DATA MINING 2
Time Series – Matrix Profile, Motifs & Discords

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Slides edited from Keogh Eamonn’s tutorial
Time Series Motif Discovery

• Finding repeated patterns, i.e., pattern mining.
• Are there any repeated patterns, of length $m$ in the TS?
Why Finding Motifs?

• Mining **association rules** in TS requires the discovery of motifs. These are referred to as primitive shapes and frequent patterns.

• Several **TS classifiers** work by constructing typical prototypes of each class. These prototypes may be considered motifs.

• Many **TS anomaly detection** algorithms consist of modeling normal behavior with a set of typical shapes (which we see as motifs) and detecting future patterns that are dissimilar to all typical shapes.
How do we find Motifs?

• Given a predefined motif length $m$, a brute-force method searches for motifs from all possible comparisons of subsequences.

• It is obviously very slow and computationally expensive.

• The most referenced algorithm is based on a hot idea from bioinformatics, random projection* and the fact that SAX allows use to lower bound discrete representations of TSs.

Example of the Motif Discovery Algorithm

- Assume that we have a time series T of length 1,000, and a motif of length 16, which occurs twice, at time $T_1$ and time $T_{58}$.  

\[ \text{a c b a} \]

$S = \{a, b, c\}$ alphabet

- $n = 16$ motif length
- $w = 4$ sax window
Example of the Motif Discovery Algorithm

- A mask \{1,2\} was randomly chosen, so the values in columns \{1,2\} were used to project matrix into buckets.
- Collisions are recorded by incrementing the appropriate location in the collision matrix.
Example of the Motif Discovery Algorithm

- A mask \{2, 4\} was randomly chosen, so the values in columns \{2, 4\} were used to project matrix into buckets.
- Once again, collisions are recorded by incrementing the appropriate location in the collision matrix.
Example of the Motif Discovery Algorithm

• At the end of the random perturbations consider the motifs observing the matrix in decreasing order of occurrences.

• For instance, this matrix indicates a high chance of having a motif staring at positions 1 and 58.

• The problem with this approach is that it is highly dependent from the approximation technique adopted.
Matrix Profile

• The Matrix Profile (MP) is a data structure that annotates a TS and can be exploited for many purposed: e.g. efficient Motif Discovery.

• Given a time series, T and a desired subsequence length, m.
We can use sliding window of length $m$ to extract all subsequences of length $m$. 
Matrix Profile

We can then compute the pairwise distance among these subsequences.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>7.6952</th>
<th>7.7399</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.6952</td>
<td>0</td>
<td>7.7106</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>7.7399</td>
<td>7.7106</td>
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<tr>
<td>...</td>
<td>...</td>
<td>...</td>
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<td>...</td>
</tr>
</tbody>
</table>

Matrix Profile

- For each subsequence we keep only the distance with the closest nearest neighbor.

<table>
<thead>
<tr>
<th>Set of all subsequences</th>
<th>Set of corresponding nearest neighbor</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>6.04</td>
</tr>
<tr>
<td></td>
<td>2.88</td>
</tr>
<tr>
<td></td>
<td>0.90</td>
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<tr>
<td></td>
<td>1.61</td>
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<td></td>
<td>5.69</td>
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<tr>
<td></td>
<td>1.23</td>
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<tr>
<td></td>
<td>1.40</td>
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<td>(</td>
<td>T</td>
</tr>
</tbody>
</table>

\(\ldots\)
Matrix Profile

• The distance to the corresponding nearest neighbor of each subsequence can be stored in a vector called matrix profile $P$.

The matrix profile value at location $i$ is the distance between $T_i$ and its nearest neighbor.
Matrix Profile

- The index of corresponding nearest neighbor of each subsequence is also stored in a vector called matrix profile index.

The matrix profile value at location $i$ is the distance between $T_i$ and its nearest neighbor.

It turns out that $T_i$'s nearest neighbor is $T_{194}$. 
Matrix Profile

- The MP index allows to find the nearest neighbor to any subsequence in constant time.
- Note that the pointers in the matrix profile index are not necessarily symmetric.
- If A points to B, then B may or may not point to A
- The classic TS motif: the two smallest values in the MP must have the same value, and their pointers must be mutual.
How to “read” a Matrix Profile

• For relatively low values, you know that the subsequence in the original TS must have (at least one) relatively similar subsequence elsewhere in the data (such regions are “motifs”)

• For relatively high values, you know that the subsequence in the original TS must be unique in its shape (such areas are anomalies).
How to Compute Matrix Profile?

