Chapter 2

A short note on unification

2.1 Signatures and terms

Let us consider:

- an infinite set of variables $X = \{ x, y, z, ... \}$,
- a set of function symbols $\Sigma = \{ c, f, g, ... \}$, called a *signature*, such that each symbol in Σ is assigned an *arity*, that is the number of arguments it takes; a symbol with arity zero is called a *constant*; a symbol with arity one is called *unary*; a symbol with arity two is called *binary*, etc.
- the set of terms $T_{\Sigma,X}$ of terms over Σ and X, i.e., the set of all terms generated according to the following rules:
  - each variable $x \in X$ is a term (i.e., $x \in T_{\Sigma,X}$),
  - each constant $c \in \Sigma$ is a term (i.e., $c \in T_{\Sigma,X}$),
  - if $f \in \Sigma$ is a function symbol whose arity is n, and $t_1, ..., t_n$ are terms (i.e., $t_1, ..., t_n \in T_{\Sigma,X}$, then also $f(t_1, ..., t_n)$ is a term (i.e., $f(t_1, ..., t_n) \in T_{\Sigma,X}$).

We denote by $\mathrm{vars}(t)$ the set of variables occurring in t.

For example, take $\Sigma = \{ 0, succ, plus \}$ with 0 a constant, *succ* unary and *plus* binary. Then all the following are terms:

- 0
- x
- $succ(0)$

- $succ(x)$
- $plus(succ(x), 0)$
- $plus(plus(x, succ(y)), plus(0, succ(x)))$

The set of variables of the above terms are respectively:

- $\mathrm{vars}(\, 0 \,) = \emptyset$
- $\mathrm{vars}(\, x \,) = \{\, x \,\}$
- $\mathrm{vars}(\, succ(0) \,) = \emptyset$
- $\mathrm{vars}(\, succ(x) \,) = \{\, x \,\}$
- $\mathrm{vars}(\, plus(succ(x), 0) \,) = \{\, x \,\}$
- $\mathrm{vars}(\, plus(plus(x, succ(y)), plus(0, succ(x))) \,) = \{\, x, y \,\}$

Instead $succ(plus(0), x)$ is not a term: can you see why?

2.2 Substitutions

A *substitution* $\tau : X \to T_{\Sigma,X}$ is a function assigning terms to variables.

Since the set of variables is infinite while we are interested only in terms with a finite number of variables, we consider only substitutions that are defined as identity everywhere except on a finite number of variables. Such substitutions are written

$$\tau = [\, x_1 = t_1 \,,\, \ldots \,,\, x_n = t_n \,]$$

meaning that

$$\tau(x) = \begin{cases} t_i & \text{if } x = x_i \\ x & \text{otherwise} \end{cases}$$

We denote by $t\tau$ the term obtained from t by simultaneously replacing each variable x with $\tau(x)$.

For example, for $t = plus(succ(x), succ(y))$ and $\tau = [\, x = succ(y) \,,\, y = 0 \,]$ we get

$$t\tau \;=\; plus(succ(x), succ(y))[\, x = succ(y) \,,\, y = 0 \,] \;=\; plus(succ(succ(y)), succ(0))$$

We say that the term t is *more general* than the term t' if there exists a substitution τ such that $t\tau = t'$.

We say that the substitution τ is more general than the substitution τ' if there exists a substitution ρ such that for any variable x we have that $\rho(\tau(x)) = \tau'(x)$ (i.e., $\tau(x)$ is more general that $\tau'(x)$ as witnessed by ρ).

2.3 Unification problem

The unification problem, in its simplest formulation (syntactic, first-order unification), consists of finding a substitution τ that identifies some terms.

