(x) \triangleq \begin{cases} 0 & x \text{ even} \\ \perp_{\mathbb{Z}_\perp} & \text{otherwise} \end{cases}$$

$$f \overset{?}{\sqsubseteq}_{[\mathbb{Z}_\perp \to \mathbb{Z}_\perp]} g$$

✖

Example

$$f \sqsubseteq_{[D \to E]} g \quad \text{iff} \quad \forall d \in D.\ f(d) \sqsubseteq_E g(d)$$

$$f, g : \mathbb{Z}_\perp \to \mathbb{Z}_\perp$$

$$f(x) \triangleq \begin{cases} x! & 1 \le x \le 10 \\ \perp_{\mathbb{Z}_\perp} & \text{otherwise} \end{cases} \qquad g(x) \triangleq \begin{cases} x! & 1 \le x \le 15 \\ \perp_{\mathbb{Z}_\perp} & \text{otherwise} \end{cases}$$

$$f \overset{?}{\sqsubseteq}_{[\mathbb{Z}_\perp \to \mathbb{Z}_\perp]} g$$

✔

Example

$$f \sqsubseteq_{[D \to E]} g \quad \text{iff} \quad \forall d \in D.\ f(d) \sqsubseteq_E g(d)$$

$$f, g : \mathbb{Z}_\perp \times \mathbb{Z}_\perp \to \mathbb{Z}_\perp$$

$$f(x,y) \triangleq \begin{cases} x * y & x, y \neq \perp_{\mathbb{Z}_\perp} \\ \perp_{\mathbb{Z}_\perp} & \text{otherwise} \end{cases} \quad g(x,y) \triangleq \begin{cases} (x * y)^2 & x, y \neq \perp_{\mathbb{Z}_\perp} \\ \perp_{\mathbb{Z}_\perp} & \text{otherwise} \end{cases}$$

$$f \overset{?}{\sqsubseteq}_{[\mathbb{Z}_\perp \times \mathbb{Z}_\perp \to \mathbb{Z}_\perp]} g$$

✗

Example

$$f \sqsubseteq_{[D \to E]} g \quad \text{iff} \quad \forall d \in D.\ f(d) \sqsubseteq_E g(d)$$

$$f, g : \mathbb{Z}_\perp \times \mathbb{Z}_\perp \to \mathbb{Z}_\perp$$

$$f(x,y) \triangleq \begin{cases} x * y & x, y \neq \perp_{\mathbb{Z}_\perp} \\ \perp_{\mathbb{Z}_\perp} & \text{otherwise} \end{cases} \qquad g(x,y) \triangleq \begin{cases} x * y & x, y \neq \perp_{\mathbb{Z}_\perp} \\ 0 & x = \perp_{\mathbb{Z}_\perp} \\ \perp_{\mathbb{Z}_\perp} & \text{otherwise} \end{cases}$$

$$f \overset{?}{\sqsubseteq}_{[\mathbb{Z}_\perp \times \mathbb{Z}_\perp \to \mathbb{Z}_\perp]} g$$

yes (as functions)

but is g continuous?

$$g(\perp, \perp) = 0 \qquad g(1,1) = 1$$

not even monotone!

Example

$$f \sqsubseteq_{[D \to E]} g \quad \text{iff} \quad \forall d \in D.\ f(d) \sqsubseteq_E g(d)$$

$$f, g : \mathbb{Z}_\perp \to \mathbb{Z}_\perp \times \mathbb{Z}_\perp$$

$$f(x) \triangleq (\perp_{\mathbb{Z}_\perp}, x) \qquad f \overset{?}{\sqsubseteq}_{[\mathbb{Z}_\perp \to \mathbb{Z}_\perp \times \mathbb{Z}_\perp]} g \qquad g(x) \triangleq (x, x)$$

Example

$$f \sqsubseteq_{[D \to E]} g \quad \text{iff} \quad \forall d \in D.\ f(d) \sqsubseteq_E g(d)$$

$$f, g : \mathbb{Z}_\bot \to \mathbb{Z}_\bot \times \mathbb{Z}_\bot$$

$$f(x) \triangleq (\bot_{\mathbb{Z}_\bot}, x) \qquad f \overset{?}{\sqsubseteq}_{[\mathbb{Z}_\bot \to \mathbb{Z}_\bot \times \mathbb{Z}_\bot]} g \qquad g(x) \triangleq (x, \bot_{\mathbb{Z}_\bot})$$

✖

$$f(0) = (\bot_{\mathbb{Z}_\bot}, 0) \not\sqsubseteq_{\mathbb{Z}_\bot \times \mathbb{Z}_\bot} (0, \bot_{\mathbb{Z}_\bot}) = g(0)$$

Functional CPO

$$[\mathcal{D} \to \mathcal{E}] = (\ [D \to E]\ ,\ \sqsubseteq_{[D\to E]}\ )$$

is it a partial order?

reflexivity, antisymmetry, transitivity of $\sqsubseteq_{[D\to E]}$ follow immediately from those of $\sqsubseteq_E$

is there a bottom element?

let $\bot_{[D\to E]} = \lambda d.\ \bot_E$

take any function $f \in [D \to E]$

for any $d \in D$ we have $\bot_{[D\to E]}\ d = \bot_E \sqsubseteq_E f(d)$

Functional CPO (ctd)

$$[\mathcal{D} \to \mathcal{E}] = (\ [D \to E]\ ,\ \sqsubseteq_{[D \to E]}\ )$$

is it complete?