• Given a time series, $T$ and a desired subsequence length, $m$.

Matrix profile is initialized as inf vector

This is just a toy example, so the values and the vector length does not fit the time series shown above
How to Compute Matrix Profile?

- Given a time series, \( T \) and a desired subsequence length, \( m \).

At the first iteration, a subsequence \( T_i \) is randomly selected from \( T \).
How to Compute Matrix Profile?

- Given a time series, $T$ and a desired subsequence length, $m$.

We compute the distances between $T_i$ and every subsequence from $T$ (time complexity = $O(|T| \log(|T|)))$. We then put the distances in a vector based on the position of the subsequences.

$$
\begin{array}{cccccccccccccccc}
\text{inf} & \text{inf} & \text{inf} & \text{inf} & \text{inf} & \text{inf} & \text{inf} & \text{inf} & \text{inf} & \text{inf} & \text{inf} & \text{inf} & \text{inf} & \text{inf} & \text{inf} & \text{inf} \\
3 & 2 & 0 & 5 & 3 & 4 & 5 & 1 & 2 & 9 & 8 & 4 & 2 & 3 & 4 & 8 & 6 & 2 & 1
\end{array}
$$

The distance between $T_i$ and $T_1$ (first subsequence) is 3.
How to Compute Matrix Profile?

• Given a time series, $T$ and a desired subsequence length, $m$.

Let's say $T_i$ happen to be the third subsequences, therefore the third value in the distance vector is 0.

We compute the distances between $T_i$ and every subsequences from $T$ (time complexity = $O(|T|\log(|T|)))$.

We then put the distances in a vector based on the position of the subsequences:

$\inf\ \inf\ \inf\ \inf\ \inf\ \inf\ \inf\ \inf\ \inf\ \inf\ \inf\ \inf\ \inf\ \inf\ \inf\ \inf$
How to Compute Matrix Profile?

- Given a time series, $T$ and a desired subsequence length, $m$.

Matrix profile is updated by applying element-wise minimum to these two vectors:

\[
\begin{array}{cccccccccccccccc}
\text{inf} & \text{inf} & \text{inf} & \text{inf} & \text{inf} & \text{inf} & \text{inf} & \text{inf} & \text{inf} & \text{inf} & \text{inf} & \text{inf} & \text{inf} & \text{inf} & \text{inf} & \text{inf} \\
3 & 2 & 0 & 5 & 3 & 4 & 5 & 1 & 2 & 9 & 8 & 4 & 2 & 3 & 4 & 8 & 6 & 2 & 1
\end{array}
\]
How to Compute Matrix Profile?

• Given a time series, $T$ and a desired subsequence length, $m$.

Matrix profile is updated by apply elementwise minimum to these two vectors.

<table>
<thead>
<tr>
<th>3</th>
<th>inf</th>
<th>inf</th>
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</tr>
</thead>
</table>

| 3 | 2 | 0 | 5 | 3 | 4 | 5 | 1 | 2 | 9 | 8 | 4 | 2 | 3 | 4 | 8 | 6 | 2 | 1 |
How to Compute Matrix Profile?

- Given a time series, $T$ and a desired subsequence length, $m$.

After we finish update matrix profile for the first iteration

3 2 inf 5 3 4 5 1 2 9 8 4 2 3 4 8 6 2 1

3 2 0 5 3 4 5 1 2 9 8 4 2 3 4 8 6 2 1
How to Compute Matrix Profile?

• Given a time series, $T$ and a desired subsequence length, $m$.

In the second iteration, we randomly select another subsequence $T_j$ and it happens to be the 12\textsuperscript{th} subsequences
How to Compute Matrix Profile?

• Given a time series, \( T \) and a desired subsequence length, \( m \).

\[
\begin{array}{c}
3 & 2 & \text{inf} & 5 & 3 & 4 & 5 & 1 & 2 & 9 & 8 & 4 & 2 & 3 & 4 & 8 & 6 & 2 & 1 \\
\hline
m
\end{array}
\]

Once again, we compute the distance between \( T_j \) and every subsequences of \( T \)

\[
\begin{array}{c}
2 & 3 & 1 & 4 & 4 & 3 & 6 & 2 & 1 & 5 & 8 & 0 & 2 & 3 & 5 & 9 & 4 & 2 & 2 \\
\end{array}
\]
How to Compute Matrix Profile?

• Given a time series, $T$ and a desired subsequence length, $m$.