Formally, given a set of potential equalities

$$G = \{\, l_1 \stackrel{?}{=} r_1, ..., l_n \stackrel{?}{=} r_n \,\}$$

where $l_i, r_i \in T_{\Sigma,X}$, we want to find the *most general* substitution τ such that

$$\forall i \in [1, n].\ l_i\tau = r_i\tau$$

We denote by vars(G) the set of variables occurring in G, i.e.:

$$\text{vars}(\{\, l_1 \stackrel{?}{=} r_1, ..., l_n \stackrel{?}{=} r_n \,\}) = \bigcup_{i=1}^{n} (\text{vars}(l_i) \cup \text{vars}(r_i))$$

Note that the solution does not necessarily exists, and when it exists it is not necessarily unique.

The first unification algorithm was given by Robinson in 1965, but was rather inefficient.

Linear-time algorithms were discovered independently by Martelli and Montanari in 1976 and by Paterson and Wegman in 1978.

Below we report a description of Martelli-Montanari's algorithm.

The algorithm take as input a set of potential equalities G as the one above and applies some transformations until:

- either it terminates (no transformation can be applied any more) after having transformed the set G to an equivalent set of equalities

$$\{\, x_1 \stackrel{?}{=} t_1, ..., x_k \stackrel{?}{=} t_k \,\}$$

  where $x_1, ..., x_k$ are all distinct variables and $t_1, ..., t_k$ are terms where $x_1, ..., x_k$ do not occur, i.e., such that $\{x_1, ..., x_k\} \cap \bigcup_{i=1}^{k} \text{vars}(t_i) = \emptyset$

- or it fails, meaning that the potential equalities cannot be unified.

In the following we denote by $G\tau$ the set of potential equalities obtained by applying the substitution τ to all terms in G. Formally:

$$\{\, l_1 \stackrel{?}{=} r_1, ..., l_n \stackrel{?}{=} r_n \,\}\tau = \{\, l_1\tau \stackrel{?}{=} r_1\tau, ..., l_n\tau \stackrel{?}{=} r_n\tau \,\}$$

The unification algorithm tries to apply the following steps (the order is not important), to transform G:

delete : $G \cup \{\, t \stackrel{?}{=} t \,\}$ is transformed to G

decompose : $G \cup \{\, f(t_1, ..., t_m) \stackrel{?}{=} f(u_1, ..., u_m) \,\}$ is transformed to $G \cup \{\, t_1 \stackrel{?}{=} u_1, ..., t_m \stackrel{?}{=} u_m \,\}$

swap : $G \cup \{\, f(t_1, ..., t_m) \stackrel{?}{=} x \,\}$ is transformed to $G \cup \{\, x \stackrel{?}{=} f(t_1, ..., t_m) \,\}$

eliminate : $G \cup \{\, x \stackrel{?}{=} t \,\}$ is transformed to $G[x = t] \cup \{\, x \stackrel{?}{=} t \,\}$ if $x \in \mathrm{vars}(G) \wedge x \notin \mathrm{vars}(t)$

conflict : $G \cup \{\, f(t_1, ..., t_m) \stackrel{?}{=} g(u_1, ..., u_h) \,\}$ leads to failure if $f \neq g \vee m \neq h$

occur chek : $G \cup \{\, x \stackrel{?}{=} f(t_1, ..., t_m) \,\}$ leads to failure if $x \in \mathrm{vars}(\, f(t_1, ..., t_m)\,)$

For example, if we start from

$$G = \{\, plus(succ(x), x) \stackrel{?}{=} plus(y, 0) \,\}$$

by applying rule **decompose** we obtain

$$\{\, succ(x) \stackrel{?}{=} y\,,\ x \stackrel{?}{=} 0 \,\}$$

by applying rule **eliminate** we obtain

$$\{\, succ(0) \stackrel{?}{=} y\,,\ x \stackrel{?}{=} 0 \,\}$$

finally, by applying rule **swap** we obtain

$$\{\, y \stackrel{?}{=} succ(0)\,,\ x \stackrel{?}{=} 0 \,\}$$

Since not further transformation is possible, we conclude that

$$\tau = [\, y = succ(0)\,,\ x = 0 \,]$$

is the most general unifier for G