first we show that any chain of functions
(not necessarily continuous)
has a limit in $D \to E$

then we show that the limit in $D \to E$
of any chain of continuous functions
is also continuous

Functional CPO (ctd)

$\{f_n : D \to E\}_{n\in\mathbb{N}}$ a chain of functions (not necessarily continuous)

we prove its lub is $h \triangleq \lambda d.\ \bigsqcup_{n\in\mathbb{N}} f_n(d)$

i.e. $h(d) \triangleq \bigsqcup_{n\in\mathbb{N}} f_n(d)$

1. it is an upper bound of the chain
2. it is smaller than or equal to any other upper bound

Functional CPO (ctd)

take a chain $\{f_n : D \to E\}_{n\in\mathbb{N}}$ (not necessarily continuous)

1. $h \triangleq \lambda d. \bigsqcup_{n\in\mathbb{N}} f_n(d)$ is an upper bound of the chain

take any $n \in \mathbb{N}$

for any $d \in D$ $\quad f_n(d) \sqsubseteq_E \bigsqcup_{n\in\mathbb{N}} f_n(d) = h(d)$

therefore $\quad f_n \sqsubseteq_{D\to E} h$

Functional CPO (ctd)

take a chain $\{f_n : D \to E\}_{n\in\mathbb{N}}$ (not necessarily continuous)

2. $h \triangleq \lambda d.\ \bigsqcup_{n\in\mathbb{N}} f_n(d)$ is the least among upper bounds

take any g such that $\forall n.\ f_n \sqsubseteq_{D\to E} g$

we want to prove $h \sqsubseteq_{D\to E} g$

take any $d \in D$ $\forall n.\ f_n(d) \sqsubseteq_E g(d)$

thus $g(d)$ is an u.b. of $\{f_n(d)\}_{n\in\mathbb{N}}$

and therefore $h(d) = \bigsqcup_{n\in\mathbb{N}} f_n(d) \sqsubseteq_E g(d)$

Functional CPO (ctd)

TH. take a chain $\{f_n : D \to E\}_{n\in\mathbb{N}}$ of continuous functions

then $h \triangleq \lambda d. \bigsqcup_{n\in\mathbb{N}} f_n(d)$ is continuous

proof. let $\{d_i\}_{i\in\mathbb{N}}$ a chain in D

we prove $h\left(\bigsqcup_{i\in\mathbb{N}} d_i\right) = \bigsqcup_{i\in\mathbb{N}} h(d_i)$

$$h\left(\bigsqcup_{i\in\mathbb{N}} d_i\right) = \bigsqcup_{n\in\mathbb{N}} f_n\left(\bigsqcup_{i\in\mathbb{N}} d_i\right) \quad \text{by def of } h$$

$$= \bigsqcup_{n\in\mathbb{N}} \bigsqcup_{i\in\mathbb{N}} f_n(d_i) \quad \text{by continuity of } f_n$$

$$= \bigsqcup_{i\in\mathbb{N}} \bigsqcup_{n\in\mathbb{N}} f_n(d_i) \quad \text{by switch lemma (applicable?)}$$

$$= \bigsqcup_{i\in\mathbb{N}} h(d_i) \quad \text{by def of } h$$

Functional CPO (ctd)

if $n \leq m \wedge i \leq j$ then $f_n(d_i) \sqsubseteq_E f_m(d_j)$? ✓

$$\Downarrow$$

$$f_n \sqsubseteq_{[D \to E]} f_m \wedge d_i \sqsubseteq d_j$$

$$f_n(d_i) \sqsubseteq_E f_n(d_j) \sqsubseteq_E f_m(d_j)$$

f_n monotone | $f_n \sqsubseteq_{[D \to E]} f_m$

$$= \bigsqcup_{n \in \mathbb{N}} \bigsqcup_{i \in \mathbb{N}} f_n(d_i)$$

✓

$$= \bigsqcup_{i \in \mathbb{N}} \bigsqcup_{n \in \mathbb{N}} f_n(d_i)$$ by switch lemma (applicable?)

Functional CPO (ctd)

TH. $[\mathcal{D} \to \mathcal{E}] = ([D \to E] , \sqsubseteq_{[D \to E]})$ is complete

proof. take a chain $\{f_n : [D \to E]\}_{n \in \mathbb{N}}$

$h \triangleq \lambda d. \bigsqcup_{n \in \mathbb{N}} f_n(d)$ is the lub in $D \to E$

is continuous $h \in [D \to E]$

since $[D \to E] \subseteq D \to E$

h is the lub in $[D \to E]$

Functional CPO: recap

$$[\mathcal{D} \to \mathcal{E}] = (\ [D \to E]\ ,\ \sqsubseteq_{[D \to E]}\ )$$

$$f \sqsubseteq_{[D \to E]} g \quad \text{iff} \quad \forall d \in D.\ f(d) \sqsubseteq_E g(d)$$

$$\bot_{[D \to E]} \triangleq \lambda d.\ \bot_E$$

$$\bigsqcup_{n \in \mathbb{N}} f_n \triangleq \lambda d.\ \bigsqcup_{n \in \mathbb{N}} f_n(d)$$

$$f \in [D \to E], g \in [E \to F] \quad \Rightarrow \quad g \circ f \in [D \to F]$$

the composition of continuous functions is continuous