![Diagram of a time series with a sliding window of length $m$ and a computation of the minimum value within each window.](image)

<table>
<thead>
<tr>
<th>3</th>
<th>2</th>
<th>inf</th>
<th>5</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>1</th>
<th>2</th>
<th>9</th>
<th>8</th>
<th>4</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>8</th>
<th>6</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
</table>

$T_j$

min

The same elementwise minimum

| 2 | 3 | 1 | 4 | 4 | 3 | 6 | 2 | 1 | 5 | 8 | 0 | 2 | 3 | 5 | 9 | 4 | 2 | 2 |
How to Compute Matrix Profile?

• Given a time series, $T$ and a desired subsequence length, $m$.

2 2 inf 5 3 4 5 1 2 9 8 4 2 3 4 8 6 2 1

min

$\sqrt{7}$

The same elementwise minimum

2 3 1 4 4 3 6 2 1 5 8 0 2 3 5 9 4 2 2
How to Compute Matrix Profile?

• Given a time series, $T$ and a desired subsequence length, $m$.

| 2 | 2 | inf | 5 | 3 | 4 | 5 | 1 | 2 | 9 | 8 | 4 | 2 | 3 | 4 | 8 | 6 | 2 | 1 |

The same elementwise minimum

| 2 | 3 | 1 | 4 | 4 | 3 | 6 | 2 | 1 | 5 | 8 | 0 | 2 | 3 | 5 | 9 | 4 | 2 | 2 |
How to Compute Matrix Profile?

• Given a time series, $T$ and a desired subsequence length, $m$.

\[
\begin{array}{cccccccccccccccc}
2 & 2 & 1 & 5 & 3 & 4 & 5 & 1 & 2 & 9 & 8 & 4 & 2 & 3 & 4 & 8 & 6 & 2 & 1 \\
\end{array}
\]

The same elementwise minimum

\[
\begin{array}{cccccccccccccccc}
2 & 3 & 1 & 4 & 4 & 3 & 6 & 2 & 1 & 5 & 8 & 0 & 2 & 3 & 5 & 9 & 4 & 2 & 2 \\
\end{array}
\]
How to Compute Matrix Profile?

- Given a time series, \( T \) and a desired subsequence length, \( m \).

We repeat the two steps (distance computation and update) until we have used every subsequences. The different indexes are analyzed in parallel and the distance is calculated using the Mueen’s Algorithm for Similarity Search (MASS) https://www.cs.unm.edu/~mueen/FastestSimilaritySearch.html
How to Compute Matrix Profile?

• Given a time series, $T$ and a desired subsequence length, $m$.

There are $|T|$ subsequences and the distance computation is $O(|T|\log(|T|))$

The overall time complexity is $O(|T|^2\log(|T|))$
Motif Discovery From Matrix Profile

time series, \( T \)

matrix profile, \( P \)

Local minimums are corresponding to motifs
Motif Discovery From Matrix Profile

• It is sometime useful to think of time series subsequences as points in m-dimensional space.

• In this view, dense regions in the m-dimensional space correspond to regions of the time series that have a low corresponding MP.
• We need a parameter $R$.
• $1 < R < \text{(small number, say 3)}$
• Let's make $R = 2$ for now.
• We begin by finding the nearest pair of points, the *motif pair*....
• This the pair of subsequences corresponding to lowest pair of values in the MP
We find the nearest pair of points are D1 apart.

Let's draw a circle, D1 times R, around both points.

Any points that are within either of these circles, are added to this motif, in this case just one.

The Top-1 motif has three members, it is done.
Now let's find the Top-2 motif. We find the nearest pair of points, excluding anything from the top motif.

The nearest pair of points are D2 apart.

Let's draw a circle D2 times R, around both points.

Any points that are within either of these circles, is added to this motif, in this case there are two for a total of four items in the Top-2 Motif.
Top-K Motifs

- We are done with the Top-2 Motif
- Note that we will always have:
  - $D_1 < D_2 < D_3 \ldots D_K$
- **When to stop?** (what is $K$?)
- We could use MDL or a predefined $K$. 
Anomaly Discovery From Matrix Profile

• We need a parameter E of subsequences to exclude in the vicinity of the anomaly.
• Let's make E = 2 for now.
• We begin by finding the subsequence with the highest distance in the MP.
• This corresponding to biggest anomaly.
Top-K Anomaly

• Then we look for the $E$ closest subsequences to the anomaly.
• We remove all of them.
• We can use a predefined $K$ or the MDL to stop.
References

References


• Compression-based data mining of sequential data. Eamonn Keogh et al. 2007.
Exercises Matrix Profile
Matrix Profile

Given the TS $x = <2,1,3,4,7,5,3,4,7,5>$

1. Build the Matrix Profile for $x$ with $m=4$ using the Manhattan distance as distance function between subsequences.

2. Draw the Matrix Profile

3. Identify the motifs with distance equals 0 and length equals to $m$

4. Which is a correct value for $m$ that would have retrieved more motifs with distance equals to 0?
<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>1</th>
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<th>4</th>
<th>7</th>
<th>5</th>
<th>3</th>
<th>4</th>
<th>7</th>
<th>5</th>
</tr>
</thead>
<tbody>
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\[ m = 4 \]
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m = 4
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<td>7</td>
<td>5</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

\[ m = 4 \]
m = 4
\begin{tabular}{cccc}
inf & 7 & 9 & 11 & 9 \\
\end{tabular}

\[ m = 4 \]
<table>
<thead>
<tr>
<th>inf</th>
<th>7</th>
<th>9</th>
<th>11</th>
<th>9</th>
<th>9</th>
<th>9</th>
</tr>
</thead>
</table>

\[ m = 4 \]
\[
\begin{array}{cccccccc}
2 & 1 & 3 & 4 & 7 & 5 & 3 & 4 & 7 & 5 \\
2 & 1 & 3 & 4 & 7 & 5 & 3 & 4 & 7 & 5
\end{array}
\]

\[
\begin{array}{ccccccc}
\text{inf} & 7 & 9 & 11 & 9 & 9 & 9
\end{array}
\]

\[m = 4\]
<table>
<thead>
<tr>
<th>inf</th>
<th>7</th>
<th>9</th>
<th>11</th>
<th>9</th>
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<th>9</th>
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<tr>
<td>7</td>
<td></td>
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</tr>
</tbody>
</table>

\[m = 4\]
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m = 4
\]
\[
\begin{array}{ccccccccc}
2 & 1 & 3 & 4 & 7 & 5 & 3 & 4 & 7 & 5 \\
2 & 1 & 3 & 4 & 7 & 5 & 3 & 4 & 7 & 5
\end{array}
\]

\[
\begin{array}{cccccccc}
\text{inf} & 7 & 9 & 11 & 9 & 9 & 9 \\
7 & \text{inf} & 8 & 12 & 12 \\
\end{array}
\]

\(m = 4\)
\[
\begin{array}{cccccccc}
2 & 1 & 3 & 4 & 7 & 5 & 3 & 4 & 7 & 5 \\
2 & 1 & 3 & 4 & 7 & 5 & 3 & 4 & 7 & 5 \\
\end{array}
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<table>
<thead>
<tr>
<th>inf</th>
<th>7</th>
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<tbody>
<tr>
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<td>inf</td>
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<td>12</td>
<td>12</td>
<td>4</td>
<td>8</td>
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</tbody>
</table>

\[m = 4\]
\[
\begin{array}{cccccccc}
2 & 1 & 3 & 4 & 7 & 5 & 3 & 4 & 7 & 5 \\
2 & 1 & 3 & 4 & 7 & 5 & 3 & 4 & 7 & 5 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
\text{inf} & 7 & 9 & 11 & 9 & 9 & 9 & 9 \\
7 & \text{inf} & 8 & 12 & 12 & 4 & 8 \\
9 & 10 & \text{inf} & 8 & 9 & 8 & 0 \\
\end{array}
\]

\[
m = 4
\]
Matrix Profile

- \( x = \langle 2, 1, 3, 4, 7, 5, 3, 4, 7, 5 \rangle \)
- \( mp = \langle 7, 4, 0, 8, 8, 4, 0 \rangle \)
Matrix Profile

- $x = <2, 1, 3, 4, 7, 5, 3, 4, 7, 5>$
- $mp = <7, 4, 0, 8, 8, 4, 0>$
Matrix Profile

• $x = <2, 1, 3, 4, 7, 5, 3, 4, 7, 5>$
• $m_p = < 4, 4, 0, 0, 5, 4, 0, 0 >$

$m=3$
Matrix Profile

- $x = <2, 1, 3, 4, 7, 5, 3, 4, 7, 5>$
- $mp = <4, 4, 0, 0, 5, 4, 0, 0>$

$m=3$
Matrix Profile

Given the TS $x = <5, 5, 3, 5, 5, 1>$

1. Build the Matrix Profile for $x$ with $m=2$ using the Manhattan distance as distance function between subsequences.

2. Draw the Matrix Profile

3. Identify the motifs with distance equals 0 and length equals to $m$